Florian Cajori

AN

ELEMENTARY TREATISE

ON

PLANE AND SPHERICAL

TRIGONOMETRY,

AND ON THE

APPLICATION OF ALGEBRA TO GEOMETRY;

FROM THE

MATHEMATICS OF LACROIX AND BÉZOUT.

TRANSLATED FROM THE FRENCH

FOR THE USE OF THE STUDENTS OF THE UNIVERSITY

AT

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ADVERTISEMENT.

Of the following work, the Treatise on *Plane and Spherical Trigonometry* is from Lacroix's Course of Mathematics. The new division of the circle being adopted in the original, a few alterations have been found necessary to adapt it to the sexagesimal notation in use in this country. Where there has been occasion to add any thing on this account, or to supply any thing by way of illustration, it is given in the form of a note, and the reference is made by an obelisk, the author's being always distinguished by an asterisk.

The chapter on the *Application of Algebra to Geometry* is selected from the Algebra of Bézout. It was the intention of the compiler to have made use of the more improved treatise of Lacroix or that of Biot upon this subject; but as analytical geometry has hitherto made no part of the mathematics taught in the public seminaries of the United States, and as only a small portion of time is allotted to such studies, and this in many instances at an age not sufficiently mature for inquiries of an abstract nature, it was thought best to make the experiment with a treatise distinguished for its simplicity and plainness. The original being prepared for the use of the Marine and Artillery, those parts have been suppressed, which were not adapted to the purpose of general instruction. Where it was apprehended that the student would meet with any difficulty in the course of an investigation, new steps have been supplied, and references are often made to theorems and processes, applicable to the case in question; the figures also, particularly those relating to conic sections, have been simplified. Moreover, such alterations have been introduced, as were found necessary to make this treatise conform to the other parts of the course of mathematics compiled for the use of the Students of the University.

Cambridge, March 1820.

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ON

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CHAPTER I.

Of Plane Trigonometry.

1. In a plane triangle there are six things to be considered, namely, three angles and three sides. But it is sufficient to know a certain number of these in order to determine the rest. It follows, indeed, from what has been proved with respect to equal triangles, that we may always construct a triangle, when we know three of the six parts, which constitute it, provided that one at least of these three parts be a side. In order to render the theory of triangles complete, we must be able to apply the calculus to geometrical figures, the exactness of which is limited by the imperfection of instruments, while there is nothing to prevent the calculus being carried to any degree of precision we choose. Such is the object which we propose to ourselves in *Plane Trigonometry.*

Those, who first undertook to develop, by a series of numerical operations, or by algebraic formulas, the relations which subsist between the different parts of a triangle, must have found themselves embarrassed by the difficulty of introducing into the calculus the magnitude of angles, which, being measured by the arcs of a circle, cannot be compared with right lines; but they must have soon perceived, that if they could, by any means whatever, calculate a series of triangles, the angles of which should be of all possible values, this series would necessarily contain a triangle similar to the one to be determined, whatever it might be; and that the parts of this last might be deduced from those of the former by a simple proportion. This will be rendered plainer by the following example.

Fig. 1. 2. I suppose that in the triangle ABC (*fig.* 1), we know the angle B, the angle C, and the side BC, we find in the series of computed triangles that, which has two angles, b and c, respectively equal to the angles B and C; it will necessarily be similar (*Geom.* 203) to the proposed triangle ABC; and since all the parts, $a\,b$, $a\,c$, $b\,c$, are known, we have the proportions,

$$b\,c : a\,b :: BC : AB, \quad b\,c : a\,c :: BC : AC,$$

in each of which the first three terms are given, whence we obtain

$$AB = \frac{BC \times a\,b}{b\,c}, \quad AC = \frac{BC \times a\,c}{b\,c};$$

and since from all other considerations we deduce $A = a$, all the parts of the triangle ABC are determined.

3. Now that we know the use which may be made of a series of triangles having all possible angles, and the sides of which are calculated, we are led to inquire into the method of constructing such a series. To take the most simple case first, I will suppose that the triangles to be determined are right-angled; it is evident that they may all be formed in the quadrant of a circle by letting
Fig. 2. fall from each of the points of the arc AB (*fig.* 2), perpendiculars MP, $M'P'$, $M''P''$, &c., upon the radius AC, and drawing the radii MC, $M'C$, $M''C$, &c., the triangles, MPC, $M'P'C'$, $M''P''C$, &c., thus formed, are right-angled at P, P', P'', &c., and the angles, MCP, $M'CP'$, $M''CP''$, &c., have successively all possible values; the angles, CMP, $CM'P'$, $CM''P''$, &c., which, with the preceding, make a right angle (*Geom.* 62), will be such, as is required by the nature of right-angled triangles, and there cannot be a right-angled triangle, which is not equiangular with some one of those furnished by a table, constructed as above described. It may be remarked, that these triangles have each the same hypothenuse which is equal to the radius of the arc AB.

4. We may form also a series of right-angled triangles, each having one of the sides comprehending the right angle equal to the radius of the circle; it is sufficient, for this purpose, to raise the indefinite tangent AT from the extremity of the radius AC, and to draw from the centre C, through the points, M, M', M'',

&c., the secants CN, CN', CN'', &c. It is evident, that the triangles CAN, CAN', CAN'', &c. must have all the combinations of angles, which can exist in a right-angled triangle, and among these triangles there will necessarily be found one similar to any right-angled triangle that can be proposed.

5. In the triangles CPM, $CP'M'$, $CP''M''$, &c. the hypothenuse of which does not change, the sides PM, $P'M'$, $P'M''$, &c., which increase with the angles ACM, ACM', ACM'', &c., or with the arcs AM, AM', AM'', &c. which measure these angles, have received a name on account of this dependence; the line PM is called the *sine* of the arc AM, the line $P'M'$, is also the sine of the arc AM', and so of the others. It follows from this, that *the sine of an arc is the perpendicular let fall, from one extremity of this arc upon the radius, which passes through the other extremity.* The lines CP, CP', CP'', &c. which diminish, as the arcs AM, AM', AM'', &c. increase, are respectively equal, being parallels comprehended between parallels to the perpendiculars MQ, $M'Q'$, $M''Q''$, &c. let fall from the points M, M', M'', &c. upon the radius CB, perpendicular to the radius CA; and it is evident that the lines MQ, $M'Q'$, $M''Q''$, &c., are, with respect to the arcs BM, BM', BM'', &c., what PM, $P'M'$, $P''M''$, &c. are, with respect to the arcs AM, AM', AM'', &c., and that, consequently, MQ is the sine of BM, $M'Q'$ of BM', and $M''Q''$ of BM'', &c.

Two arcs, the sum or difference of which is the fourth part of the circumference of a circle are called *complements* the one of the other. The arcs BM, BM', BM'', &c. are respectively the complements of AM, AM', AM'', &c. We designate the lines MQ, $M'Q'$, $M''Q''$, &c., as well as their equals CP, CP', CP'', &c. under the name of *cosines* of the arcs AM, AM', AM'', &c. Whence *the cosine of an arc is the sine of the complement of this arc, and is equal to that part of the radius comprehended between the centre and the foot of the sine.*

The right-angled triangles CPM, $CP'M$, $CP''M'$, &c., which have all the same hypothenuse, are formed, therefore, by the radius of the circle, and the sine and cosine of the acute angle, which has its vertex at the centre.*

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* The part AP of the radius AC, comprehended between the foot

6. I pass to the triangles CAN, CAN', CAN'', &c. The hypothenuses of these are the *secants* of the arcs AM, AM', AM'', &c., since *we call the secant of an arc the radius drawn through one extremity of this arc and produced till it meets the tangent, drawn through the other extremity.* The portions AN, AN', AN'', &c., taken upon the tangent AT, are the tangents of the arcs AM, AM', AM'', &c., since we call *the tangent of an arc, that part which is intercepted, on the tangent drawn through one extremity of this arc, by the two radii which terminate it.**

Fig. 3. 7. If, through the extremity B of the arc AB (*fig.* 3), we draw the tangent $B\,n$, and produce it till it meets the secant CN, the line $C\,n$ is the secant of the arc BM, the complement of AM, and is called the *cosecant* of AM; the line $B\,n$, the tangent of BM, is the *cotangent* of AM, since we understand by the *cotangent and cosecant of an arc the tangent and secant of the complement of this arc.* The cotangent and tangent, and the cosecant and secant do not respectively make a part of the same triangle, as we have observed with respect to the sine and cosine.

8. Tangents and secants have with sines and cosines relations, that are very simple, by means of which the one may be found from the other. The triangles CPM and CAN being similar, give $CP : PM :: CA : AN$; whence $AN = \frac{PM \times CA}{CP}$; putting, instead of the lines CP, PM, AN, what they denote, namely, $\cos AM$, $\sin AM$, and $\text{tang}\, AM$, and expressing the radius CA by R, we have $\text{tang}\, AM = \frac{R \sin AM}{\cos AM}$.

From the same triangles CPM and CAN, we deduce also, $CP : CM :: CA : CN$; whence $CN = \frac{CM \times CA}{CP}$; but

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of the sine and the extremity of the arc is called the *versed sine.* This line, however, is not used in trigonometry.

* It will be perceived, that the words *secant* and *tangent* are here taken in a different sense, from what they are in the *Elements of Geometry*, where they are considered, as indefinite right lines, one of which cuts the circle and the other touches it. But in trigonometry these terms are always used to denote lines of a determinate magnitude. Where any doubt might otherwise exist, the latter are called *trigonometrical secants and tangents.*

$$CN = \sec AM,\ CM = CA = R,\ CP = \cos AM;$$

whence $\sec AM = \frac{R^2}{\cos AM}$.

9. If we compare the triangles CAN and CBn, which are also similar, since they are both right-angled, and the angle $ACN = CnB$ (*Geom.* 60), we have the proportion

$$AN : CA :: CB \text{ or } CA : Bn,$$

which gives

$$Bn = \frac{\overline{CA}^2}{AN}, \text{ which becomes } \cot AM = \frac{R^2}{\text{tang } AM}.$$

From this proportion and that for the secant it is evident, that *the radius is a mean proportional between the secant and the cosine, and between the tangent and the cotangent;* since

$$\cos AM \times \sec AM = R^2, \quad \text{tang } AM \times \cot AM = R^2.$$

10. With what precedes, we shall require, in order to be able to construct the tables necessary in trigonometry, merely the methods of calculating the sines and cosines; and even the cosine is deduced immediately from the sine, for the right-angled triangle CPM, which contains them, and the hypothenuse of which is radius, gives

$\overline{PM}^2 + \overline{CP}^2 = \overline{CM}^2$ (*Geom.* 186), or $(\sin AM)^2 + (\cos AM)^2 = R^2$, that is, *the square of radius is equal to the sum of the squares of the sine and cosine;* whence

$$\cos AM = \sqrt{R^2 - (\sin AM)^2}.$$

The following proposition, which gives the expression of the sine and cosine of the sum and of the difference of two arcs, merits the greatest attention, as it involves all the properties of sines and cosines.

11. *Let there be any two arcs* a *and* b; *we have*

$$\sin(\text{a} \pm \text{b}) = \frac{\sin \text{a}. \cos \text{b} \pm \sin \text{b}. \cos \text{a}}{R},$$

$$\cos(\text{a} \pm \text{b}) = \frac{\cos \text{a}. \cos \text{b} \mp \sin \text{a}. \sin \text{b}}{R}.$$

In order to demonstrate this, I take in the circle AMB', (*fig.* 4), Fig. 4.
the arc $AM = a$; and on each side of the point M, the arcs MN and MN', each equal to b; I draw the chord NN'; from the points N, M, N', I let fall upon the radius AC, the perpendiculars NQ, MP, $N'Q'$; through the point M, I draw the radius MC, and from the point E, where it meets the chord NN', I let

fall upon AC the perpendicular EF; through the points E and N', I draw the right lines ED, $N'G$, parallel to AC.

This being done, I remark 1. that NQ is the sine of the arc $AN = AM + MN = a + b$, and that CQ is the cosine of the same arc; 2. that $N'Q'$ is the sine of the arc

$$AN' = AM - MN' = a - b,$$

and that CQ' is the cosine. But the chord NN' being divided into two equal parts in the point E (*Geom.* 106), by the radius CM, which, by construction, bisects the arc NN', we infer from the similar triangles NED, $NN'G$, that NG is also divided into two equal parts in the point D, and that $DN = DG$. Moreover, $DQ = EF$, $GQ = N'Q'$, $DE = FQ$; and as DE is half of $N'G$, FQ will be half of $Q'Q$; so that $Q'F = QF = DE$. Lastly,

$$NQ = DQ + DN = EF + DN,$$
$$N'Q' = GQ = DQ - DG = EF - DN,$$
$$CQ = CF - FQ = CF - DE,$$
$$CQ' = CF + FQ' = CF + DE;$$

putting for NQ, $N'Q'$, CQ, CQ', what they respectively denote, namely,

$$\sin(a+b),\ \sin(a-b),\ \cos(a+b),\ \cos(a-b),$$

I have

$$\sin(a+b) = EF + DN,\ \cos(a+b) = CF - DE,$$
$$\sin(a-b) = EF - DN,\ \cos(a-b) = CF + DE.$$

It only remains to determine the four lines, EF, CF, DN, and DE. The first two are obtained by the similar triangles CMP and CEF, which give

$$CM : PM :: CE : EF, \qquad CM : CP :: CE : CF.$$

Now, since $AM = a$, I have $PM = \sin a$, $CP = \cos a$; it follows also from the definition of a sine and cosine (5), that EN is the sine of arc MN, that CE is the cosine of it, and that, consequently, $EN = \sin b$, $CE = \cos b$; also $CM = R$. Substituting these values in the above proportions, we have

$$EF = \frac{PM \times CE}{CM} = \frac{\sin a \cos b}{R},$$

$$CF = \frac{CP \times CE}{CM} = \frac{\cos a \cos b}{R}.$$

I next compare the triangles CPM, DEN, which are similar, because the sides of the second are respectively perpendicular to those of the first (*Geom.* 209), and I deduce from them

$$CM : EN :: CP : DN, \quad CM : EN :: PM : DE.$$

substituting, in the first three terms of each of these proportions, what they respectively denote, as above stated, we obtain

$$DN = \frac{EN \times CP}{CM} = \frac{\sin b \cos a}{R},$$

$$DE = \frac{PM \times EN}{CM} = \frac{\sin a \sin b}{R}.$$

Uniting these values to the preceding, in order to form those of $\sin(a+b)$ and $\sin(a-b)$, we have the four equations,

$$\begin{cases} \sin(a+b) = \dfrac{\sin a \cos b + \sin b \cos a}{R}, \\ \sin(a-b) = \dfrac{\sin a \cos b - \sin b \cos a}{R}, \end{cases}$$

$$\begin{cases} \cos(a+b) = \dfrac{\cos a \cos b - \sin a \sin b}{R}, \\ \cos(a-b) = \dfrac{\cos a \cos b + \sin a \sin b}{R}, \end{cases}$$

which may be reduced to the two, that compose the enunciation of the proposition.

With these equations, we may find the sine and cosine of an arc, which is double, triple, or any multiple of the arc, whose sine and cosine are given. Indeed, if we take, successively, $b = a$, $b = 2\,a$, we have

$$\begin{cases} \sin 2\,a = \dfrac{2 \sin a \cos a}{R}, \\ \cos 2\,a = \dfrac{\cos a^2 - \sin a^2}{R}, \end{cases}$$

$$\begin{cases} \sin 3\,a = \dfrac{\sin a \cos 2\,a + \sin 2\,a \cos a}{R}, \\ \cos 3\,a = \dfrac{\cos a \cos 2\,a - \sin a \sin 2\,a}{R}; \end{cases}$$

and we may deduce from the last two equations the $\sin 3\,a$ and $\cos 3\,a$, when $\sin 2\,a$ and $\cos 2\,a$ are calculated.

12. The equation $\sin 2\,a = \dfrac{2 \sin a \cos a}{R}$, leads also from the side of an arc a to the expression of the sine of half of this arc. If we substitute for the $\cos a$ its value $\sqrt{R^2 - \sin a^2}$*, the equa-

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* I would apprise the learner, that hereafter I shall designate the square of the sine of an arc a by $\sin a^2$, an expression which would otherwise be understood as the sine of the square of the arc a, thus $\sin a^2 = (\sin a)^2$.

tion becomes

$$\sin 2a = \frac{2 \sin a \sqrt{R^2 - \sin a^2}}{R},$$

and, by raising this to the second power, we find

$$R^2 \sin 2a^2 = 4 R^2 \sin a^2 - 4 \sin a^4;$$

taking $\sin a$ for the unknown quantity in this equation, which may be resolved after the manner of those of the second degree, we obtain

$$\sin a = \pm \sqrt{\tfrac{1}{2} R^2 \pm \tfrac{1}{2} R \sqrt{R^2 - \sin 2a^2}}.$$

If we make $2a = a'$, we have $a = \frac{1}{2} a'$, and consequently

$$\sin \tfrac{1}{2} a' = \pm \sqrt{\tfrac{1}{2} R^2 \pm \tfrac{1}{2} R \sqrt{R^2 - \sin a'^2}},$$

or $$\sin \tfrac{1}{2} a' = \pm \tfrac{1}{2} \sqrt{2 R^2 \pm 2 R \cos a'},$$

putting $\cos a'^2$ instead of $R^2 - \sin a'^2$ (10), multiplying the quantities under the radical by 4, and dividing that without by 2, which does not alter the value of the expression. Such is the formula which gives the sine of half an arc, when that of the whole arc is known.

13. We may arrive at the same result by a very simple construction.

Fig. 5. If we divide the arc AM (*fig.* 5) into two equal parts, the chord AQM will also be divided into two equal parts, and QM will be the sine of MN, or of the half of AM; the triangle AMP, right-angled at P, will give

$$AM = \sqrt{\overline{PM}^2 + \overline{AP}^2};$$

and, as $AP = AC - CP = R - \cos AM = R \cos a'$, and $PM = \sin AM = \sin a'$, we have

$$AM = \sqrt{\sin a'^2 + R^2 - 2R \cos a' + \cos a'^2} = \sqrt{2R^2 - 2R \cos a'},$$

since $\sin a'^2 + \cos a'^2 = R^2$ (10); hence

$$QM = \tfrac{1}{2} AQM = \tfrac{1}{2} \sqrt{2 R^2 - 2 R \cos a'}.$$

We find in this manner only the second value of $\sin \frac{1}{2} a'$; the other is MQ'; for the arc $MN'A'$, which with the arc AM makes the half circumference, has PM also for its sine, since this line is in fact a perpendicular let fall from the extremity M upon the radius CA', which passes through the other extremity (5); and since there is nothing, in the equation from which we set out, to show, which of these two arcs it is proposed to divide, we ought to find the sine of the half of each. According to the construction we have

$$A'M = \sqrt{\overline{PM}^2 + \overline{A'P}^2} = \sqrt{\overline{PM}^2 + (A'C + CP)^2}$$
$$= \sqrt{\sin a'^2 + (R + \cos a')^2}$$
$$= \sqrt{\sin a'^2 + R^2 + 2R \cos a' + \cos a'^2}$$
$$= \sqrt{2R^2 + 2R\cos a'},$$

and consequently,

$$MQ' = \sin \tfrac{1}{2} a' = \tfrac{1}{2}\sqrt{2R^2 + 2R\cos a'},$$

a result, which is the first value of $\sin \frac{1}{2} a'$.

It should be carefully observed, that although $\sin a'$ is the same in the two values of $\sin \frac{1}{2}a'$ the arc a' is different. For one of these the arc is AM, and for the other $A'M$, which is the supplement of AM, for *we understand by the supplement of an arc or angle, that which must be added to this arc or angle in order to make two right angles or a half circumference.* We infer, therefore, from what precedes, that *the sine of the supplement of an arc is the same as the sine of this arc.* I shall hereafter treat of the different arcs, which may have the same sine, the same tangent, &c.

14. It follows from what has been said, that the sine of any arc AN is half of the chord AM of the double arc ANM, and that the chord AM is double the sine of the arc AN, which is half of ANM; so that when the sines are known, the chords may be deduced from them, and the reverse.

15. It is not the absolute values of the sines, which we have occasion to calculate, but only the ratio they have to radius; since it is sufficient to know in the triangles CPM, $CP'M'$, &c. (*fig.* 2), the ratio which the sides have amongst themselves. We may, therefore, on account of the greater simplicity, consider radius as unity, and express the sines, PM, $P'M'$, &c. in decimal parts of unity, or, as was formerly done, suppose radius divided into 100 000 parts. Fig. 2.

16. It may be observed, that the length of an arc is always less than that of its tangent, and greater than that of its sine. Indeed, if we take below the radius AC (*fig.* 6), the arc $AM' = AM$, and draw the chord MM' and the tangents MT, $M'T$, it is easy to see, that these tangents must both meet the radius AC in the same point, since the triangles CMT and $CM'T$ are equal. The lines MT and $M'T$ being equal, as well as the lines PM and PM', and the arcs AM and AM', we have Fig. 6.

$2\,AM < 2\,MT$ and $2\,AM > 2\,PM$ (*Geom.* 283). Whence we conclude that $AM < MT$, $AM > PM$.

I will further remark, in this place, that the ratio between the tangent and the sine of an arc tends continually toward unity, according as the arc diminishes; indeed from $\text{tang}\, a = \frac{\sin a}{\cos a}$ we deduce $\frac{\sin a}{\text{tang}\, a} = \cos a$; and as $\cos a$ approaches continually towards unity, it follows that the tangent and sine approach also more and more nearly to an equality, since the *limit* (*Alg.* 234) of their ratio is unity.

17. From this it may clearly be inferred, that if the value of the tangent and that of the sine do not differ for a certain number of figures, these same figures will express also the approximate value of the arc. Taking, for example, $PM = 0{,}0001$, we find

$$CP = \sqrt{\overline{CM}^2 - \overline{PM}^2} = 0{,}999\ 999\ 995,$$

and $$MT = \frac{CM \times PM}{CP} = 0{,}000\ 100\ 000\ 000\ 5,$$

a value which does not differ from PM, except in the thirteenth figure; we may then take this number for the value of the arc AM expressed in parts of radius.*

18. In order to apply the formulas of art. 12, 11, and 10, we must know at least the sine of one of the arcs comprehended in the fourth part of a circle. Now there are two of these arcs the sines of which are easily known, namely, the quadrant and its third part. *Indeed the sine of a quadrant is simply radius, and the sine of a third part of a quadrant is equal to half of radius.*

Fig. 2. The first of these values is evident from inspection (*fig.* 2);

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* The same thing may be proved by reducing the expression of the tangent to a series. Indeed we have

$$\text{tang}\, a = \frac{\sin a}{\cos a} = \frac{\sin a}{\sqrt{1 - \sin a^2}} = \frac{\sin a}{1 - \frac{1}{2}\sin a^2 - \frac{1}{8}\sin a^4 - \&c.} =$$
$$\sin a + \tfrac{1}{2}\sin a^3 + \tfrac{3}{8}\sin a^5 + \&c.$$

It is evident, that while $\sin a$ is a small decimal fraction, the term $\frac{1}{2}\sin a^3$ can affect only the last figures in the expression of tang a, and that for the first we have $\text{tang}\, a = \sin a$.

For $\sin a = 0{,}0001$, we have $\frac{1}{2}\sin a^3 = 0{,}000\ 000\ 000\ 000\ 5$, result which can affect only the thirteenth figure.

and the second results from this, that the side of an inscribed hexagon, or, what amounts to the same thing, the chord of two thirds of a quadrant, (*fig.* 8) is equal to radius (*Geom.* 271); the half of this chord then will be the sine of one third of a quadrant (14). Fig. 8.

Beginning with an entire quadrant, the formula

$$\sin \tfrac{1}{2}\, a' = \tfrac{1}{2}\sqrt{2R^2 - 2R\cos a'}$$

gives the sine of the half, then that of the half of this half, or of the fourth of a quadrant, and thus, successively, of all the fractions of this arc comprehended in the series

$$\tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \tfrac{1}{32},\ \&c.$$

The same formula, if we set out with the third of a quadrant, leads in like manner to the sine of

$$\tfrac{1}{6},\ \tfrac{1}{12},\ \tfrac{1}{24},\ \tfrac{1}{48},\ \tfrac{1}{96},\ \&c.$$

of this arc.

We see by this, that if an arc were divided into a number of parts equal to any one of the denominators of the above fractions, we might find directly the sine of each of these parts, and thus form a table by writing them against the arcs, to which they would belong. But we have not proceeded thus. It is only among the Indian astronomers, it seems, that the quadrant is divided into twentyfour parts for the purpose of calculating the sines. Very ancient usage, together with other reasons, has led to the adoption of a different division from the progressions above given.

19. The custom has been to divide the entire circumference into 360 parts, called *degrees*, and each degree into 60 parts, called *minutes*, and each minute into 60 parts, called *seconds*, and each second into 60 parts, called *thirds*, &c. The character used to denote degrees is $^\circ$, placed on the right of the number and a little above it, that for minutes is $'$, that for seconds is $''$, that for thirds is $'''$, &c., so that $42^\circ\ 31'\ 14''\ 5'''$ signifies 42 degrees, 31 minutes, 14 seconds and 5 thirds.

Since, in the measure of angles, we have no regard to the absolute value of the arcs, but only to their ratio with an entire circumference, it would seem very natural to consider this circumference as unity, and to express the arcs by fractions, either decimals or others. Still there are certain considerations, which induced those who were charged with the reformation of the system of weights and measures, to take the right angle for the unit of angles, and, consequently, the fourth part of a circle, or quadrant, for the unit of arcs. This they divided into a hundred equal parts, which they called degrees, and which they substituted in the place of the ancient degrees; each of these degrees was divided into a hundred equal parts, which took the place of minutes; these last were to be subdivided, as far as occasion might require, according to the decimal progression.*

20. The radius of the circle, upon which it is proposed to construct the tables, being 1, and the circumference being denot-
Fig. 9. ed by 2π, the sine of AB (*fig.* 9), or $\sin \frac{1}{2}\pi = 1$; we have, besides, $\cos \frac{1}{2}\pi = 0$; taking therefore $a' = \frac{1}{2}\pi$, from the formula $\sin \frac{1}{2} a' = \frac{1}{2}\sqrt{2R^2 - 2R\cos a'}$ (13), it will be seen, that the sine of half a quadrant, or of $\frac{1}{4}\pi$, is $\frac{1}{2}\sqrt{2}$.†

The arc $AB = \frac{1}{2}\pi$ being taken for unity, AM will be $0^q,5$; we have then

$$\sin 0^q,5 = \cos 0^q,5 = \tfrac{1}{2}\sqrt{2} = 0,707106781186.$$

Now if we make $0^q,5 = a'$ we find

$$\sin \tfrac{1}{2} a' = \sin 0^q,25 = 0,382683432365,$$
$$\cos \tfrac{1}{2} a' = \cos 0^q,25 = 0,923879532511;$$

But by continuing thus to divide each arc into two equal parts, we do not fall upon any of the decimal parts of a quadrant; we only arrive at arcs, that become smaller and smaller, and which accordingly approach continually to an equality with their sines. At the fourteenth division, for example, we come to an arc, which is only $\frac{1}{16384}$ of a quadrant, the sine of which is 0,000 095 873 799, less consequently than 0,0001; the smallness of this arc

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* The principal reasons for selecting the right angle as the unit seem to be, 1. that the entire circle does not, properly speaking,
Fig. 2. measure an angle, since the movable radius CM (*fig.* 2) in this case returns to a coincidence with CA; 2. that the sine, to which all trigonometrical lines are referred, takes in the fourth part of a circle, or right angle, all the values of which it is susceptible.

† This may be demonstrated *a priori*, since the triangle CMP,
Fig. 9. (*fig.* 9), which is isosceles, gives

$$2\,\overline{PM}^2 = \overline{CM}^2 = 1,$$

whence

$$\overline{PM}^2 = \tfrac{1}{2}, \text{ and } PM = \sqrt{\tfrac{1}{2}} = \sqrt{2 \times \tfrac{1}{4}} = \tfrac{1}{2}\sqrt{2}.$$

is then such, that it does not differ from its sine in the first twelve decimal figures.

In smaller arcs the difference will be still less; and it is evident, that all arcs, which are confounded with their sines and tangents, are proportional to these lines; therefore,

$$\sin \frac{1^q}{16384} : \sin \frac{1^q}{100000} :: \frac{1^q}{16384} : \frac{1^q}{100000},$$

$$\text{or} :: 100000 : 16384,$$

whence

$$\sin 0^q,00001 = \frac{16384 \sin \frac{1^q}{16384}}{100000} = 0{,}000\ 015\ 707\ 963,$$

at least in the first twelve decimal figures.* For the same reason

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* A similar proportion may be used in forming a table for the ancient division of the circle. Since

$$1'' = \frac{1^q}{90.60.60} = \frac{1^q}{324000} < \frac{1^q}{100000}$$

$$\sin \frac{1^q}{16384} : \sin \frac{1^q}{324000} :: \frac{1^q}{16384} : \frac{1^q}{324000} :: 324000 : 16384;$$

whence

$$\sin 1'' = \frac{16384 \sin \frac{1^q}{16384}}{324000} = 0{,}000\ 004\ 848\ 136,$$

at least for the first twelve decimal figures; and multiples of this may be taken for the sine of 2″, 3″, &c., subject to the limitation pointed out in the text. But since, in the case of such small arcs, the approximate value of the sine given in most tables does not differ from that of the arc itself, the above results may be obtained more simply thus. The circumference of a circle, when the diameter is 1, being 3,141592653, this number will express the semicircumference when the diameter is 2, or radius 1. Hence,

$$\sin 0^q,00001 = \frac{\frac{1}{2}\pi}{100000} = \frac{\frac{1}{2}3{,}141592653}{100000} = 0{,}000\ 015\ 707\ 963,$$

so also,

$$\sin 1'' = \frac{\frac{1}{2}\pi}{324000} = \frac{3{,}141592623}{648000} = 0{,}000\ 004\ 848\ 136,$$

as obtained above.

It is usual in the *computation of the canon*, as this operation for constructing a table of sines, cosines, &c., is called, to begin with the sine of 1′. As the value even of this does not differ from that of the arc in the first ten decimal figures, it may be found in a similar man-

$\sin 0^q{,}00002 = 2 \sin 0^q{,}00001$
$\sin 0^q{,}00003 = 3 \sin 0^q{,}00001$
$\sin 0^q{,}00004 = 4 \sin 0^q{,}00001$
&c.

If we take care at the same time to calculate the cosine and tangent of each of these arcs, it is evident, that we may proceed in this manner, so long as the arc has its sine and tangent still confounded in the first twelve decimal figures.

If we would obtain the approximate values to the eighth decimal only, we may extend this process to the arc $0^q{,}001$.

In order afterwards to rise to greater arcs, we may make use of the equation

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ner by calculating the length of the arc, that is, by dividing the semicircumference 3,141592653 by 180.60, which gives 0,0002908882 for sine 1′. Then

$$\cos 1' = \sqrt{R^2 - (\sin 1')^2} = \sqrt{1 - \overline{0{,}0002908882}^2} = 0{,}99999996$$
(10).

The operation may be continued for the sexagesimal division, as indicated in the text for the centesimal. It may be observed, moreover, that as the sine of any arc is the cosine of its complement and the reverse, the sines and cosines being computed from 1′ to 30°, we have also the sines and cosines from 60° to 90°.

The sines of arcs between 30° and 60° may be found by simple subtraction. If we add the equation

$$\sin (a + b) = \sin a \cos b + \sin b \cos a$$

to the equation

$$\sin (a - b), = \sin a \cos b - \sin b \cos a,$$

and from each member of the sum

$$\sin (a + b) + \sin (a - b) = 2 \sin a \cos b$$

subtract $+ \sin (a - b)$,
we have $\sin (a + b) = 2 \sin a \cos b - \sin (a - b)$;
then putting $a = 30°$, and $b = 1', 2', 3'$, &c., successively, we obtain

$$\sin (30°\ 1') = \cos 1' - \sin (29°\ 59')$$
$$\sin (30°\ 2') = \cos 2' - \sin (29°\ 58')$$
$$\sin (30°\ 3') = \cos 3' - \sin (29°\ 57'),$$
&c.

The sines of arcs from 30° to 60° being calculated, they will be the cosines of arcs from 60° to 30°.

$$\sin 2a = 2 \sin a \cos a,$$
$$\cos 2a = \cos a^2 - \sin a^2,$$
$$\sin (a \pm b) = \sin a \cos b \pm \sin b \cos a,$$
$$\cos (a \pm b) = \cos a \cos b \mp \sin a \sin b;$$

making, successively, $a = 0^q,001$, $a = 0^q,002$, &c., in the first two of these equations, we deduce from them

$$\sin 0^q,002, \cos 0^q,002, \sin 0^q,004, \cos 0^q,004, \&c.,$$

and taking $a = 0^q,001$, $b = 0^q,002$, $a = 0^q,002$, $b = 0^q,003$, &c., we obtain by means of the last two equations

$$\sin 0^q,003, \cos 0^q,003, \sin 0^q,005, \cos 0^q,005, \&c.$$

It will be perceived from what has been said, how a set of trigonometrical tables may be formed. There are other methods, more convenient for calculating the sines of any arcs whatever, by means of converging series, which are deduced from the equations of art. 11. They may be found in the introduction to my *Treatise on the Differential and Integral Calculus.*

21. To render the calculation more easy, the custom has been for a long time to use logarithms instead of the values of the sines, cosines, tangents, and cotangents; and in most tables the latter are not to be found. The questions, therefore, which present themselves, are of the following nature.

1. *An arc being given, to find the logarithm of its sine, that of its cosine, tangent, or cotangent.*

2. *The lorgarithm of the sine, that of the cosine, tangent, or cotangent of an arc being known, to find this arc.*

In solving these questions, regard must be had to the particular disposition of the tables, that are used, as they are not all alike, and each is usually accompanied with the necessary directions. I shall omit, therefore, giving the instruction, that the student may want on this subject. I will merely mention the tables of Callet, as the best for the ancient division, and those of Borda, or those of Hobert and Ideler for the new.

Trigonometrical tables extend only to the fourth part of a circle; but they give, notwithstanding, the sines and cosines, the tangents and cotangents, for all arcs however great. This I shall now proceed to show, by tracing the progress of the trigonometrical lines with respect to the different degrees of magnitude, which an arc of a circle is capable of assuming.

In order to comprehend fully what I am about to offer, we must first understand the continuity, which prevails among the

different results obtained from the same algebraic expression, or from the same geometrical construction, and which consists in this, that each value, which the expression in question assumes, is preceded and followed by values, which differ as little as we please, from the first, and that in describing a line, each point is preceded and followed by points, which are immediately contiguous.
Fig. 10. This being supposed, if we conceive the radius MC (*fig.* 10), at first coinciding with AC, to turn about the point C, as upon a pivot, this radius will form, successively, with AC, all possible angles; and the point M, situated at the extremity, will pass over all the points of the circumference of the circle $ABA'B'A$, or, which is the same thing, will describe it. By following with attention this motion, we see in the first place, that at the point A, where the arc is nothing, the sine is also nothing, and the cosine does not differ from the radius AC. When the radius CM moves off from AC, the sine PM increases, as the point M, which I shall call the *describing* point, advances toward B, and when it has reached it, PM becomes equal to CB, or to radius. Under the same circumstances, the cosine PC diminishes continually and becomes nothing, when the point M is in B; the angle ACB is then a right angle, and the arc $AB = \frac{1}{2}\pi$. The point M being continued beyond B, the sine decreases, and the cosine, which falls now upon the diameter AA' on the side of the point C opposite to that in which it was before, increases. This is evident from the figure; $P'M'$, the sine of ABM', is less than BC, the sine of AB, and CP', the cosine of the first of these arcs, exceeds the cosine of the second, which is nothing. It may be remarked, that $P'M'$ and CP' are, respectively, the sine and cosine of the arc $A'M'$, counted from A', and the supplement of ABM'; whence it follows, that *an obtuse angle has the same sine and the same cosine, as its supplement.*

22. When the point M has arrived at A', the sine is nothing, as at the point A, and the cosine is again equal to radius. At the point A' the arc ABA' is equal to the semicircumference π; the angle ACM has attained its greatest magnitude, but there is nothing to prevent the radius CM and the describing point, being continued below the diameter AA'. The sine, which then becomes $P''M''$, falls also below the diameter, and increases, according as the point M'' approaches to B', while the cosine CP'' diminishes. At the point B', where the arc $ABA'B'$ is $\frac{3}{4}$

of the circumference, or $\frac{3}{2}\pi$, the sine is equal to the radius CB' and the cosine is nothing. Lastly, from B' to A the sine $B'''M'''$, constantly below AA', diminishes continually, and the cosine, CP''', which is now on the same side of C, that it was in the first quadrant AB, increases and becomes equal to radius in A. At this point the sine is nothing; the describing point has completed a revolution, but we may suppose it to begin another, and by considering, as a single arc, the whole course passed over by this point from the commencement of its motion, we have arcs that exceed a circumference and which have the same sines, cosines, tangents, cotangents, as those which are described in the first revolution. These considerations lead to consequences, that are of the greatest importance in analysis, and which I have developed in my treatise on the *Differential and Integral Calculus.*

23. It may be well now to see how the algebraic expressions for the sine and cosine correspond with the different circumstances, which we have been considering. In order to this, I make, in the first place, $a = \frac{1}{2}\pi$ in the equations

$$\left.\begin{array}{l}\cos\,(a \pm b) = \cos a \cos b \mp \sin a \sin b \\ \sin\,(a \pm b) = \sin a \cos b \pm \sin b \cos a\end{array}\right\} \dots\dots (A).$$

Recollecting that the $\cos \frac{1}{2}\pi = 0$, and that the $\sin \frac{1}{2}\pi = 1$, we have

$$\cos\,(\tfrac{1}{2}\pi \pm b) = \mp \sin b,$$
$$\sin\,(\tfrac{1}{2}\pi \pm b) = \cos b.$$

There are two things to be attended to in these expressions, namely, their absolute value, and the sign with which it is affected.

This value is verified by the figure; for AB being $\frac{1}{2}\pi$, if we take the arc BM' for b, the arc AM' will be $\frac{1}{2}\pi + b$; but $P'M'$ being the sine of $A'M'$, as well as of AM', will be the cosine of BM' or of b; while CP' will be the sine of b.

As to the sign —, which affects $\cos\,(\frac{1}{2}\pi + b)$, it signifies, that if we regard, as positive, the sine and cosine of an arc less than the fourth of the circumference, the cosine of a greater arc will be negative, while its sine will be positive. If we make $b = \frac{1}{2}\pi$, we have $\cos \pi = -1$, $\sin \pi = 0$.

Again, if we suppose, that in the equations (A), $a = \pi$, we shall obtain, according to what precedes,

$$\cos\,(\pi \pm b) = -\cos b,$$
$$\sin\,(\pi \pm b) = \mp \sin b.$$

The absolute value of these formulas may be verified, as easily as that of the preceding; the sign shows, that every arc comprehended between π and $\frac{3}{2}\pi$, has its sine and cosine negative; and when $b = \frac{1}{2}\pi$, we have

$$\cos \tfrac{3}{2}\pi = 0, \qquad \sin \tfrac{3}{2}\pi = -1.$$

Lastly, when $a = \frac{3}{2}\pi$, the equations (A) are reduced by means of the values just found to

$$\cos (\tfrac{3}{2}\pi \pm b) = \pm \sin b,$$
$$\sin (\tfrac{3}{2}\pi \pm b) = - \cos b,$$

from which it follows, that every arc comprehended between $\frac{3}{2}\pi$ and $\frac{4}{2}\pi$, or 2π, has its cosine positive and its sine negative.

The results, then, to which we have arrived, are

1. That from the point A to the point A', at which the arc $ABA' = \pi$, the sines are positive;

2. That from the point A' to the point A, at which the arc $ABA'B'A = 2\pi$, that is from π to 2π, the sines are negative;

3. That from the point A to the point B, at which the arc $AB = \frac{1}{2}\pi$, the cosines are positive;

4. That from the point B to the point B', at which the arc $ABA'B' = \frac{3}{2}\pi$, that is, from $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$, the cosines are negative;

5. Lastly, that from the point B' to the point A, at which the arc $ABA'B'A = 2\pi$, that is, from $\frac{3}{2}\pi$ to 2π, the cosines are positive.

It will be readily observed, that the sines change their sign, when they pass below the diameter AA', and the cosines, when they pass from one side to the other of the point C, or according as they fall on this side or that of the diameter BB' perpendicular to AA'.

By attending to these things we shall be able to extend the formulas of art. 11 to all possible magnitudes of the arcs AM and MN (*fig.* 4); and the values deduced from these formulas will agree with those which are derived from the construction and reasoning employed in the article referred to, if we apply them immediately to the proposed arcs. The application of these formulas would be a useful exercise for the learner.

Fig. 4.

24. By following the course of the tangents we find, that they increase continually from the point A (*fig.* 10), to the point B, at which the arc AM becomes equal to $\frac{1}{2}\pi$. At this point the secant NC, coinciding with CB, is parallel to the tangent AN, and therefore no longer meets it; so that the arc AB has not,

Fig. 10.

properly speaking, a trigonometrical tangent. We say, nevertheless, that its tangent is infinite; but the meaning of this expression is, that by taking the point M sufficiently near to the point B, we may make the tangent AN greater than any assignable quantity. It is in this manner, that we show the truth of the equation $\text{tang } a = \frac{\sin a}{\cos a}$, which gives for tang a a value, so much the greater, as cos a becomes smaller, or as we approach nearer to the point B.

When $a = 0^q{,}5$ it becomes $\cos a = \sin a$ (20), and, consequently, $\text{tang } 0^q{,}5 = 1$.

This may be shown also by the triangle CAN (*fig.* 9), which becomes isosceles in this case, since the angle ACN, being equal to half a right angle, is necessarily equal to the angle ANC; the tangent AN is then equal to radius (*Geom.* 48). Fig. 9.

When the arc AM (*fig.* 10), is greater than $\frac{1}{2}\pi$, the radius CM will no longer meet the line AN above the diameter but below it. The true tangent AN' is equal, as may be easily shown, to $A'n'$, the tangent of the arc $A'M'$, the supplement of AM', but it lies in an opposite direction. In the third quarter of the circle the tangent which has nothing at the point A', returns above the diameter AA', and AN is the tangent of $AA'M''$. The radius becomes again parallel to AN at the point B', and the tangent is infinite; beyond this point it falls below the diameter, and the arc $AA'M'''$ has AN' for its tangent. Fig. 10.

25. I proceed now to point out what results from the algebraic expression, $\text{tang } a = \frac{\sin a}{\cos a}$.

It is evident, that its value will be positive in all those cases, where the sine and cosine have the same sign, or from 0 to $\frac{1}{2}\pi$, and from π to $\frac{3}{2}\pi$; it will, consequently, be negative from $\frac{1}{2}\pi$ to π, and from $\frac{3}{2}\pi$ to 2π; whence it follows, that for the tangents, as well as for the sines and cosines, a change of sign corresponds to a change of situation; we find likewise, that the cotangents are positive from 0 to $\frac{1}{2}\pi$, from π to $\frac{3}{2}\pi$; and negative from $\frac{1}{2}\pi$ to π, and from $\frac{3}{2}\pi$ to 2π.

26. We sometimes meet with negative arcs in a calculation, the sines and cosines of which may be determined by the formulas of art. 11. As the expression $\sin(a - b)$ changes its sign, when we change a into b, and b into a, it is manifest, that

$$\sin(b - a) = -\sin(a - b);$$

thus, when $a > b$, the negative arc $b - a$ has a negative sign.

If we construct fig. 4* on this supposition by taking $AM = b$, $MN = a$, and carrying this last arc below the point M, in order to represent the operation to be performed according to art. 11, the arc AN will be found below AC instead of being above it; the sine $Q'N$ then will change its direction, as well as the arc. As to the cosine, it will remain in the same direction; and we find also by the formulas, that $\cos(b - a) = \cos(a - b)$.

27. There are many other conclusions to be drawn from the proposition demonstrated in art. 11, some of which will be necessary in the subsequent part of this treatise; I will therefore put them down in this place.

1. By adding together the two equations

$$\sin(a + b) = \frac{\sin a \cos b + \sin b \cos a}{R},$$

$$\sin(a - b) = \frac{\sin a \cos b - \sin b \cos a}{R},$$

we have

$$\sin(a + b) + \sin(a - b) = \frac{2 \sin a \cos b}{R},$$

whence

$$\sin a \cos b = \frac{R}{2} \sin(a + b) + \frac{R}{2} \sin(a - b).$$

2. By subtracting the second equation from the first, we obtain

$$\sin(a + b) - \sin(a - b) = \frac{2 \sin b \cos a}{R},$$

whence

$$\sin b \cos a = \frac{R}{2} \sin(a + b) - \frac{R}{2} \sin(a - b).$$

When $a = b$, this formula and the preceding give

$$\cos a \sin a = \frac{R}{2} \sin 2a.$$

3. By adding the two equations

$$\cos(a + b) = \frac{\cos a \cos b - \sin a \sin b}{R},$$

$$\cos(a - b) = \frac{\cos a \cos b + \sin a \sin b}{R},$$

we have

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$$\cos(a+b)+\cos(a-b)=\frac{2\cos a\cos b}{R},$$

whence

$$\cos a\cos b=\frac{R}{2}\cos(a+b)+\frac{R}{2}\cos(a-b).$$

When $a=b$, this formula gives

$$\cos a^2=\frac{R}{2}\cos 2a+\frac{R^2}{2}=\tfrac{1}{2}R(R+\cos 2a),$$

it being recollected, that the cosine is equal to radius, when the arc is nothing.

4. By subtracting the first equation from the second, the result becomes

$$\cos(a-b)-\cos(a+b=\frac{2\sin a\sin b}{R},$$

whence

$$\sin a\sin b=\frac{R}{2}\cos(a-b)-\frac{R}{2}\cos(a+b).$$

When $a=b$, this formula gives

$$\sin a^2=\frac{R^2}{2}-\frac{R}{2}\cos 2a=\tfrac{1}{2}R(R-\cos 2a).$$

5. If we make $a+b=a'$, $a-b=b'$, we find, by adding these two equations, $2a=a'+b'$, and by subtracting the second from the first, $2b=a'-b'$; it follows from this that

$$a=\frac{a'+b'}{2},\qquad b=\frac{a'-b'}{2}.$$

Putting these values of a and b in the expression for $\sin a\cos b$, $\sin b\cos a$, $\cos a\cos b$, $\sin a\sin b$, obtained above, we find

$$\sin\tfrac{1}{2}(a'+b')\cos\tfrac{1}{2}(a'-b')=\frac{R}{2}(\sin a'+\sin b')$$

$$\cos\tfrac{1}{2}(a'+b')\sin\tfrac{1}{2}(a'-b')=\frac{R}{2}(\sin a'-\sin b')$$

$$\cos\tfrac{1}{2}(a'+b')\cos\tfrac{1}{2}(a'-b')=\frac{R}{2}(\cos a'+\cos b')$$

$$\sin\tfrac{1}{2}(a'+b')\sin\tfrac{1}{2}(a'-b')=\frac{R}{2}(\cos b'-\cos a').$$

Dividing the second of these formulas by the first, we have

$$\frac{\cos\frac{1}{2}(a'+b')\sin\frac{1}{2}(a'-b')}{\sin\frac{1}{2}(a'+b')\cos\frac{1}{2}(a'-b')}=\frac{\sin a'-\sin b'}{\sin a'+\sin b'},$$

or

$$\frac{\sin\frac{1}{2}(a'-b')}{\cos\frac{1}{2}(a'-b')}\times\frac{\cos\frac{1}{2}(a'+b')}{\sin\frac{1}{2}(a'+b')}=\frac{\sin a'-\sin b'}{\sin a'+\sin b'}.$$

Observing then that $\frac{\sin A}{\cos A}=\frac{\text{tang}\,A}{R}$ (8), and that, consequently,

$$\frac{\cos A}{\sin A} = \frac{R}{\tang A},$$

we obtain

$$\frac{\tang \frac{1}{2}(a' - b')}{\tang \frac{1}{2}(a' + b')} = \frac{\sin a' - \sin b'}{\sin a' + \sin b'}.$$

We infer in like manner from the last two formulas above given, that

$$\frac{\cos b' - \cos a'}{\cos a' + \cos b'} = \frac{\tang \frac{1}{2}(a' + b') \tang \frac{1}{2}(a' - b')}{R^2}.$$

6. By dividing the expression for the sin $(a \pm b)$ by that for the cos $(a \pm b)$, we have

$$\frac{\sin (a \pm b)}{\cos (a \pm b)} = \frac{\sin a \cos b \pm \sin b \cos a}{\cos a \cos b \mp \sin a \sin b};$$

then, dividing the numerator and denominator of the second member by cos a cos b, it becomes

$$\frac{\frac{\sin a}{\cos a} \pm \frac{\sin b}{\cos b}}{1 \mp \frac{\sin a}{\cos a} \cdot \frac{\sin b}{\cos b}};$$

and since, universally, $\frac{\sin A}{\cos A} = \frac{\tang A}{R}$ (8), we hence deduce

$$\frac{\tang (a \pm b)}{R} = \frac{\frac{\tang a}{R} \pm \frac{\tang b}{R}}{1 \mp \frac{\tang a}{R} \cdot \frac{\tang b}{R}} = \frac{R(\tang a \pm \tang b)}{R^2 \mp \tang a \tang b};$$

and, lastly,

$$\tang (a \pm b) = \frac{R^2(\tang a \pm \tang b)}{R^2 \mp \tang a \tang b}.$$

And since $\cot A = \frac{R^2}{\tang A}$ (9), we have

$$\cot (a \pm b) = \frac{R^2}{\tang (a \pm b)};$$

hence, dividing R^2 by tang $(a \pm b)$ and by its equal in the above equation, we obtain

$$\frac{R^2}{\tang (a \pm b)} = \frac{R^2 \mp \tang a \tang b}{\tang a \pm \tang b} =$$

$$\frac{R^2 \mp \frac{R^2}{\cot a} \cdot \frac{R^2}{\cot b}}{\frac{R^2}{\cot a} \pm \frac{R^2}{\cot b}};$$

which being reduced becomes

$$\cot(a \pm b) = \frac{\cot a \cot b \mp R^2}{\cot b \pm \cot a}.$$

28. The equation $\frac{\text{tang} \frac{1}{2}(a' - b')}{\text{tang} \frac{1}{2}(a' + b')} = \frac{\sin a' - \sin b'}{\sin a' + \sin b'}$, from which we infer, that *the sum of the sines of two arcs is to their difference, as the tangent of half the sum of these arcs is to the tangent of half their difference,* is obtained immediately by a very elegant geometrical construction.

AM and AN (*fig.* 11), being two arcs represented by a' and b', we have $MP = \sin a'$, $NQ = \sin b'$; drawing NC parallel to the diameter AB, and producing MP to M', we deduce Fig. 11.

$$MR = MP - NQ = \sin a' - \sin b',$$
$$M'R = M'P + NQ = \sin a' + \sin b' \ (14).$$

This being done, if from the point C, as a centre, and with a radius CD equal to that of the circle ACB, we describe an arc EDG, and draw, through the point D of this arc, a tangent meeting the straight lines CM and CM', it is evident, that DF and DH will be the tangents of the arcs DE and DG, which measure the angles MCN, NCM'; and as these angles have their vertex in the circumference of the circle ACB, they will have for their measure, respectively, (*Geom.* 126),

$$\tfrac{1}{2} NM = \tfrac{1}{2}(AM - AN) = \tfrac{1}{2}(a' - b'),$$
$$\tfrac{1}{2} NM' = \tfrac{1}{2}(AM' + AN) = \tfrac{1}{2}(a' + b');$$

we have then

$$DF = \text{tang} \tfrac{1}{2}(a' - b'), \quad DH = \text{tang} \tfrac{1}{2}(a' + b').$$

But on account of the parallels MM' and FH, we have this proportion,

$$MR : M'R :: DF : DH,$$

that is,

$$\sin a' - \sin b' : \sin a' + \sin b' :: \text{tang} \tfrac{1}{2}(a' - b') : \text{tang} \tfrac{1}{2}(a' + b'),$$

which is the same as the equation above given.

It would be easy to modify the construction, so as to deduce from it the different equations analogous to that just demonstrated.

29. As we have often occasion to make use of the formulas, which we have already obtained, I have put them together in the following table with others, which may be deduced by a process easy to be imagined. The number against each formula, marks the article, in which it may be found, or from which it may be obtained.

Trigonometrical Formulas.

$$\sin a^2 + \cos a^2 = R^2 \quad (10)$$

$$\left.\begin{aligned}\sin(a \pm b) &= \frac{\sin a \cos b \pm \sin b \cos a}{R} \\ \cos(a \pm b) &= \frac{\cos a \cos b \mp \sin a \sin b}{R}\end{aligned}\right\} (11)$$

$$\left.\begin{aligned}\sin a \cos b &= \tfrac{1}{2} R\,[\sin(a+b) + \sin(a-b)] \\ \cos a \sin b &= \tfrac{1}{2} R\,[\sin(a+b) - \sin(a-b)] \\ \cos a \cos b &= \tfrac{1}{2} R\,[\cos(a+b) + \cos(a-b)] \\ \sin a \sin b &= -\tfrac{1}{2} R\,[\cos(a+b) - \cos(a-b)]\end{aligned}\right\} (27)$$

$$\left.\begin{aligned}\sin a + \sin b &= \frac{2}{R} \sin\tfrac{1}{2}(a+b) \cos\tfrac{1}{2}(a-b) \\ \sin a - \sin b &= \frac{2}{R} \cos\tfrac{1}{2}(a+b) \sin\tfrac{1}{2}(a-b) \\ \cos a + \cos b &= \frac{2}{R} \cos\tfrac{1}{2}(a+b) \cos\tfrac{1}{2}(a-b) \\ \cos a - \cos b &= -\frac{2}{R} \sin\tfrac{1}{2}(a+b) \sin\tfrac{1}{2}(a-b)\end{aligned}\right\} (27)$$

$$\sin 2a = \frac{2 \sin a \cos a}{R} (11), \sin\tfrac{1}{2}a = \tfrac{1}{2}\sqrt{2R^2 - 2R\cos a} \quad (13)$$

$$\cos 2a = \frac{\cos a^2 - \sin a^2}{R} = \frac{2\cos a^2 - R^2}{R} \quad (11)$$

$$\sin a^2 = \tfrac{1}{2} R\,(R - \cos 2a) \quad (27)$$

$$\cos a^2 = \tfrac{1}{2} R\,(R + \cos 2a) \quad (27)$$

$$\sin a^2 - \sin b^2 = \cos b^2 - \cos a^2 = \sin(a+b)\sin(a-b) (11, 10)$$

$$\cos a^2 - \sin b^2 = \cos(a+b)\cos(a-b) \; (11, 10)$$

$$\operatorname{tang} a = \frac{R \sin a}{\cos a} \; (8), \; \cot a = \frac{R^2}{\operatorname{tang} a} = \frac{R \cos a}{\sin a} \; (9)$$

$$\sec a = \frac{R^2}{\cos a}, \; \operatorname{cosec} a = \frac{R^2}{\sin a} \; (8)$$

$$\operatorname{tang}(a \pm b) = \frac{R \sin(a \pm b)}{\cos(a \pm b)} = \frac{R^2(\operatorname{tang} a \pm \operatorname{tang} b)}{R^2 \mp \operatorname{tang} a \operatorname{tang} b} \; (27)$$

Trigonometrical Formulas.

$$\left.\begin{aligned}
\text{tang}\, a + \text{tang}\, b &= \frac{R^2 \sin(a+b)}{\cos a \cos b} \\
\text{tang}\, a - \text{tang}\, b &= \frac{R^2 \sin(a-b)}{\cos a \cos b} \\
\cot a + \cot b &= \frac{R^2 \sin(a+b)}{\sin a \sin b} \\
\cot a - \cot b &= -\frac{R^2 \sin(a-b)}{\sin a \sin b}
\end{aligned}\right\} (8, 11)$$

$$\left.\begin{aligned}
\text{tang}\, a^2 - \text{tang}\, b^2 &= \frac{R^4 \sin(a+b)\sin(a-b)}{\cos a^2 \cos b^2} \\
\cot a^2 - \cot b^2 &= -\frac{R^4 \sin(a+b)\sin(a-b)}{\sin a^2 \sin b^2}
\end{aligned}\right\} (8, 11)$$

$$\left.\begin{aligned}
\frac{\sin a + \sin b}{\sin a - \sin b} &= \frac{\text{tang}\, \frac{1}{2}(a+b)}{\text{tang}\, \frac{1}{2}(a-b)} \\
\frac{\sin a + \sin b}{\cos a + \cos b} &= \frac{\text{tang}\, \frac{1}{2}(a+b)}{R}, \quad \frac{\sin a}{R + \cos a} = \frac{\text{tang}\, \frac{1}{2} a}{R} \\
\frac{\sin a + \sin b}{\cos a - \cos b} &= -\frac{\cot \frac{1}{2}(a-b)}{R}, \quad \frac{\sin a}{R - \cos a} = \frac{\cot \frac{1}{2} a}{R} \\
\frac{\sin a - \sin b}{\cos a + \cos b} &= \frac{\text{tang}\, \frac{1}{2}(a-b)}{R} \\
\frac{\sin a - \sin b}{\cos a - \cos b} &= -\frac{\cot \frac{1}{2}(a+b)}{R} \\
\frac{\cos a + \cos b}{\cos a - \cos b} &= -\frac{\cot \frac{1}{2}(a-b)}{\text{tang}\, \frac{1}{2}(a+b)} = -\frac{\sec a + \sec b}{\sec a - \sec b}
\end{aligned}\right\} (27)$$

$$\sin a = \frac{R\, \text{tang}\, a}{\sqrt{R^2 + \text{tang}\, a^2}}, \quad \cos a = \frac{R^2}{\sqrt{R^2 + \text{tang}\, a^2}} \quad (8, 10)$$

$$R = \sin 1^q = \cos 0^q = \text{tang}\, \tfrac{1^q}{2} = \cot \tfrac{1^q}{2} = \sec 0^q = \text{cosec}\, 1^q$$

$$= \tfrac{1}{2} \sec \tfrac{2^q}{3} \quad (23, 24)$$

$$\sin a = \tfrac{1}{2}\, \text{chord}\, 2a \quad (14)$$

$$\left.\begin{aligned}
\sin(1^q \pm b) &= +\cos b, & \cos(1^q \pm b) &= \mp \sin b \\
\sin(2^q \pm b) &= \mp \sin b, & \cos(2^q \pm b) &= -\cos b \\
\sin(3^q \pm b) &= -\cos b, & \cos(3^q \pm b) &= \pm \sin b \\
\sin(4^q \pm b) &= \pm \sin b, & \cos(4^q \pm b) &= +\cos b
\end{aligned}\right\} (23)$$

30. I proceed now to treat of the application of trigonometrical tables to the resolution of triangles. It must be recollected, that, by means of these tables, when an angle is known, the value of its sine, that of its cosine, tangent, and cotangent are also known, and that, reciprocally, when the value of one of these lines is given, that of the arc is to be regarded as given.

Fig. 12. Let CDE (*fig.* 12) be a triangle, right-angled at D; from one of the acute angles C, we describe, with a radius equal to that of the tables, the arc AM, and let fall the perpendicular PM upon AC; we then raise the tangent AN in order to form the two triangles of the tables, namely, CPM, which will be that of the sine and cosine, and CAN, that of the tangent and secant. These will be each similar to the triangle proposed; and by comparing them successively with this, we obtain the following proportions;

$$\left.\begin{array}{l} CM : PM :: CE : DE \\ CM : CP :: CE : CD \\ CA : AN :: CD : DE \end{array}\right\} \text{ or } \left\{\begin{array}{l} R : \sin C :: CE : DE \\ R : \cos C :: CE : CD \\ R : \text{tang}\, C :: CD : DE. \end{array}\right.$$

The angle E being the complement of the angle C, we have $\cos C = \sin E$; and the first two propositions admit of being united in one, and may be enunciated thus; *in any right-angled triangle, radius is to the sine of one of the acute angles, as the hypothenuse is to the side opposite to this angle.*

The third shows, that *radius is to the tangent of one of the acute angles, as the side of the right angle adjacent to this acute angle is to the side opposite.*

Radius being always given, it is sufficient to know two of the three other terms of each of the proportions, which I have just stated, in order to find the remaining one. Thus by the first proportion when two of these three things, namely, *the hypothenuse, a side, and an acute angle are known*, the third is readily determined.

I say simply an acute angle, although the proportion requires, that this angle should be opposite to the side given, or to that required, because one of the acute angles enables us to find the other immediately; therefore, if that which is known, or that which is sought, do not fulfil the condition, we may employ its complement.

By the second proportion when two of these three things are known, namely, *the two sides of a right angle and an acute angle*, the third is readily determined.

It follows from this, 1. that knowing a side and an angle of a right-angled triangle, we can calculate the two other sides; 2. that any two sides whatever being known, we can calculate the acute angles.

These two cases do not comprehend that in which any two sides being given to find the third; but this is immediately resolved by the known property of a right-angled triangle, which gives $\overline{CD}^2 + \overline{DE}^2 = \overline{CE}^2$, from which we deduce

$$CE = \sqrt{\overline{CD}^2 + \overline{DE}^2}.$$

If we have given the hypothenuse CE and one of the sides of the right angle DE, for example, we have

$$CD = \sqrt{\overline{CE}^2 - \overline{DE}^2}.$$

Recollecting that $\overline{CE}^2 - \overline{DE}^2 = (CE + DE)\ (CE - DE)$ (*Alg.* 34), if we take the logarithms of the two members of the equation $CD = \sqrt{(CE + DE)\ (CE - DE)}$, we shall have

$$l\ CD = \tfrac{1}{2}\ [(l\ CE + DE) + l\ (CE = DE)].$$

When we construct formulas to be used in numerical calculations, we should endeavor to prepare them in such a manner that logarithms may be conveniently applied to them, that is, so that it will be necessary to pass from logarithms to numbers and from numbers to logarithms, as little as possible. By applying logarithms to the determination of CD, by means of the first expression above given, we shall perceive very clearly the object of this remark.

I will conclude this exposition of the principles, that are employed in the resolution of right-angled triangles, by observing, that the two cases last treated may be resolved also by the two propositions given at the commencement of this article. For 1. if, having CD and DE, we would determine CE, we can calculate one of the acute angles, C, for example, by the proportion R : tang C : : $CD : DE$; having found this angle, we calculate the hypothenuse CE by the proportion R : sin C : : $CE : DE$, in which the three terms R, sin C and DE are known. 2. When the hypothenuse CE and one of the other sides, CD, for example, are known, we calculate the acute angle opposite to the side sought, by the proportion, R : sin E or cos C : : $CE : CD$; then the side DE is found by the proportion R : sin C : : $CE : DE$.

31. What has been said upon right-angled triangles may be put into a convenient form, by using the letters A, B, C, to denote the angles, A being the right angle, and a, b, c, to denote the sides respectively opposite to these angles, as is shown by figure
Fig. 13. 13; we have then by the first principle

$$R : \sin C :: a : c, \quad R : \sin B :: a : b,$$

whence

$$\frac{c}{a} = \frac{\sin C}{R}, \quad \frac{b}{a} = \frac{\sin B}{R}.$$

Eliminating a from these two equations, which is done by dividing the two members of the first each by the corresponding member of the second, we find

$$\frac{c}{b} = \frac{\sin C}{\sin B};$$

and as $\sin B = \cos C$, and $\frac{\sin C}{\cos C} = \frac{\text{tang } C}{R}$, we obtain $\frac{c}{b} = \frac{\text{tang } C}{R}$, an equation, which represents the second principle enunciated in the preceding article.

Lastly, if we square each member of the first two equations, and add the results, member to member, observing, at the same time, that

$$\sin C^2 + \sin B^2 = \sin C^2 + \cos C^2 = R^2 \ (10),$$

we have

$$\frac{c^2}{a^2} + \frac{b^2}{a^2} = 1, \quad \text{or } b^2 + c^2 = a^2.$$

It follows from this, that the two equations

$$\frac{c}{a} = \frac{\sin C}{R}, \quad \frac{b}{a} = \frac{\sin B}{R},$$

are sufficient, together with the relation subsisting between the angles B and C, for the resolution of all cases of right-angled triangles.

32. The principle upon which the resolution of right-angled triangles is founded, leads also to that of triangles of whatever kind. By letting fall from the angle B of the triangle ABC
Fig. 14. (*fig.* 14) a perpendicular BD, we form two triangles ABD, BDC right-angled at D; we have in the first

$$R : \sin A :: AB : BD,$$

and in the second

$$R : \sin C :: BC : BD,$$

which gives

$$R \times BD = \sin A \times AB, R \times BD = \sin C \times BC,$$

whence

$$\sin A \times AB = \sin C \times BC, \text{ or } \sin A : \sin C :: BC : AB.$$

When the perpendicular falls without the triangle, the angle C is not common to the two angles ABC, BCD; but the angles BCD, BCA, being together equal to two right angles, have the same sine (22).

The proportion just given admits of a general application, and may be enunciated thus; *In any triangle whatever the sines of the angles are to each other as the sides opposite to these angles.*

33. The same proposition may be demonstrated in the following manner, which may appear more conformable to the idea I have given of trigonometry in art. 1 and 2.

Having inscribed the triangle ABC (*fig.* 15) in a circle, if from the centre O of this circle, and with a radius Oa, equal to that of the tables, we describe a circle $a\,b\,c$, and then join by the lines $a\,b$, $b\,c$, and $a\,c$, the points where the radii AO, BO, CO, meet the circle of the tables, we form a triangle $a\,b\,c$ similar to the triangle proposed, the sides of which are deduced from the tables. Fig. 15.

The similarity of the two triangles ABC, $a\,b\,c$, (*Geom.* 209), becomes evident, when we consider that the right lines $a\,O$, $b\,O$, and $c\,O$, being equal, as radii of the same circle, as well as the straight lines AO, BO, and CO, the triangles AOB, BOC, and AOC, have their sides AO and BO, BO and CO, AO and CO cut proportionally in the points a and b, b and c, a and c, and consequently the right lines AB and $a\,b$, BC and $b\,c$, AC and $a\,c$, are respectively parallel; we have then

$$AB : BC : AC :: a\,b : b\,c : a\,c,$$

or

$$:: \tfrac{1}{2}\,a\,b : \tfrac{1}{2}\,b\,c : \tfrac{1}{2}\,a\,c.$$

This being the case, the angles of the triangle $a\,b\,c$, having their vertex in the circumference, are measured by half of the arc subtending the side opposite to them, and each of these arcs has evidently for its sine half of the same side (14); whence

$$\tfrac{1}{2}\,a\,b = \sin c = \sin C,$$
$$\tfrac{1}{2}\,b\,c = \sin a = \sin A,$$
$$\tfrac{1}{2}\,a\,c = \sin b = \sin B,$$

and consequently,

$$AB : BC : AC :: \sin C : \sin A : \sin B.$$

We shall perceive moreover by comparing the triangles AOB and $a\,O\,b$, that $AB : a\,b : : AO : a\,O$, or that

$$AB : 2 \sin C : : AO : a\,O;$$

that is, *each side of the triangle* ABC, *is to double the sine of the opposite angle, as the radius of the circumscribed circle, is to that of the tables.**

34. Designating, as in art. 31, the three angles by A, B, C,
Fig. 16. and the sides respectively opposite by a, b, c, (*fig.* 16), we have, according to what precedes, the following proportions,

$$\sin A : \sin B : : a : b,$$
$$\sin A : \sin C : : a : c,$$
$$\sin B : \sin C : : b : c,$$

from which we deduce the equations

$$\frac{b}{a} = \frac{\sin B}{\sin A}, \quad \frac{c}{a} = \frac{\sin C}{\sin A}, \quad \frac{c}{b} = \frac{\sin C}{\sin B}.$$

We may resolve a triangle immediately by these proportions; 1. *when two of the angles and one of the sides are known*, since in this case all the angles will be known, and the sides sought will necessarily be opposite to two of these angles; if, for example, a and the angles B and C are given, we subtract the sum of these angles from two right angles in order to obtain the angle A, and we find by the two first proportions the sides sought b and c; 2. *when we have one angle and two sides, one of which is opposite to the given angle;* if, for example, we have the angle A and the sides a and b, we determine the angle B, by the first proportion, and then two angles being known, the question falls within the preceding case.

There are two cases, which are not comprehended within the rules now enunciated, and which seem to elude this method, one is, *when two sides and the included angle are given*, and the other, *when the three sides are given*. I shall treat of these in order.

35. I will suppose, in the first place, that the two sides a and b and the included angle C are known. If we give to the equations

---

* We may consider the lines $a\,b$, $b\,c$, and $a\,c$, as the sines of the angles A, B, C, by taking for unity the diameter of the circle $a\,b\,c$. It is thus that Carnot has presented them in a work entitled *Géométrie de Position*, where may be found, according to this definition a very simple and elegant demonstration of the proposition given in art. 11, and of the most important conclusions deduced from it.

$$\frac{c}{a} = \frac{\sin C}{\sin A}, \qquad \frac{c}{b} = \frac{\sin C}{\sin B},$$

the form

$$a \sin C = c \sin A, \qquad b \sin C = c \sin B,$$

and then add them, member to member, and subtract the one from the other, we have

$$(a + b) \sin C = c (\sin A + \sin B),$$
$$(a - b) \sin C = c (\sin A - \sin B);$$

dividing the second by the first, the unknown side c will disappear, and we find

$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

But we have seen (27), that

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\text{tang} \frac{1}{2} (A - B)}{\text{tang} \frac{1}{2} (A + B)};$$

we conclude therefore, that

$$\frac{a - b}{a + b} = \frac{\text{tang} \frac{1}{2} (A - B)}{\text{tang} \frac{1}{2} (A + B)},^*$$

whence we obtain the proportion,

$$a + b : a - b :: \text{tang} \tfrac{1}{2} (A + B) : \text{tang} \tfrac{1}{2} (A - B),$$

which may be enunciated thus; *The sum of two sides of a triangle is to their difference, as the tangent of half the sum of the opposite angles to the tangent of half their difference.*

Every term in this proportion is known but $A - B$, for if we subtract the angle C from two right angles, the remainder will be $A + B$; whence

$$\text{tang} \tfrac{1}{2} (A - B) = \frac{a - b}{a + b} \times \text{tang} \tfrac{1}{2} (A + B).$$

Knowing then $\frac{1}{2} (A + B)$ and $\frac{1}{2} (A - B)$, we have by adding them together

$$\tfrac{1}{2} (A + B) + \tfrac{1}{2} (A - B) = A,$$

and by subtracting the second from the first,

$$\tfrac{1}{2} (A + B) - \tfrac{1}{2} (A - B) = B;$$

---

* We may also, for the sake of conciseness, take the proportion

$$a : b :: \sin A : \sin B \ (32),$$

from which we deduce immediately

$$a + b : a - b :: \sin A + \sin B : \sin A - \sin B;$$

and thence by art. 28,

$$a + b : a - b :: \text{tang} \tfrac{1}{2} (A + B) : \text{tang} \tfrac{1}{2} (A - B).$$

that is, *the greater angle is obtained by adding the half difference to the half sum, and the less by subtracting the half difference from the half sum.*

When all the angles are determined, we find the third side by the rule given in art. 32.

36. We may also find immediately the third side, by letting fall a perpendicular upon one of the given sides, from the angle

Fig. 14. B, for example, upon the side AC (*fig.* 14). We have, by the known property of oblique-angled triangles (*Geom.* 191, 192),

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 \mp 2\,AC \times DC,$$

the upper sign being used when the perpendicular falls within the triangle, and the lower when it falls without; moreover, in the right-angled triangle BDC, we have (30)

$$DC = BC \times \sin DBC = BC \times \cos C,$$

making $R = 1$.

Whence $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 - 2\,AC \times BC \times \cos C$,

and consequently

$$AB = \sqrt{\overline{AC}^2 + \overline{BC}^2 - 2\,AC.BC.\cos C},$$

or, adopting the notation of art. 31,

$$c = \sqrt{a^2 + b^2 - 2\,a\,b - \cos C},$$

which gives the side c by means of the two other sides, a and b, and the angle C. One sign is sufficient for the term $2\,a\,b\cos C$, because when the angle C is obtuse, its cosine is negative, and consequently changes — into +, as is required by the geometrical construction.

37. This formula is not suited to calculation by logarithms; but since we have,

$$\cos 2\,C = 1 - 2\sin C^2 \text{ (27)†},$$

we have also by writing $\frac{1}{2}\,C$ in the place of C,

$$\cos C = 1 - 2\,(\sin \tfrac{1}{2}\,C)^2,$$

and by this transformation we obtain

---

† In article 27, section 4, the formula

$$\sin a^2 = \frac{R^2}{2} - \frac{R}{2}\cos 2\,a,$$

when $R = 1$, becomes

$$\sin a^2 = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\,a.$$

Transposing $\sin a^2$ and $\frac{1}{2}\cos 2\,a$ and multiplying by 2, we have

$$\cos 2\,a = 1 - 2\sin a^2.$$

$$c = \sqrt{a^2 + b^2 - 2ab + 4ab(\sin\tfrac{1}{2}C)^2}$$
$$= \sqrt{(a-b)^2 + 4ab(\sin\tfrac{1}{2}C)^2}.$$

If we make $\frac{2\sin\frac{1}{2}C}{a-b}\sqrt{ab} = \text{tang}\,\alpha$, there will result from the substitution of this last,

$$c = (a-b)\sqrt{1+\text{tang}\,\alpha^2} = \frac{a-b}{\cos\alpha},$$

since $\cos\alpha = \frac{1}{\sqrt{1+\text{tang}\,\alpha^2}}$. Tang α is easily calculated by the first of the above equations, and when the angle α is obtained, we have by the second $c = \frac{a-b}{\cos\alpha}$ †.

38. By the equation $c = \sqrt{a^2+b^2-2ab\cos C}$, we can determine the angle C, when the three sides a, b, c, are given, for by raising each member to the square we obtain

$$a^2 + b^2 - c^2 = 2ab\cos C,$$

---

† By making $\frac{2\sin\frac{1}{2}C}{a-b}\sqrt{ab} = \text{tang}\,\alpha$, we are to understand, that the value of this expression is to be considered as the tangent of an angle. If we resolve it into a proportion, we shall have

$$a-b : 2\sin\tfrac{1}{2}C :: \sqrt{ab} : \frac{2\sin\frac{1}{2}C}{a-b}\sqrt{ab},$$

and this fourth term may evidently be considered as the tangent of some angle, and the first three terms being known, this is also known.

If in the equation $c = (a-b)\sqrt{1+\text{tang}\,\alpha^2}$, we substitute the square of $\frac{2\sin\frac{1}{2}C}{a-b}\sqrt{ab}$, namely, $\frac{4ab(\sin\frac{1}{2}C)^2}{(a-b)^2}$, for tang α^2, we shall have $c = (a-b)\sqrt{1+\frac{4ab(\sin\frac{1}{2}C)^2}{(a-b)^2}}$, then, by squaring the second member and putting it under the sign of the square root, it becomes

$$c = \sqrt{(a-b)^2 + 4ab(\sin\tfrac{1}{2}C)^2},$$

which is the same as the preceding expression for c.

To show that $\cos\alpha = \frac{1}{\sqrt{1+\text{tang}\,\alpha^2}}$, let C (*fig.* 12) $= \alpha$, then Fig. 12.

$$CN \text{ or } \sqrt{\overline{CA}^2 + \overline{AN}^2} : CA :: CM : CP,$$

or, radius being 1, $\sqrt{1+\text{tang}\,\alpha^2} : 1 :: 1 : \cos\alpha = \frac{1}{\sqrt{1+\text{tang}\,\alpha^2}}$.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab};$$

but this expression not being well adapted to calculation by logarithms, another is to be sought.

If we write $2C'$ for C, and put $1 - 2\sin C'^2$ instead of $\cos C$ (27), we have this expression,

$$2\sin C'^2 = 1 + \frac{c^2 - a^2 - b^2}{2ab} = \frac{c^2 - a^2 - b^2 + 2ab}{2ab}$$

$$= \frac{c^2 - (a-b)^2}{2ab} = \frac{(c+a-b)(c-a+b)}{2ab},$$

and consequently,

$$\sin C'^2 = \frac{(c+a-b)(c-a+b)}{4ab} =$$

$$= \frac{\frac{(c+a-b)}{2} \; \frac{(c-a+b)}{2}}{ab};$$

but it is easy to see that

$$\frac{c+a-b}{2} = \frac{c+a+b}{2} - b,$$

$$\frac{c-a+b}{2} = \frac{c+a+b}{2} - a;$$

if then we make $c + a + b = f$, we have, by extracting the square root, and restoring $\frac{1}{2}C$ in the place of C',

$$\sin \tfrac{1}{2} C = \sqrt{\frac{(\frac{1}{2}f - a)(\frac{1}{2}f - b)}{ab}},$$

a formula, which leads to the following rule;

To find an angle of a triangle, when the three sides are known; from the half sum of the three sides subtract successively each of those which contain the angle sought; multiply the two remainders together; and divide the product by the product of the sides which contain the angle sought, and the square root of the quotient will be the sine of half of this angle.

39. All cases of oblique-angled triangles are solved by the three rules given in art. 32, 35, 38, which depend upon a principle deduced from right angled triangles in art. 30; it will be easy then with a little attention to retain these rules, and the examples which I am about to give, will be sufficient to enable the learner to apply them.

Examples of the solution of right-angled triangles.

1. In the triangle BAC (*fig.* 13), the hypothenuse a and a side c being given, to find the angle C, opposite to this side; let the hypothenuse $a = 13^{chains},178$, and the side $c = 7^{ch},357$. We have, in order to determine sin C, the proportion Fig. 13.

$$a : c :: R : \sin C,$$

whence

$$\sin C = \frac{R \times c}{a},$$

or by logarithms,

$$\mathrm{l} \sin C = \mathrm{l}\, R + \mathrm{l}\, c - \mathrm{l}\, a.$$

To render the calculation more simple we almost always make radius equal to unity; its logarithm is then zero, and no account need be taken of it; and, instead of performing the subtractions, we employ the arithmetical complements, the theory of which has already been explained (*Alg.* 248). Thus

l c = l 7,357 = ... 0,8667008
arith. comp. l a = arith. comp. l 13,178 = 8,8801505

sum or l sin C = ... 9,7468513
which in the tables answers to 33° 56′ 13″ = C.

2. The angle $C = 52°\ 21'\ 59''$, the hypothenuse $a = 33^{ch},253$, being given, to find the side b. We have

$$R : \sin B \text{ or } \cos C :: a : b, (31),$$

whence

$$b = \frac{a \times \cos C}{R},$$

$$\mathrm{l}\, b = \mathrm{l}\, a + \mathrm{l} \cos C - \mathrm{l}\, R = \mathrm{l}\, a + \mathrm{l} \cos C.$$

But l a = l 33,253 = ... 1,5218308
l cos C = l cos 52° 31′ 59″ = ... 9,7841204

sum or l b = ... 1,3059512
which answers in the tables to $20^{ch},228 = b$ to within one thousandth.

3. The side $c = 5^{ch},391$, the angle $B = 31°\ 30'\ 44''$, being given, to find the side b. We have

$$R : \text{tang } B :: c : b,$$

whence

$$b = \frac{c \times \text{tang } B}{R},$$

$$\text{l}\, b = \text{l}\, c + \text{l tang}\, B - \text{l}\, R.$$

But $\text{l}\, c = 15{,}391 =$ 0,7316693
tang $B = \text{l tang } 31°\ 30'\ 44'' =$ 9,7875272

sum or $\text{l}\, c =$ 0,5191965
which answers in the tables to $3^{ch},305 = c$.

Examples of the solution of oblique-angled triangles.

Fig. 16. 1. In the triangle ABC (*fig.* 16), the side c, and the angles A and B, being given, to find the side b.

Let $A = 112°\ 30'\ 21''$, $B = 52°\ 54'\ 40''$, $c = 27^{ch},348$; the angle C will be

$180° - (A + B) = 180° - 165°\ 25'\ 01'' = 14°\ 34'\ 59''$,
and we have

$$\sin C : \sin B :: c : b,$$

whence

$$b = \frac{c \times \sin B}{\sin C};$$

$$\text{l}\, b = \text{l}\, c + \text{l} \sin B - \text{l} \sin C.$$

But $\text{l}\, c = \text{l}\, 27{,}348 =$ 1,4369256
$\text{l} \sin B = \text{l} \sin 52°\ 54'\ 40''$ 9,9018401
arith. comp. $\text{l} \sin C =$ arith. comp. $\text{l} \sin 14°\ 34'\ 59'' =$ 0,599633

sum or $\text{l}\, b =$ 1,9377290
which answers in the tables to $86^{ch},642 = b$.

2. In the triangle ABC the two sides a, b, and the included angle C, being given, to find the third side c.

Let $a = 28^{ch},442$, $b = 17^{ch},803$, $C = 75°\ 50'$. We begin with finding the other angles by means of the proportion (35)

$$a + b : a - b :: \text{tang}\,\frac{A+B}{2} : \text{tang}\,\frac{A-B}{2},$$

whence

$$\text{tang}\,\frac{A-B}{2} = \frac{\left(\text{tang}\,\frac{A+B}{2}\right)(a-b)}{a+b},$$

and

$$\text{l tang}\,\frac{A-B}{2} = \text{l tang}\,\frac{A+B}{2} + \text{l}\,(a-b) - \text{l}\,(a+b);$$

but $\quad A + B = 180° - 75°\ 50' = 104°\ 10'$ and

$$\frac{A+B}{2} = 52°\ 05'$$

$$a + b = 28{,}442 + 17{,}803 = 46{,}245,$$

$$a - b = 28{,}442 - 17{,}803 = 10{,}639.$$

$\text{l tang}\,\frac{A+B}{2} = \text{l tang } 52^\circ\, 05' \ldots\ldots\ldots\ldots$ 10,1084926
$\text{l}\,(a-b) = \text{l } 10{,}639 = \ldots\ldots\ldots\ldots\ldots\ldots$ 1,0269008
arith. comp. $\text{l}\,(a+b) =$ arith. comp. l 46,245 = . . 8,3349352

sum or log tang $\frac{A-B}{2} = \ldots\ldots\ldots\ldots\ldots\ldots$ 9,4703286

which answers to 16° 27′ 15″;

therefore $$\frac{A+B}{2} + \frac{A-B}{2} = A = 68^\circ\, 32'\, 15'',$$

and $$\frac{A+B}{2} - \frac{A-B}{2} = 35^\circ\, 37'\, 45''.$$

In order to determine the side c, we have the proportion

$$\sin B : \sin C :: b : c,$$

whence

$$c = \frac{b \times \sin C}{\sin B},$$

and $$\text{l}\,c = \text{l}\,b + \text{l} \sin C - \text{l} \sin B;$$

but $\text{l}\,b = \text{l } 17{,}803 = \ldots\ldots\ldots\ldots\ldots\ldots$ 1,2504932
$\text{l} \sin C = \text{l} \sin 75^\circ\, 50' = \ldots\ldots\ldots\ldots\ldots$ 9,9865872
arith. comp. $\text{l} \sin B =$ arith. comp. sin 35° 37′ 45″ = 0,2346766

sum or $\text{l}\,c = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ 1,4717570
which answers in the tables to $29^{ch},632 = c$.

3. In the triangle ABC, the three sides a, b, c, being known, to find the angle A.

Let $a = 29^{ch},037$, $b = 18^{ch},743$, $c = 13^{ch},782$.

According to art. 38, we add the three sides a, b, c, together, which gives 61,562; and from half of the sum 30,781, we subtract successively b, c; the remainders are 12,038 and 16,999, we have then

l 16,999 = ...	1,2304234
l 12,038 = ...	1,0805543
arith. comp. l 18,743 = ...	8,7271609
arith. comp. l 13,782 = ...	8,8606878
sum	19,8988264
the half sum, or $\text{l} \sin \frac{1}{2} A =$	9,9494132

which answers in the table to $62^\circ\, 52'\, 55'' = \frac{1}{2} A$, therefore $A = 125^\circ\, 45'\, 30''$.

40. A work of this nature does not admit of a particular ac-

count of the applications of plane trigonometry. I shall confine myself to the solution of three questions, which may be regarded as the basis of the art of drawing plans.

The first is, *having given in magnitude and position upon a* Fig. 17. *plane a right line* AB (fig. 17), *to determine the position of a point* C, *situated in the same plane*, or which is the same thing, *to find the distances* AC *and* BC.

In order to resolve this question, the side AB, which is the *base* of the operation, must be measured, as also the angles CAB, CBA, comprehended between this base and the lines, which connect the extremities with the point C; the distances sought, AC and BC, may be calculated by the rule laid down in art. 32; and these being known, the triangle ABC may be constructed by means of a scale of equal parts, and the relative position of the three points A, B, C, may be calculated.*

We can then, by the resolution of the right-angled triangle ACP, in which the side AC and the angle CAP are known, find the length of the perpendicular CP let fall upon AB, or of the shortest distance of the point C from AB, and the length of the segment AP. By means of these the position of the point C with respect to the line AB is determined. The situation of the point D may also be found, if it can be perceived from any two of the points A, B, and C.

41. When we have determined immediately the point D with respect to the line AB, by measuring the angles DAB, DBA, we have every thing, which is necessary in order to compute the distance of the points C and D with respect to each other; for, having resolved the triangle DAB, as also the triangle CAB, by subtracting the angle DAB, from the angle CAB, we have, in the triangle CAD, the two sides AC and AD, and the included angle CAD; by applying the rules of art. 35, we shall obtain the two other angles DCA, CDA, and the third side CD, which

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* I do not insist upon the angles being measured, since more might be learned by a sight of the instruments, which are employed for this purpose, than by anything which I can say on the subject. To conceive of the possibility of performing this operation, it is sufficient to imagine, that there is placed at the centre C, a sector of a circle, the radii of which correspond to the direction of the sides AB and AC, which contain the angle to be measured.

is the distance sought. The angle DCA gives the position of the right line CD; and if we consider AC, as secant, by comparing the angles DCA and CAB, we shall be able to find the inclination of CD with respect to AB.

If we set out from the points C and D, taking CD as a new base, we may determine other points not visible from the first two, A and B, and by proceeding in this manner we can determine the relative position of all the points of a country. It is in this manner that the map of France was constructed under the direction of Cassini.

42. The second question, which I am to consider, is merely the first rendered more general by supposing the point to be determined, situated without the plane, in which the line AB is found. Let C (*fig.* 18) be this point, and ABC' the plane in which Fig. 18. AB is situated. The position of the point C will be known, if we have that of the foot C' of the perpendicular let fall from this point upon the plane ABC', and the length of the perpendicular CC', which shows how much the point C is elevated above C', its *projection*. In this case the angles $C'AB$ and $C'BA$ are not the angles to be measured, but we take, instead of them, the angles CAB and CBA, situated in the plane CAB passing through the lines AC and BC, which are drawn from the given points A and B to the point required; and in order to fix the position of this plane, we measure also the angle DBC, which the line CB makes with the line BD perpendicular to the plane ABC', and consequently parallel to the right line CC'.* We resolve the triangle CBA, as in the preceding article, the same things being given; then in the triangle $C'BC$ right-angled at C', knowing the hypothenuse CB and the angle $C'BC$, which is the difference between the right angle DBC' and the measured angle DBC, we calculate the sides CC', $C'B$. The first is the height of the point C above the plane $C'AB$, and is used in connexion

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* When the question relates to points situated on the earth's surface, we take the plane of the *horizon* for the plane ABC', the lines CC' and BD are then *vertical*, and their direction is given by the *plumb-line;* the plane $C'CB$, which passes through these lines, is vertical, and is determined by the line DB and the point C, which is seen from the point B. The line $C'B$ is horizontal and comprehended in the same plane.

with the side AC, to determine AC', by means of the triangle CAC', right-angled at C'. This being done, we have the three sides of the triangle $C'AB$, and the point C' is therefore given.

43. It is for the sake of greater simplicity that I have supposed the line AB in the plane, to which the points to be determined are referred; when this is not the case, it is necessary to
Fig. 19. measure also the angle DBA (*fig.* 19), which this line makes with the line DB perpendicular to the plane $A'BC'$, to which we would refer the point C. This being done, we calculate in the first place, as before, the sides AC and BC, of the triangle ABC, the sides CC' and $C'B$ of the right-angled triangle $C'BC$; then, in the triangle $BA'A$, right-angled at A', knowing AB and the angle ABA', the complement of the observed angle DBA, we calculate BA' and AA'.

If now we conceive AC'' parallel to $A'C'$, there will result from it the triangle $AC''C$, right-angled at C'', in which we have given AC, the calculated side of the triangle ABC, and CC'', the difference between the lines CC' and $C'C''$, or AA', before calculated; we can therefore calculate AC'', or $A'C'$. The triangle $BA'C'$ then becomes determinate by means of the three sides, as was the case with respect to the triangle BAC' in the preceding article.

44. If we take arbitrarily the sides BC and BA, and pursue the course I have just pointed out, we can calculate the triangle $A'C'B$, with the view to find the angle $C'BA'$ formed by the lines BC' and BA', which are, upon the plane $A'BC'$, the *projections* of the visual rays BC and BA, drawn from the point B to the points A and C.

The angle $C'BA'$, comprehended between these projections, is the angle CBA, *reduced* from the inclined plane to the plane $A'BC'$, on which the objects are considered with reference to each other, and which is generally horizontal. I shall give hereafter (62) a second method of reducing an angle from one plane to another; but for the most part, as these planes are but little inclined to each other, the reduction may be made by approximate methods, that are much shorter. There are also tables constructed for this purpose.

I will merely add, for the present, that if we observe at the point A the angles EAC, EAB, and reduce by means of them the angle CAB to the angle $C'A'B$, and then calculate $A'B$, by multiplying AB by the cosine of the angle ABA' or the sine of the angle DBA, then knowing immediately the angles $C'BA'$, $C'A'B$, and the right line $A'B$, the determination of the point C falls within the method laid down in art. 40.

Reduction to the plane of the horizon is not the only one, which we have occasion to apply to the observed angles. It seldom happens, that the observer can place himself at the remarkable points selected, as the vertices of the angles, which are ordinarily the tops of steeples and towers; hence arises a new reduction, which is called *reduction of the angles to the centre of the station.* The student may consult upon this subject, as well as upon every other connected with great trigonometrical operations, the work of M. Delambre, entitled *Méthodes analytiques pour la determination d'un arc du méridien*, and the treatises of M. Puissant.

45. The third question, which I proposed to resolve, has for its object, *the determination of a point by means of the angles comprehended between three straight lines drawn from this point to three given points;* it presents itself as one of the most convenient methods of placing, upon a plan or map, a point which is not marked.

When we consider this case in its most general point of view, we shall find that it belongs to geometry in space; but, when the angles are in the same plane, there is always one which is the sum or difference of the other two, so that it is sufficient to observe these two in order to obtain the third, and the other cases may be reduced to this, by the method of reducing angles to a horizontal plane explained in article 62.

The graphical solution of this case consists simply in describing upon the lines AB and AC (*fig.* 20), which connect the three given points A, B, C, two segments of a circle containing the angles BDA, CDA, observed at the required point D, between the points A and B, A and C. The circles cut each other, in the first place at the point which becomes common to them by construction, and afterward at the point D, which will evidently be the point sought. Fig. 20.

I shall not enter into a discussion of the different cases which this problem admits of, arising from the different situations of the given points A, B, C, with respect to the point sought, D; I will merely remark, that the sum of the observed angles BDA, CDA, must show whether it is situated in the triangle ABC, or without it. In the first case it will be greater than two right angles, in the second it will be less; and, if it be exactly equal to two right angles, it will fall upon the line BC. This is too obvious to require proof.

I shall here give one of the methods of applying the trigonometrical calculus to this problem. The things given are the parts of the triangle BAC, and the observed angles BDA and CDA; I put therefore

$$AB=a,\ AC=b,\ BDA=\alpha,\ CDA=\beta,\ BAC=\gamma,$$

and I take for the unknown quantities

$$ABD=x,\ ACD=y;$$

because, these angles being found, we shall have in each of the triangles BAD and DAC, two angles and one side, from which the other parts may be determined (34). This being supposed, the triangles BAD and DAC will give

$$\sin BDA : \sin ABD :: AB : AD,$$
$$\sin CDA : \sin ACD :: AC : AD,$$

or

$$\sin\alpha : \sin x :: a : AD = \frac{a\sin x}{\sin\alpha},$$
$$\sin\beta : \sin y :: b : AD = \frac{b\sin y}{\sin\beta};$$

whence we obtain the equation

$$\frac{a\sin x}{\sin\alpha} = \frac{b\sin y}{\sin\beta},$$

which becomes, by multiplication and transposition,

$$a\sin\beta\sin x - b\sin\alpha\sin y = 0.$$

But, in the quadrilateral $ABDC$, we have

$$ACD = 4 \text{ right angles} - ABD - ADC - BAC - ABD,$$

whence $y = 4$ right angles $-\alpha-\beta-\gamma-x$;
if, for the sake of conciseness, we make

$$4 \text{ right angles} - \alpha - \beta - \gamma = \delta,$$

it will become $y=\delta-x$, and consequently,

$$a\sin\beta\sin x - b\sin\alpha(\sin\delta\cos x - \cos\delta\sin x) = 0;$$

dividing the whole by $\sin x$, we obtain

$$a\sin\beta - b\sin\alpha\left(\sin\delta\frac{\cos x}{\sin x} - \cos\delta\right) = 0;$$

whence

$$\frac{\cos x}{\sin x} = \cot x = \frac{a\sin\beta + b\sin\alpha\cos\delta}{b\sin\alpha\sin\delta}.$$

If we divide this expression into two parts, we shall have

$$\cot x = \frac{a \sin \beta}{b \sin \alpha \sin \delta} + \frac{\cos \delta}{\sin \delta},$$

or rather

$$\cot x = \frac{\cos \delta}{\sin \delta}\left(\frac{a \sin \beta}{b \sin \alpha \cos \delta} + 1\right),$$

or lastly,

$$\cot x = \cot \delta \left(\frac{a \sin \beta}{b \sin \alpha \cos \delta} + 1\right).$$

The question is then resolved, since by having the angle x we can find that of y.

CHAPTER II.

Of Spherical Trigonometry.

46. The spherical triangles usually calculated are those formed on the surface of a sphere, by the intersection of three great circles. A triangle of this description always determines a triple solid angle; and reciprocally, from a triple solid angle,
Fig. 22. we may always deduce a spherical triangle. Let ABC (*fig.* 22) be any spherical triangle, and let the radii AS, BS, CS, be drawn from its angles to the centre of the sphere to which it belongs; ABS, ACS, BCS, will be the planes of the great circles, in which the arcs AB, AC, BC, the sides of the proposed triangle, are taken; and these arcs measure the plane angles comprehended on the respective faces of the solid angle $SABC$, between the edges SA and SB, SA and SC, SB and SC. The inclination of two planes is measured by the rectilineal angle contained by two straight lines drawn from the same point perpendicularly to the common intersection, the one being in one of the planes and the other in the other (*Geom.* 349); it follows from this, that, if from the point A, the straight lines AI, and AK, be drawn perpendicularly each to AS, the former in the plane CAS, and the latter in the plane BAS, the rectilineal angle IAK will measure the inclination of these two planes. It is moreover evident, that the line AI will be a tangent to the arc AC, and that AK will be a tangent to the arc AB; and since, for the angle formed by two curved lines, we take that contained by the tangents drawn to the point where these lines meet (*Geom.* 471), the angle IAK will also be the measure of the angle contained by the arcs AC and AB. The same may be shown with respect to each of the two other angles of the spherical triangle ABC; the inclination of any two faces of the solid angle $SABC$ has then the same measure as the corresponding angle of the spherical triangle ABC. The spherical triangle, therefore, and the solid angle, consist respectively of six parts which correspond to each other, namely; the three sides of the triangle, answering to the angles formed by the edges of the solid angle, and the three angles of the triangle, answering to the several inclinations of the faces of the solid angle.

Euler repeatedly turned his attention to the subject of spherical trigonometry, and in order to exhibit it under new points of view he published in 1779* a memoir, that may be regarded as a complete treatise upon this branch of mathematics. Its form being entirely analytical, I have been induced to present it to my readers, with such alterations as make it to depend on a single principle. I have also simplified some of the results.

47. All that I have to offer on the subject of spherical trigonometry depends simply upon the following construction, which ought therefore to be well understood.

From the angle C of the triangle ABC, let fall a perpendicular CD upon ASB, the plane of the side BA opposite to this angle; from the point D draw the lines DE and DF perpendicular respectively to SA and SB; join CE, CF, which will be perpendicular respectively to SA, SB, (*Geom.* 332). It follows from what has been said, that the angles CED, CFD, measure the inclinations of the planes CSA, CSB, to the plane ASB, or, which is the same thing, give the values of the angles A and B of the spherical triangle ABC. I shall in future distinguish the angles of these triangles by capital letters, placed at the vertex, and the opposite sides by the corresponding small letters; thus the side BC opposite the angle A, I shall call a, &c., as in art. 31. Then, radius of the tables being supposed equal to unity, we shall have

$$CE = \sin\ CA = \sin b, \qquad SE = \cos\ CA = \cos b,$$
$$CF = \sin\ CB = \sin a, \qquad SF = \cos\ CB = \cos a.$$

In the plane triangle CDE, right-angled at D, and whose angle $CED = A$, we shall find

$$CD = CE \sin\ CED = \sin b \sin A; \ (30)$$
$$DE = CE \cos\ CED = \sin b \cos A.$$

From the plane triangle CDF, also right-angled at D, and of which the angle $CFD = B$, we obtain

$$CD = CF \sin\ CFD = \sin a \sin B;$$
$$DF = CF \cos\ CFD = \sin a \cos B.$$

The two expressions for the line CD, being put equal to each other, give directly

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* *Acta Academiæ Scientiarum Petropolitanæ, anno* 1779, *pars prior;* see also *Développement de la partie élémentaire des Mathematiques, par Bertrand. Genève,* 1778, (Tom. II. pag. 576).

$$\sin b \sin A = \sin a \sin B \ldots\ldots (A),$$

a result which is, with regard to spherical triangles, analogous to that of art. 32 in plane trigonometry.†

It will be seen that from a similar construction we may derive the two following equations;

$$\sin c \sin A = \sin a \sin C;$$
$$\sin c \sin B = \sin b \sin C.$$

Now from the point E draw EG perpendicular to SB, and through the point D draw DH parallel to SB; we thus form a right-angled triangle HDE, in which $HED = ASB$, since by taking the angle GES from the right angle SED, we have remaining the angle HED, and since the angle ASB, or ESG, is also the difference between a right angle and the angle GES. From the resolution of the triangle EHD, we shall consequently deduce $HD = DE \sin DEH = DE \sin c = \cos A \sin b \sin c$;

but $\qquad SF = \cos a = SG + GF = SG + HD,$

and $\qquad SG = SE \cos ESG = \cos b \cos c;$

we shall then have,

$$\cos a = \cos b \cos c + \cos A \sin b \sin c,$$

an equation that expresses the relation between the side a, the two other sides b and c, and the angle which they contain.

It is evident that, by making a similar construction on the planes of the other sides, we shall find two equations similar to the preceding; and we shall, in this way, form among the six parts of the triangle ABC, the three equations;

$$\left.\begin{array}{l}\cos a = \cos b \cos c + \cos A \sin b \sin c \\ \cos b = \cos a \cos c + \cos B \sin a \sin c \\ \cos c = \cos a \cos b + \cos C \sin a \sin b\end{array}\right\} \ldots\ldots (B).$$

48. These three equations involve the equation (A). To be convinced of this, it is sufficient to take the values which they give for $\cos A$, $\cos B$, $\cos C$, and substitute them in the equations

$$\sin A^2 = 1 - \cos A^2 \;(10),$$
$$\sin B^2 = 1 - \cos B^2,$$
$$\sin C^2 = 1 - \cos C^2.$$

We find by the first of these,

$$\sin A^2 = 1 - \frac{\cos a^2 - 2 \cos a \cos b \cos c + \cos b^2 \cos c^2}{\sin b^2 \sin c^2}$$
$$= \frac{\sin b^2 \sin c^2 - \cos a^2 + 2 \cos a \cos b \cos c - \cos b^2 \cos c^2}{\sin b^2 \sin c^2}$$

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† It is evident that the equation (A) is equivalent to the proportion

$$\sin A : \sin B :: \sin a : \sin b.$$

$$=\frac{(1-\cos b^2)(1-\cos c^2)-\cos b^2\cos c^2-\cos a^2+2\cos a\cos b\cos c\dagger}{\sin b^2\sin c^2}$$

$$=\frac{1-\cos a^2-\cos b^2-\cos c^2+2\cos a\cos b\cos c}{\sin b^2\sin c^2};$$

multiplying the two terms of this fraction by sin a^2, and then taking the square root, we shall obtain

$$\sin A=\sin a\times\frac{\sqrt{1-\cos a^2-\cos b^2-\cos c^2+2\cos a\cos b\cos c}}{\sin a\sin b\sin c}.$$

If, in order to abridge the expression, we represent by M, the quantity multiplied by sin a in the second member of this equation, we shall have $\sin A=M\sin a$.

We shall find, in like manner,

$$\sin B=M\sin b,\qquad \sin C=M\sin c;$$

and, by the elimination of M, we shall fall upon the equations (A)††. It may be observed that the three sides a, b, c, enter all in the same manner into the expression M; for it is in consequence of this that M is common to the values of the sines of all the angles.

The equations (B) then will serve to resolve any spherical triangle, when we know three of its parts, it being observed, that a sine and cosine are to be regarded as only a single unknown quantity, since one may always be expressed by the other.

The application of the equations (B) to the different cases which may occur, becomes more easy by means of certain transformations, which I proceed to make.

49. We may change angles into their opposite sides, and sides into the opposite angles, by giving the sign — to the cosines. To prove this, we substitute in each of the last two of the equations (B) for cos a, its value, $\cos b\cos c+\cos A\sin b$; we shall have

$$\cos b=\cos b\cos c^2+\cos A\sin b\sin c\cos c+\cos B\sin a\sin c;$$
$$\cos c=\cos b^2\cos c+\cos A\sin b\sin c\cos b+\cos C\sin a\sin b.$$

By substituting, in these results, $1-\sin c^2$ for $\cos c^2$, $1-\sin b^2$

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† By developing $(1-\cos b^2)(1-\cos c^2)$, and cancelling those terms which destroy each other.

†† Sin $B=M\sin b$, gives $\frac{\sin B}{\sin b}=M$; and, substituting $\frac{\sin B}{\sin b}$ in the place of M in the equation $\sin A=M\sin a$, we have

$$\sin A=\sin a\frac{\sin B}{\sin b},$$

or

$$\sin b\sin A=\sin a\sin B.$$

for $\cos b^2$, they may be reduced, the first becoming divisible by $\sin c$, the second by $\sin b$; and they may be written thus,

$$\left.\begin{array}{l}\cos B \sin a = \cos b \sin c - \cos A \sin b \cos c \\ \cos C \sin a = \sin b \cos c - \cos A \cos b \sin c\end{array}\right\} \ldots (C).$$

If the second of these equations be multiplied by $\cos A$, and added to the first, and $1 - \sin A^2$ be substituted in the place of $\cos A^2$, we shall obtain

$$\sin a\,(\cos B + \cos A \cos C) = \sin A^2 \cos b \sin c;$$

but it follows from the equations (A), that $\sin c \sin A = \sin a \sin C$; substituting this value in the second member of the above equation, it will become divisible by $\sin a$, and we shall have for the result

$$\cos B + \cos A \cos C = \cos b \sin A \sin C,$$

or, which is the same thing,

$$\cos B = -\cos A \cos C + \cos b \sin A \sin C.$$

Comparing this equation with the equations (B), we see that it might be deduced immediately from the second of the equation (B), by changing capital into small letters, and small into capital, and giving to all the cosines the sign $-$. Indeed, by proceeding thus, we have

$$-\cos B = \cos A \cos C - \cos b \sin A \sin C,$$

an equation, which is transformed into the preceding by changing all the signs.

The relation which the angle B has to the two angles A, C, and the included side b, necessarily exists in each of the similar combinations of angles and sides; we have therefore at the same time the three equations.

$$\left.\begin{array}{l}\cos A = -\cos B \cos C + \cos a \sin B \sin C \\ \cos B = -\cos A \cos C + \cos b \sin A \sin C \\ \cos C = -\cos A \cos B + \cos c \sin A \sin B\end{array}\right\} \ldots (B').$$

50. It must be remarked, that by taking the cosines negatively, we pass from the arcs a, b, c, and the angles A, B, C, to their supplements, since $-\cos A = \cos(2^q - A)$, $-\cos a = \cos(2^q - a)$, and so of the rest (art. 23). If we substitute these values in the above equations, making, in order to abridge the expressions, $2^q - A = A'$, $2^q - a = a'$, &c., they will take the form

$$\left.\begin{array}{l}\cos A' = \cos B' \cos C' + \cos a' \sin B' \sin C' \\ \cos B' = \cos A' \cos C' + \cos b' \sin A' \sin C' \\ \cos C' = \cos A' \cos B' + \cos c' \sin A' \sin B'\end{array}\right\};$$

equations perfectly similar to the equations (B), and which consequently belong to a spherical triangle, whose sides are A', B', C', and angles a', b', c'. Such a triangle has its angles measured by the supplements of the sides of the triangle ABC, and its own sides measure the supplements of the angles ABC; it is called the *supplemental* or *polar triangle;* and it is shown that the vertices of its angles are the poles of the sides of the first triangle, and *vice versa*, (*Geom.* 476).

51. The equations obtained in article 49, and designated by (C), which involve five parts of the spherical triangle ABC, may be transformed into others containing only four. In order to this, we must substitute for $\sin a$, in the first, $\frac{\sin b \sin A}{\sin B}$, and in the second, $\frac{\sin c \sin A}{\sin C}$ (47), and as $\frac{\cos p}{\sin p} = \cot p$, we shall find

$$\left.\begin{aligned} \cot B &= \frac{\cos b \sin c - \cos A \sin b \cos c}{\sin A \sin b} \\ \cot C &= \frac{\sin b \cos c - \cos A \cos b \sin c}{\sin A \sin c} \end{aligned}\right\} \ldots . (D).$$

If we examine these values, we shall readily see that others analogous to them may be readily formed by proper permutations of the letters; but it is particularly important to remark, that, as these are deduced from the equations (B), we may, in like manner as in these last, change sides into angles and angles into sides, by giving to cosines and cotangents the contrary sign; they will thus become

$$\left.\begin{aligned} \cot b &= \frac{\cos B \sin C + \cos a \sin B \cos C}{\sin a \sin B} \\ \cot c &= \frac{\sin B \cos C + \cos a \cos B \sin C}{\sin a \sin C} \end{aligned}\right\} \ldots . (D').$$

52. The five sets of equations (A), (B), (B′), (D), (D′), are sufficient to resolve immediately all the cases which can occur among spherical triangles. The first expresses the relation between the angles and their opposite sides.

53. From the second we deduce the following formulas;

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \cos A \sin b \sin c \\ \cos b &= \cos a \cos c + \cos B \sin a \sin c \\ \cos c &= \cos a \cos b + \cos C \sin a \sin b \end{aligned}\right\},$$

$$\left.\begin{aligned} \cos A &= \frac{\cos a - \cos b \cos c}{\sin b \sin c} \\ \cos B &= \frac{\cos b - \cos a \cos c}{\sin a \sin c} \\ \cos C &= \frac{\cos c - \cos a \cos b}{\sin a \sin b} \end{aligned}\right\},$$

Of these the first three make known a side by means of the two other sides and the angle contained by them; and the last three give the angles by means of the sides.

54. The third set, like the preceding, furnishes six formulas, which are,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$
$$\cos B = -\cos A \cos C + \sin A \sin C \cos b,$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c,$$
$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$
$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C},$$
$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

By means of the first three we can find an angle, when the two other angles and the included side are known; the last three will give each of the sides, when all the angles are known.

55. From the fourth set, after making all the possible permutations, we obtain the six following formulas,

$$\cot A = \frac{\cos a \sin b - \cos C \sin a \cos b}{\sin C \sin a},$$
$$\cot B = \frac{\sin a \cos b - \cos C \cos a \sin b}{\sin C \sin b},$$
$$\cot A = \frac{\cos a \sin c - \cos B \sin a \cos c}{\sin B \sin a},$$
$$\cot C = \frac{\sin a \cos c - \cos B \cos a \sin c}{\sin B \sin c},$$
$$\cot B = \frac{\cos b \sin c - \cos A \sin b \cos c}{\sin A \sin b},$$
$$\cot C = \frac{\sin b \cos c - \cos A \cos b \sin c}{\sin A \sin c},$$

by means of which we may determine two of the angles of a spherical triangle, when we know the third angle and the sides which contain it.

56. The fifth set leads in like manner to six formulas, namely,

$$\cot a = \frac{\cos A \sin B + \cos c \sin A \cos B}{\sin c \sin A},$$
$$\cot b = \frac{\sin A \cos B + \cos c \cos A \sin B}{\sin c \sin B},$$
$$\cot a = \frac{\cos A \sin C + \cos b \sin A \cos C}{\sin b \sin A},$$

$$\cot c = \frac{\sin A \cos C + \cos b \cos A \sin C}{\sin b \sin C},$$

$$\cot b = \frac{\cos B \sin C + \cos a \sin B \cos C}{\sin a \sin B},$$

$$\cot c = \frac{\sin B \cos C + \cos a \cos B \sin C}{\sin a \sin C};$$

these will serve to determine two of the sides of a triangle, when the third and the two adjacent angles are known.

57. The formulas deduced from the sets (B), (B′), (D), and (D′), (53—56), merit the greatest attention, as well for their elegance as the property they have, of making known whether the arc or angle, which they express, is less or greater than a quadrant or a right angle, a property which does not belong to the expressions for the sines of the same arcs. Indeed, the sine of an arc, and the sine of its supplement being the same both as to its value and sign, whenever we know the sine only of an arc, it is impossible thence to determine whether the arc be less or greater than a quadrant. But, when we have the cosine or cotangent of an arc, and know also that this arc cannot be equal to a semicircumference, which is the case with the sides of all spherical triangles, and with the arcs which measure their angles (*Geom.* 492), we are able to determine by the sign of the result, whether the arc sought is, or is not contained between 1^q and 2^q. Both the cosine and the cotangent have the sign — in the former case, and + in the latter.

If then we are careful to give to the known quantities, which enter into the formulas above mentioned, the signs with which they ought to be affected, according to the value of the arcs to which they belong, the sign of the result will make known the *species* of the side or angle sought; that is to say, whether the side be less or greater than a quadrant, and whether the angle be acute or obtuse.

58. These formulas are very much simplified, when the triangle proposed is right-angled; that is, when one of its angles is a right angle. Indeed, if we suppose $C = 1^q$, we shall have

$$\sin C = 1, \qquad \cos C = 0;$$

also $\cos c = \cos a \cos b$ (53),

$$\cos c = \frac{\cos A \cos B}{\sin A \sin B} = \cot A \cot B \text{ (54) (9)},$$

$$\left.\begin{array}{l}\cos A = \sin B \cos a \\ \cos B = \sin A \cos b\end{array}\right\} \text{ (54)},$$

$\sin a = \sin c \sin A$, $\sin b = \sin c \sin B$ (47),

$$\left.\begin{array}{l} \cot b = \dfrac{\cos B}{\sin a \sin B} \\ \cot a = \dfrac{\cos A}{\sin b \sin A} \\ \cot c = \dfrac{\cos b \cos A}{\sin b} \\ \cot c = \dfrac{\cos a \cos B}{\sin a} \end{array}\right\} (56,) \text{ whence } \left\{\begin{array}{l} \text{tang } b = \sin a \text{ tang } B \, (9, 8) \\ \text{tang } a = \sin b \text{ tang } A \\ \text{tang } b = \cos A \text{ tang } c \\ \text{tang } a = \cos B \text{ tang } c, \end{array}\right.$$

taking among these formulas, those only which essentially differ, we shall have the six following;

$$\begin{aligned} \cos c &= \cos a \cos b, \\ \cos c &= \cot A \cot B, \\ \sin a &= \sin c \sin A, \\ \text{tang } a &= \sin b \text{ tang } A, \\ \text{tang } a &= \cos B \text{ tang } c, \\ \cos A &= \sin B \cos a. \end{aligned}$$

These, by the changes of which they are susceptible, will serve to resolve right-angled spherical triangles; in which the side opposite to the right-angle is called the *hypothenuse*, as in plane triangles. We might obtain analogous formulas, for the case in which the proposed spherical triangle has one of its sides equal to a quadrant; but I shall not stop to deduce them.

59. For the convenient application of logarithms to the calculus of spherical triangles, the formulas of articles 53 and 54 may be transformed into others, having their numerators and denominators decomposed into factors; Euler has done this in a manner equally simple and elegant.

1. From the expression $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$, contained among those of art. 53, we deduce

$$1 - \cos A = \frac{\cos (b - c) - \cos a}{\sin b \sin c} \quad (11)\dagger,$$

$$1 + \cos A = \frac{\cos a - \cos (b + c)}{\sin b \sin c} \dagger;$$

† The equations $\cos (a - b) = \dfrac{\cos a \cos b + \sin a \sin b}{R}$ (11), when $R = 1$, gives $\cos a \cos b = \cos (a - b) - \sin a \sin b$; whence, by substituting for $\cos b \cos c$ its equal $\cos (b - c) - \sin b \sin c$, the first of the above results is obtained; and the second is obtained in a similar manner.

whence, since $\frac{1 - \cos A}{1 + \cos A} = \text{tang } \frac{1}{2} A^2$ (27)†,

we have $\text{tang } \frac{1}{2} A^2 = \frac{\cos (b - c) - \cos a}{\cos a - \cos (b + c)}$;

but $\cos p - \cos q = -2 \sin \frac{1}{2} (p + q) \sin \frac{1}{2} (p - q)$ (27); therefore,

$$\text{tang } \frac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2} (b - c + a) \sin \frac{1}{2} (b - c - a)}{\sin \frac{1}{2} (a + b + c) \sin \frac{1}{2} (a - b - c)}}.$$

By proceeding in this manner with the other expressions of the same article we shall arrive at similar results.

2. Taking in article 54 the expression

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

we deduce from it

$$1 - \cos a = -\frac{\cos (B + C) + \cos A}{\sin B \sin C},$$

$$1 + \cos a = \frac{\cos A + \cos (B - C)}{\sin B \sin C};$$

whence $\text{tang } \frac{1}{2} a^2 = -\frac{\cos (B + C) + \cos A}{\cos (B - C) + \cos A}$;

but $\cos p + \cos q = 2 \cos \frac{1}{2} (p + q) \cos \frac{1}{2} (p - q)$ (27); therefore,

$$\text{tang } \frac{1}{2} a = \sqrt{\frac{-\cos \frac{1}{2} (B + C + A) \cos \frac{1}{2} (B + C - A)}{\cos \frac{1}{2} (B - C + A) \cos \frac{1}{2} (B - C - A)}},$$

a formula, which the sign $-$ of the numerator does not render imaginary, because the arc $\frac{1}{2} (A + B + C)$, exceeding a quadrant has its cosine negative.*

3. The expressions of art. 53 give also,

$$\cos a - \cos b \cos c = \sin b \sin c \cos A,$$
$$\cos b - \cos a \cos c = \sin a \sin c \cos B;$$

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† In art. 27, the equation

$$\frac{\cos b' - \cos a'}{\cos a' + \cos b'} = \frac{\text{tang } \frac{1}{2} (a' + b') \text{ tang } \frac{1}{2} (a' - b')}{R^2},$$

when $b' = 0$, and $R = 1$, becomes $\frac{1 - \cos a'}{\cos a' + 1} = \text{tang } \frac{1}{2} a' \text{ tang } \frac{1}{2} a'$.

* Euler, for the sake of giving greater uniformity to his results, always employs the tangents of the arcs to be determined; but we may, by what precedes, arrive at the sines in a manner somewhat more simple.

dividing the first of these equations by the second, and observing that, according to equations (A), we have

$$\frac{\sin b}{\sin a}=\frac{\sin B}{\sin A},$$

we shall find

$$\frac{\cos a - \cos b\ \cos c}{\cos b - \cos a \cos c}=\frac{\sin B \cos A}{\sin A \cos B}.$$

If we add unity to each member of this last equation, it will become

$$1+\frac{\cos a - \cos b \cos c}{\cos b - \cos a \cos c}=1+\frac{\sin B \cos A}{\sin A \cos B};$$

by reducing the two terms of each member to the same denominator, and substituting for $\sin A \cos B + \sin B \cos A$ its value $\sin (A+B)$ (11), it is transformed into

$$\frac{(\cos a + \cos b)\ (1 - \cos c)}{\cos b - \cos a \cos c}=\frac{\sin (A+B)}{\sin A\ \cos B}.$$

By taking away unity instead of adding it, we shall have

1st. We have $1 - \cos A = 2 \sin \frac{1}{2} A^2$ (27)†, and by the formula

$$\cos p - \cos q = -2 \sin \tfrac{1}{2} (p+q) \sin \tfrac{1}{2} (p-q),$$

we find

$$\cos (b-c) - \cos a = -2 \sin \tfrac{1}{2} (b - c + a) \sin \tfrac{1}{2} (b - c - a);$$

or, by changing the sign of the arc $b - c - a$ and of its sine,

$$\cos (b-c) - \cos a = 2 \sin \tfrac{1}{2} (a + b - c) \sin \tfrac{1}{2} (a + c - b);$$

substituting this value for its equal in the expression for $1 - \cos A$, and taking the square root of each member, we have

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2} (a+b-c) \sin \frac{1}{2} (a+c-b)}{\sin b \sin c}}.$$

2d. If we also observe that $1 - \cos a = 2 \sin \frac{1}{2} a^2$, and that the expression for $\cos p + \cos q$, gives

$$\cos (B+C) + \cos A = 2 \cos \tfrac{1}{2} (B+C+A) \times \cos \tfrac{1}{2} (B+C-A),$$

we shall find

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos \frac{1}{2} (A+B+C) \cos \frac{1}{2} (B+C-A)}{\sin B \sin C}},$$

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† In article 27, the equation,

$$\sin \tfrac{1}{2} (a' + b') \sin \tfrac{1}{2} (a' - b') = \frac{R}{2} (\cos b' - \cos a'),$$

when $b' = 0$, and $R = 1$, becomes $\sin \frac{1}{2} a' \sin \frac{1}{2} a' = \frac{1}{2} (1 - \cos a')$; whence the above equation is manifest.

$$\frac{\cos a - \cos b \cos c}{\cos b - \cos a \cos c} - 1 = \frac{\sin B \cos A}{\sin A \cos B} - 1;$$

whence we deduce

$$\frac{(\cos a - \cos b)(1 + \cos c)}{\cos b - \cos a \cos c} = \frac{\sin (B - A)}{\sin A \cos B}.$$

This result, divided by the preceding, becomes

$$\frac{\cos a - \cos b}{\cos a + \cos b} \times \frac{1 + \cos c}{1 - \cos c} = \frac{\sin (B - A)}{\sin (B + A)};$$

and since, by table page 24 (art. 27 and 9),

$$\frac{\cos a - \cos b}{\cos a + \cos b} = \text{tang}\, \tfrac{1}{2}(b + a)\ \text{tang}\, \tfrac{1}{2}(b - a),$$

$$\frac{1 + \cos c}{1 - \cos c} = \cot \tfrac{1}{2} c^2, \text{ (art. 27 and note to page 53)},$$

$$\sin p = 2 \sin \tfrac{1}{2} p \cos \tfrac{1}{2} p, (27),$$

we shall find

$$\text{tang}\, \tfrac{1}{2}(b - a)\ \text{tang}\, \tfrac{1}{2}(b + a) \cot \tfrac{1}{2} c^2$$
$$= \frac{\sin \tfrac{1}{2}(B - A) \cos \tfrac{1}{2}(B - A)}{\sin \tfrac{1}{2}(B + A) \cos \tfrac{1}{2}(B + A)} \ldots \text{(a)}.$$

But unity being successively added to and subtracted from each of the members of the equation $\frac{\sin b}{\sin a} = \frac{\sin B}{\sin A}$, then, one of the two results being divided by the other, we arrive at the equation

$$\frac{\sin b - \sin a}{\sin b + \sin a} = \frac{\sin B - \sin A}{\sin B + \sin A},$$

which may be transformed into the following, by the formulas of table page 24;

$$\text{tang}\, \tfrac{1}{2}(b - a) \cot \tfrac{1}{2}(b + a) = \frac{\sin \tfrac{1}{2}(B - A) \cos \tfrac{1}{2}(B + A)}{\sin \tfrac{1}{2}(B + A) \cos \tfrac{1}{2}(B - A)};$$

then, multiplying this equation and equation (a), member into member, observing that

$$\text{tang}\, \tfrac{1}{2}(b + a) \cot \tfrac{1}{2}(b + a) = 1 \ (9),$$

we shall obtain

$$(\text{tang}\, \tfrac{1}{2}(b - a))^2 \cot \tfrac{1}{2} c^2 = \frac{(\sin \tfrac{1}{2}(B - A))^2}{(\sin \tfrac{1}{2}(B + A))^2};$$

extracting the root of each member, we find

$$\text{tang}\, \tfrac{1}{2}(b - a) \cot \tfrac{1}{2} c = \frac{\sin \tfrac{1}{2}(B - A)}{\sin \tfrac{1}{2}(B + A)},$$

and dividing the equation (a) by this last, we have

$$\text{tang}\, \tfrac{1}{2}(a + b) \cot \tfrac{1}{2} c = \frac{\cos \tfrac{1}{2}(B - A)}{\cos \tfrac{1}{2}(B + A)}.$$

Recollecting that $\frac{1}{\cot p}=\text{tang } p$ (9), we shall deduce, from the two foregoing equations, the expressions

$$\text{tang } \tfrac{1}{2}(b-a)=\text{tang } \tfrac{1}{2}c\,\frac{\sin\frac{1}{2}(B-A)}{\sin\frac{1}{2}(B+A)},$$

$$\text{tang } \tfrac{1}{2}(b+a)=\text{tang } \tfrac{1}{2}c\,\frac{\cos\frac{1}{2}(B-A)}{\cos\frac{1}{2}(B+A)},$$

which will make known two sides of a spherical triangle, in which we have the third side and the two angles adjacent to it; since, if we designate by b' and a' the values of the arcs $b+a$ and $b-a$, there will result

$$b=\tfrac{1}{2}(b'+a'),\qquad a=\tfrac{1}{2}(b'-a').$$

4. Again, taking in article 54 the equations

$$\cos A+\cos B\cos C=\sin B\sin C\cos a,$$
$$\cos B+\cos A\cos C=\sin A\sin C\cos b,$$

and, dividing the former by the latter, we shall find

$$\frac{\cos A+\cos B\cos C}{\cos B+\cos A\cos C}=\frac{\sin B\cos a}{\sin A\cos b}=\frac{\sin b\cos a}{\sin a\cos b}.$$

Unity being successively added to and subtracted from each member of this equation, and one of the results being then divided by the other, we shall derive from it, as above (a),

$$\frac{\cos A-\cos B}{\cos A+\cos B}\times\frac{1-\cos C}{1+\cos C}=\frac{\sin(b-a)}{\sin(b+a)},$$

$$\text{tang } \tfrac{1}{2}(B-A)\text{ tang } \tfrac{1}{2}(B+A)\text{ tang } \tfrac{1}{2}C^2$$
$$=\frac{\sin\frac{1}{2}(b-a)\cos\frac{1}{2}(b-a)}{\sin\frac{1}{2}(b+a)\cos\frac{1}{2}(b+a)}\;\ldots\;(b);$$

and, as the equation $\frac{\sin b-\sin a}{\sin b+\sin a}=\frac{\sin B-\sin A}{\sin B+\sin A}$, employed in the preceding transformation, may be written thus,

$$\text{tang } \tfrac{1}{2}(B-A)\cot\tfrac{1}{2}(B+A)=\frac{\sin\frac{1}{2}(b-a)\cos\frac{1}{2}(b+a)}{\sin\frac{1}{2}(b+a)\cos\frac{1}{2}(b-a)},$$

by multiplying and dividing the equation (b) by this last, we shall find

$$\text{tang } \tfrac{1}{2}(B-A)=\cot\tfrac{1}{2}C\,\frac{\sin\frac{1}{2}(b-a)}{\sin\frac{1}{2}(b+a)}$$

$$\text{tang } \tfrac{1}{2}(B+A)=\cot\tfrac{1}{2}C\,\frac{\cos\frac{1}{2}(b-a)}{\cos\frac{1}{2}(b+a)},$$

formulas, which will supply the place of the preceding, when we know two sides and the angle contained by them.

60. By taking all the variations, of which the equations found above are susceptible, we have

$$\text{tang}\ \tfrac{1}{2}A = \sqrt{\frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+c-b)}{\sin\frac{1}{2}(b+c-a)\sin\frac{1}{2}(a+b+c)}}$$

$$\text{tang}\ \tfrac{1}{2}B = \sqrt{\frac{\sin\frac{1}{2}(b+c-a)\sin\frac{1}{2}(a+c-b)}{\sin\frac{1}{2}(a+c-b)\sin\frac{1}{2}(a+b+c)}}$$

$$\text{tang}\ \tfrac{1}{2}C = \sqrt{\frac{\sin\frac{1}{2}(a+c-b)\sin\frac{1}{2}(b+c-a)}{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+b+c)}}$$

$$\text{tang}\ \tfrac{1}{2}a = \sqrt{\frac{-\cos\frac{1}{2}(B+C-A)\cos\frac{1}{2}(A+B+C)}{\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A+C-B)}}$$

$$\text{tang}\ \tfrac{1}{2}b = \sqrt{\frac{-\cos\frac{1}{2}(A+C-B)\cos\frac{1}{2}(A+B+C)}{\cos\frac{1}{2}(B+C-A)\cos\frac{1}{2}(A+B-C)}}$$

$$\text{tang}\ \tfrac{1}{2}c = \sqrt{\frac{-\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A+B+C)}{\cos\frac{1}{2}(A+C-B)\cos\frac{1}{2}(B+C-A)}}\ (*).$$

$$\text{tang}\ \frac{b-a}{2} = \text{tang}\ \tfrac{1}{2}c\ \frac{\sin\frac{1}{2}(B-A)}{\sin\frac{1}{2}(B+A)}$$

$$\text{tang}\ \frac{b+a}{2} = \text{tang}\ \tfrac{1}{2}c\ \frac{\cos\frac{1}{2}(B-A)}{\cos\frac{1}{2}(B+A)}$$

$$\text{tang}\ \frac{c-b}{2} = \text{tang}\ \tfrac{1}{2}a\ \frac{\sin\frac{1}{2}(C-B)}{\sin\frac{1}{2}(C+B)}$$

$$\text{tang}\ \frac{c+b}{2} = \text{tang}\ \tfrac{1}{2}a\ \frac{\cos\frac{1}{2}(C-B)}{\cos\frac{1}{2}(C+B)}$$

$$\text{tang}\ \frac{a-c}{2} = \text{tang}\ \tfrac{1}{2}b\ \frac{\sin\frac{1}{2}(A-C)}{\sin\frac{1}{2}(A+C)}$$

$$\text{tang}\ \frac{a+c}{2} = \text{tang}\ \tfrac{1}{2}b\ \frac{\cos\frac{1}{2}(A-C)}{\cos\frac{1}{2}(A+C)}$$

$$\text{tang}\ \frac{B-A}{2} = \cot\ \tfrac{1}{2}C\ \frac{\sin\frac{1}{2}(b-a)}{\sin\frac{1}{2}(b+a)}$$

$$\text{tang}\ \frac{B+A}{2} = \cot\ \tfrac{1}{2}C\ \frac{\cos\frac{1}{2}(b-a)}{\cos\frac{1}{2}(b+a)}$$

$$\text{tang}\ \frac{C-B}{2} = \cot\ \tfrac{1}{2}A\ \frac{\sin\frac{1}{2}(c-b)}{\sin\frac{1}{2}(c+b)}$$

$$\text{tang}\ \frac{C+B}{2} = \cot\ \tfrac{1}{2}A\ \frac{\cos\frac{1}{2}(c-b)}{\cos\frac{1}{2}(c+b)}$$

$$\text{tang}\ \frac{A-C}{2} = \cot\ \tfrac{1}{2}B\ \frac{\sin\frac{1}{2}(a-c)}{\sin\frac{1}{2}(a+c)}$$

$$\text{tang}\ \frac{A+C}{2} = \cot\ \tfrac{1}{2}B\ \frac{\cos\frac{1}{2}(a-c)}{\cos\frac{1}{2}(a+c)}.$$

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* In order to deduce these formulas from those analogous to them in the preceding article, it is necessary to observe that

$$\alpha - \beta - \gamma = \alpha - (\beta + \gamma),$$

and that the $\sin(p-q) = -\sin(q-p)$, and $\cos(p-q) = \cos(q-p)$.

From the last twelve formulas we deduce the following, which will serve to find the third angle or the third side of a triangle, when two sides and the two opposite angles are known.

$$\text{tang}\ \tfrac{1}{2}\, c = \text{tang}\ \tfrac{1}{2}\,(b - a)\ \frac{\sin \frac{1}{2}\,(B + A)}{\sin \frac{1}{2}\,(B - A)}$$

$$\text{tang}\ \tfrac{1}{2}\, c = \text{tang}\ \tfrac{1}{2}\,(b + a)\ \frac{\cos \frac{1}{2}\,(B + A)}{\cos \frac{1}{2}\,(B - A)}$$

$$\text{tang}\ \tfrac{1}{2}\, a = \text{tang}\ \tfrac{1}{2}\,(c - b)\ \frac{\sin \frac{1}{2}\,(C + B)}{\sin \frac{1}{2}\,(C - B)}$$

$$\text{tang}\ \tfrac{1}{2}\, a = \text{tang}\ \tfrac{1}{2}\,(c + b)\ \frac{\cos \frac{1}{2}\,(C + B)}{\cos \frac{1}{2}\,(C - B)}$$

$$\text{tang}\ \tfrac{1}{2}\, b = \text{tang}\ \tfrac{1}{2}\,(a - c)\ \frac{\sin \frac{1}{2}\,(A + C)}{\sin \frac{1}{2}\,(A - C)}$$

$$\text{tang}\ \tfrac{1}{2}\, b = \text{tang}\ \tfrac{1}{2}\,(a + c)\ \frac{\cos \frac{1}{2}\,(A + C)}{\cos \frac{1}{2}\,(A - C)}$$

$$\cot \tfrac{1}{2}\, C = \text{tang}\ \tfrac{1}{2}\,(B - A)\ \frac{\sin \frac{1}{2}\,(b + a)}{\sin \frac{1}{2}\,(b - a)}$$

$$\cot \tfrac{1}{2}\, C = \text{tang}\ \tfrac{1}{2}\,(B + A)\ \frac{\cos \frac{1}{2}\,(b + a)}{\cos \frac{1}{2}\,(b - a)}$$

$$\cot \tfrac{1}{2}\, A = \text{tang}\ \tfrac{1}{2}\,(C - B)\ \frac{\sin \frac{1}{2}\,(c + b)}{\sin \frac{1}{2}\,(c - b)}$$

$$\cot \tfrac{1}{2}\, A = \text{tang}\ \tfrac{1}{2}\,(C + B)\ \frac{\cos \frac{1}{2}\,(c + b)}{\cos \frac{1}{2}\,(c - b)}$$

$$\cot \tfrac{1}{2}\, B = \text{tang}\ \tfrac{1}{2}\,(A - C)\ \frac{\sin \frac{1}{2}\,(a + c)}{\sin \frac{1}{2}\,(a - c)}$$

$$\cot \tfrac{1}{2}\, B = \text{tang}\ \tfrac{1}{2}\,(A + C)\ \frac{\cos \frac{1}{2}\,(a + c)}{\cos \frac{1}{2}\,(a - c)}\ (*).$$

If to these equations, we add the equations (A) which are applicable to the case in which two sides and one of the opposite angles, or two angles and one of the opposite sides, are known, we shall have everything necessary for the solution of spherical triangles. The preceding part of this chapter may therefore be regarded as a complete treatise on spherical trigonometry. By combining together the different formulas successively obtained, we may deduce from them a great many others of very frequent use in astronomical calculations. We are indebted in

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* These formulas and the preceding are known under the name of the *Analogies of Napier*, because they are deduced from the rules given by that geometer, for the solution of spherical triangles. *(Logarithmorum Canonis Descriptio.)*

this respect to M. Delambre, for very elegant and very numerous results, and for important applications of approximate methods and series, to the cases which are susceptible of them.

Recapitulation of the Formulas necessary for the solution of any Spherical Triangle.

61. Neglecting the variations which the same case may present, we find only the six following formulas,

1. *Given the three sides*, (a, b, c), *to find one of the angles* (A).

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+c-b)}{\sin\frac{1}{2}(b+c-a)\sin\frac{1}{2}(a+b+c)}}.$$

2. *Given the three angles* (A, B, C), *to find one of the sides* (a).

$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos\frac{1}{2}(B+C-A)\cos\frac{1}{2}(A+B+C)}{\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A+C-B)}}\ (*).$$

3. *Given two sides* (b, c) *and the contained angle* (A), *to find the other angles* (B, C).

$$\tan \tfrac{1}{2}(B+C) = \frac{\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}(b+c)}\cot\tfrac{1}{2}A.$$

$$\tan \tfrac{1}{2}(B-C) = \frac{\sin\frac{1}{2}(b-c)}{\sin\frac{1}{2}(b+c)}\cot\tfrac{1}{2}A.$$

To find afterwards the third side (a), see the formula for case 6.

4. *Given two angles*, (B, C), *and the included side* (a), *to find the other sides* (b, c).

$$\tan \tfrac{1}{2}(b+c) = \frac{\cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B+C)}\tan\tfrac{1}{2}a,$$

$$\tan \tfrac{1}{2}(b-c) = \frac{\sin\frac{1}{2}(B-C)}{\sin\frac{1}{2}(B+C)}\tan\tfrac{1}{2}a.$$

In order to find the third angle (A), see the formula for case 5.

5. *Given two sides* (a, c), *and one of the opposite angles* (C), *to find the other opposite angle* (A).

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* Instead of this formula and the preceding, we often make use of the following, namely,

$$\sin\tfrac{1}{2}A = \sqrt{\frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a+c-b)}{\sin b \sin c}}$$

$$\sin\tfrac{1}{2}a = \sqrt{\frac{-\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(B+C-A)}{\sin B\sin C}},$$

obtained in the note to page 53, and which are analogous to that employed for the corresponding case of plane trigonometry (38).

$$\sin A = \frac{\sin a \sin C}{\sin c}.$$

6. *Given two angles* (A, C), *and one of the opposite sides* (c), *to find the other opposite side* (a).

$$\sin a = \frac{\sin c \sin A}{\sin C}.$$

To find, in the two last cases, *the angle* (B), *and the side* (b) *contained, one by the sides, the other by the angles, given or calculated*, we may change in the formulas of case 3 and case 4, b into a, B into A, and the contrary; we shall thus have

$$\text{tang}\,\tfrac{1}{2}(A + C) = \frac{\cos\frac{1}{2}(a - c)}{\cos\frac{1}{2}(a + c)}\cot\tfrac{1}{2}B,$$

$$\text{tang}\,\tfrac{1}{2}(a + c) = \frac{\cos\frac{1}{2}(A - C)}{\cos\frac{1}{2}(A + C)}\,\text{tang}\,\tfrac{1}{2}b,$$

in which everything is known except $\cot\frac{1}{2}B$ and $\text{tang}\,\frac{1}{2}b$, which will consequently be determined.

By means of this recapitulation, and of that of article 58, it is easy to resolve any spherical triangle whatever,† by applying, according to the enunciation above given, the letters A, B, C, a, b, c, to the angles and sides given and sought. The arithmetical calculations are performed by means of the addition and subtraction of logarithms, in the manner indicated in the examples under article 39; except that we employ only the table of logarithms of trigonometrical lines, since we have to do with the arcs of a circle only.

Whenever, in the four first cases, the circumstances of the question leave it doubtful whether the arcs or angles sought are greater or less than a quadrant or a right angle, the difficulty will be removed by having recourse to the expressions for the cosines or cotangents of the unknown parts (57). But in the two last cases, it may happen that the proposed question is susceptible of two solutions; we may be easily assured of this, by observing the manner of constructing a triple solid angle, when we know two of its faces and the inclination of one of them to the third, or when we know the inclinations of two faces to the third, and one of the faces. I cannot here enter into these details,* but give the results as follows.

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† See note at the end of this chapter.

* It would be well to consult on this subject, *Développement nouveau de la partie élémentaire des Mathématiques* of Bertrand, tom. II., *Trigonométrie*, section 5, or his *Elemens de Géométrie*, third part.

1. A spherical triangle can be constructed in but one way, with a, c, and C, given,

when $C = 1^q$

$C < 1^q$, $a < 1^q$, $c > a$

$C < 1^q$, $a > 1^q$, $c > 2^q - a$

$C > 1^q$, $a < 1^q$, $c < 2^q - a$

$C > 1^q$, $a > 1^q$, $c < a$;

and it is susceptible of two forms,

when $C < 1^q$, $a < 1^q$, $c < a$

$C < 1^q$, $a > 1^q$, $c < 2^q - a$

$C > 1^q$, $a < 1^q$, $c > 2^q - a$

$C > 1^q$, $a > 1^q$, $c > a$

$C <$ or $> 1^q$, $a = 1^q$.

2 With A, C, and c, given, it admits of but one form, when

$c = 1^q$

$c > 1^q$, $A > 1^q$, $C < A$

$c > 1^q$, $A < 1^q$, $C < 2^q - A$

$c < 1^q$, $A > 1^q$, $C > 2^q - A$

$c < 1^q$, $A < 1^q$, $C > A$;

and it admits of two, when

$c > 1^q$, $A > 1^q$, $C > A$

$c > 1^q$, $A < 1^q$, $C > 2^q - A$

$c < 1^q$, $A > 1^q$, $C < 2^q - A$

$c < 1^q$, $A < 1^q$, $C < A$

$c <$ or $> 1^q$, $A = 1^q$.

62. As an example of the use of spherical trigonometry, I shall select the following problem. *Given an angle* MSN (fig. 23), *measured in an inclined plane, and the angles formed by a vertical line* SS′ *with the sides* SM *and* SN *of the former, to find the angle* M′S′N′ *formed in the plane* M′S′N′, *which is horizontal, or perpendicular to* SS′, *by the projections* M′S′ *and* N′S′, *of the lines* MS *and* NS. Fig. 23.

The three lines SS', SM, and SN, determine a triple solid angle, whose vertex is the point S, in which the three plane angles MSN, MSS', and NSS', are known; and the straight line SS' being perpendicular to each of the lines $M'S'$, $N'S'$, situated respectively in the planes $S'SM$, $S'SN$ (*Geom.* 313); therefore, the lines $M'S'$, $N'S'$, contain an angle equal to that which measures the inclination of these planes (*Geom.* 349). The proposed problem is therefore reduced to determining this inclination.

But we may obtain the angle sought by considering it as making a part of the spherical triangle BAC, formed by the circles resulting from the sections, which the three planes MSN, $S'SM$, $S'SN$, would make, of a sphere whose centre is S, and radius equal to that of the tables. We have in this triangle the sides AB, AC, BC, which are respectively the measures of the given angles NSS', MSS', MSN; and the angle required is precisely the angle A. This angle, therefore, may be found by the first rule of the preceding article.

To give an example of the operation by logarithms, I shall suppose that we have observed the angles with the proper instrument, and determined their value, as follows, namely,

$$MSN = 110^\circ\ 00'\ 00'' = BC$$
$$S'SM = 58\ \ 00\ \ 00 = AC$$
$$S'SN = 79\ \ 17\ \ 14 = AB.$$

These angles represent the sides of a spherical triangle, of which the angle A is required. I put $a = 110^\circ$, $b = 58^\circ$, $c = 79^\circ\ 17'\ 14''$, and I employ this formula.

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2}(a+b-c) \sin \frac{1}{2}(a+c-b)}{\sin b \sin c}} \text{ (note to p. 59).}$$

The expression $\frac{1}{2}(a+b-c) = \frac{1}{2}(a+b+c-2c) = \frac{a+b+c}{2} - c$ is found by adding together the three sides a, b, c, and from the half sum subtracting c, one of the sides containing the required angle. In like manner, $\frac{1}{2}(a+c-b)$ is obtained by subtracting from the half sum the other of the sides, containing the angle sought. We have then only to add together the logarithms of these remainders and the arithmetical complements of the logarithms of the above mentioned sides, as exhibited below.

	110° 00′ 00″		
	58 00 00		
	79 17 14		
Sum	247 17 14		
Half sum . .	123 38 37		
	79 17 14		
1st remainder . .	44 21 28	log. sin.	9,8445513
2nd remainder .	65 38 37	log. sin	9,9595173
b	58 00 00	ar. comp. log. sin.	0,0715795
c	79 17 14	ar. comp. log. sin.	0,0076359
			19,8832840
		log. sin. $\frac{1}{2}A$ =	9,9416420,

which in the tables answers to $60^\circ\ 57'\ 28'' = \frac{1}{2}A$, and by doubling this, we have $A = 121^\circ\ 54'\ 56''$. This is the value of the angle $M'SN'$, corresponding to the given value of the angle MSN.

NOTE.

Referred to, page 60.

NAPIER'S RULES.

Certain rules were invented by Lord Napier, which afford very important assistance to the memory in the solution of all cases of right-angled spherical triangles and all but two of oblique-angled triangles.

There being six things in a triangle which are the subject of consideration, namely, three sides and three angles, in a right-angled triangle, if we set aside the right angle, we shall have five things, or *parts*, which may be reckoned round in succession, and are thence denominated *circular* parts. Of these, three enter into every case that presents itself, two being given to find the third; and these three parts must either be contiguous to each other, or (the right angle not being considered as separating the sides which contain it,) two of them must be contiguous, while the third stands by itself. When the three parts are contiguous to each other, that which is in the middle is called the *middle* part, and the two others are called *adjacent* parts. When one of the three parts stands by itself, this is called the *middle* part, and the two others are called *opposite* parts. In applying what is here said, instead of the arcs and angles themselves *the complements are always to be used, except with respect to the sides which contain the right angle*; that is, the complement of the hypothenuse is always to be taken instead of the hypothenuse, and the complement of each of the oblique angles instead of the angles themselves.

Of the five parts of a right-angled triangle, each may become the middle part, and of the remaining four, two will be adjacent and the other two opposite parts. Thus, if *CA* (complement) (*fig.* 24) be made the middle part, the angles *C*, *A*, (or rather their complements), will be the adjacent parts, and *CP*, *PA*, the opposite parts. But if *C* (complement) (*fig.* 25), be made the middle part, then *CA* (complement), *CP*, will be the adjacent parts, and *A* (complement,) *AP*, the opposite parts. Fig. 24. Fig. 25.

This being premised, we have the following rules for the solution of every case of right-angled spherical triangles.

Rectangle of radius and sine of the middle part = Rectangle of the tangents of the adjacent parts.

Rectangle of radius and sine of the middle part = Rectangle of the cosines of the opposite parts;
or, radius being put equal to unity,

1. *Sine of the middle part = Rectangle of the tangents of the adjacent parts.*

2. *Sine of the middle part = Rectangle of the cosines of the opposite parts.*†

Fig. 22. If, now, in the triangle ABC (*fig.* 22), supposed to be right-angled at C, we take each of the five circular parts successively as the middle part, we shall have all the varieties of results that can be derived from the foregoing rules, and we shall find that they agree with the formulas of art. 58, adapted to all cases of right-angled spherical triangles.

1. If (co c) be made the middle part, we shall have

(co A), (co B), for the adjacent parts,
and a, b, for the opposite parts.

Then, by rule 1,

$$\sin(\text{co } c) = \text{tang}(\text{co } A)\ \text{tang}(\text{co } B), \text{ or } \cos c = \cot A \cot B;$$

and by rule 2,

$$(\cos c) = \cos a \cos b.$$

2. If (co B) be made the middle part, we shall have

a, (co c), for the adjacent parts,
and (co A), b, for the opposite parts.

Then the rules give

$$\sin(\text{co } B,) = \text{tang } a\ \text{tang}(\text{co } c,) \quad \text{or} \quad \cot c = \frac{\cos a \cos B}{\sin a}\ (8),$$

$$\sin(\text{co } B) = \cos(\text{co } A) \cos b, \quad \text{or} \quad \cos B = \sin A \cos b.$$

3. If a be made the middle part, we shall have

b, (co B), for the adjacent parts,
and (co c) (co A,) for the opposite parts.

Then

$$\sin a = \text{tang } b\ \text{tang}(\text{co } B,) \quad \text{or} \quad \cot b = \frac{\cos B}{\sin a \sin B}\ (8, 9),$$

$$\sin a = \cos(\text{co } c) \cos(\text{co } A), \quad \text{or} \quad \sin a = \sin c \sin A.$$

---

† It will be of some assistance, in recollecting these rules, to bear in mind, that the first vowel in the words *tangents* and *adjacent* is the same, as also in the words *cosines* and *opposite*. A similar purpose is intended to be answered by the following lines;

The product of radius and middle part's sine,
Equals that of the tangents of parts that combine,
And also the cosines of those that disjoin.

4. If b be made in the middle part, we shall have

a, (co A), for the adjacent parts,

(co c,) (co B), for the opposite parts.

Then

$$\sin b = \text{tang } a \text{ tang (co } A), \quad \text{or} \quad \cot a = \frac{\cos A}{\sin b \sin A},$$

$$\sin b = \cos (\text{co } c) \cos (\text{co } B), \quad \text{or } \sin b = \sin c \sin B.$$

5. If (co A) be made the middle part, we shall have

b, (co c), for the adjacent parts,

(co B), a, for the opposite parts.

Then

$$\sin (\text{co } A) = \text{tang } b \text{ tang (co } c), \quad \text{or} \quad \cot c = \frac{\cos b \cos A}{\sin b}.$$

$$\sin (\text{co } A) = \cos (\text{co } B) \cos a, \quad \text{or} \quad \cos A = \sin B \cos a.$$

Hence the above rules, which admit of no separate and independent proof, may be considered as demonstrated.

In order to apply the method here given to oblique-angled spherical triangles, it is necessary to divide the proposed triangle into two right-angled ones, by means of a perpendicular let fall from one of the angles upon the opposite side; the perpendicular being so chosen *as to make two of the given things fall in one of the right-angled triangles*, or in other words, *the perpendicular ought to be let fall from the end of a given side and opposite to a given angle.** Each of the triangles thus found contains, as above, five circular parts, the perpendicular being counted, and bearing the same name in each of them; consequently, the parts of each triangle, similarly situated with respect to the perpendicular, must have the same name

In every case of oblique-angled spherical triangles, there are three parts given to find a fourth; and, in making use of the method of solution by means of the perpendicular, there will in general be two of these four parts in each of the triangles, similarly situated with respect to each other; to each of these must be joined the perpendicular, and there will then be three parts in each triangle, which are to be named *middle*, *adjacent*, or *opposite*, according to the directions already given.

If now we put M for the middle part, A for the adjacent part, and B for the opposite part of the triangle APC (*fig.* 26, 27, 28, 29), m, a, b, for the corresponding parts of the triangle APB, and P for **Fig. 26, 27, 28, 29.**

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* When this can be done in two different ways, as in cases 2, 4, it will generally produce the shortest solution to make use of that perpendicular which does not divide the *required* side or angle into segments.

the perpendicular AP; then, if P be an adjacent part, we shall have, by the rules already established,

$$\sin M = \text{tang}\, P\, \text{tang}\, A, \quad \text{or} \quad \text{tang}\, P = \frac{\sin M}{\text{tang}\, A},$$

also $$\sin m = \text{tang}\, P\, \text{tang}\, a, \quad \text{or} \quad \text{tang}\, P = \frac{\sin m}{\text{tang}\, a},$$

whence $$\frac{\sin M}{\text{tang}\, A} = \frac{\sin m}{\text{tang}\, a},$$

or $$\sin M : \text{tang}\, A :: \sin m : \text{tang}\, a.$$

If P be on opposite part, we shall have

$$\sin M = \cos P \cos B, \quad \text{or} \quad \cos P = \frac{\sin M}{\cos B},$$

also $$\sin m = \cos P \cos b, \text{ or } \cos P = \frac{\sin m}{\cos b},$$

whence $$\frac{\sin M}{\cos B} = \frac{\sin m}{\cos b},$$

or $$\sin M : \cos B :: \sin m : \cos b.$$

We have, therefore, demonstrated the following additional rules for oblique-angled spherical triangles, namely,

3. *The sines of the middle parts are proportional to the tangents of the adjacent parts.*
4. *The sines of the middle parts are proportional to the cosines of the opposite parts;*

it being observed that the perpendicular, common to the two triangles and bearing the same name in each, is not to be made use of in the proportions, or counted as a middle part. This can produce no embarrassment, as the cases of oblique-angled spherical triangles may in general be solved in the shortest manner without calculating the perpendicular, as will be evident by the following examples.

Fig. 26, 27, 28, 29. 1. *Given two sides* AB, AC (fig. 26, 27, 28, 29,) *and an angle* C *opposite to one of these sides; to find* BC *and the angles* A, B.

In the right-angled triangle APC, are given AC and C; and, by marking it as in figure 25, CP may be found by rule 1, which gives

$$\sin (\text{co}\, C) = \text{tang}\, CP\, \text{tang}\, (\text{co}\, AC),$$

or $$\text{tang}\, CP = \cos C\, \text{tang}\, AC^* \quad (9).$$

Then, in the triangles ABP, ACP, are given AB, AC, and CP, to find BP. If to these is joined the perpendicular AP, it will be found that in the triangle ACP, (co AC) is the middle part (fig. 26), and CP

* In putting this or any similar expression in logarithms, the radius must be neglected in the sum of the two logarithms of the second member.

an opposite part. The triangle ABP being marked in a similar manner, we have, by rule 4,

$$\sin(\text{co } AC) : \cos CP : : \sin(\text{co } AB) : \cos BP\dagger.$$

And $$BC = BP \underset{\backsim}{+} CP\ddagger.$$

By marking the segments as in figure 27, we have, by rule 3,

$$\sin CP : \text{tang}(\text{co } C) : : \sin BP : \text{tang}(\text{co } B);$$

having found BC, the angle A may be found by the equations marked (A) (47), which are equivalent to

4. *The sines of the sides are proportional to the sines of the opposite angles.*

This rule gives

$$\sin AB : \sin C : : \sin BC : \sin A.$$

Otherwise; if the side BC be not required, the angles A, B, may be found in the following manner. Marking as in figure 24, we have, by rule 1,

$$\sin(\text{co } AC) = \text{tang}(\text{co } C)\,\text{tang}(\text{co } CAP),$$

or $$\cot CAP = \cos AC \,\text{tang}\, C \;(9);$$

and, marking as in figure 28, we have, by rule 3,

$$\sin(\text{co } CAP) : \text{tang}(\text{co } AC) : : \sin(\text{co } BAP) : \text{tang}(\text{co } AB),$$

or $$\text{tang}(\text{co } AC) : \sin(\text{co } CAP) : : \text{tang}(\text{co } AB) : \sin(\text{co } BAP).$$

Then $$A = BAP \underset{\backsim}{+} CAP.$$

Marking the segments as in figure 29, we have, by rule 4,

$$\sin(\text{co } C) : \cos(\text{co } CAP) : : \sin(\text{co } B) : \cos(\text{co } BAP),$$

or $$\cos(\text{co } CAP) : \sin(\text{co } C) : : \cos(\text{co } BAP) : \sin(\text{co } B),$$

or $$\sin CAP : \cos C : : \sin BAP : \cos B.$$

Having A, C, and AB, we can find BC by rule 5, which gives

$$\sin C : \sin AB : : \sin A : \sin BC.$$

2. *Given two sides* AC, BC (fig. 26, 27), *and the included angle* C, *to find* AB, *and the angles* A, B. Fig. 26, 27.

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† It is evident, that the same result might be obtained directly, in each of these examples, by means of the rules 1 and 2; thus, the same marking being used as above,

$$\sin(\text{co } AC) = \cos CP \cos AP, \quad \text{or} \quad \frac{\sin(\text{co } AC)}{\cos CP} = \cos AP,$$

$$\sin(\text{co } AB) = \cos BP \cos AP, \quad \text{or} \quad \frac{\sin(\text{co } AB)}{\cos BP} = \cos AP;$$

whence $$\frac{\sin(\text{co } AC)}{\cos CP} = \frac{\sin(\text{co } AB)}{\cos BP},$$

or $\sin(\text{co } AC) : \cos CP : : \sin(\text{co } AB) : \cos BP$; and so of the others.

‡ The sign $\backsim$ signifies *difference*, and by the double $\underset{\backsim}{+}$ is to be understood the sum or difference of the quantities between which it is placed.

We have by rule 1, as in the former case,

$$\text{tang } CP = \cos C \text{ tang } AC,$$

then $$BP = BC \mp CP,$$

and, marking as in figure 26, we have, by rule 4,

$$\sin (\text{co } AC) : \cos CP :: \sin (\text{co } AB) : \cos BP,$$

or $$\cos CP : \sin (\text{co } AC) :: \cos BP : \sin (\text{co } AB).$$

Marking as in figure 27, we have, by rule 3,

$$\sin CP : \text{tang } (\text{co } C) :: \sin BP : \text{tang } (\text{co } B).$$

Having found AB, we can find A by rule 5, thus,

$$\sin AB : \sin C :: \sin BC : \sin A.$$

If the angle A had been required, and not B, it would have been shorter to employ a perpendicular, let fall from the point B, by which means the required angle A would not be divided into segments. In this case the side AB and the angle A might be found in a manner similar to that by which AB and B are found above.

Fig. 26, 27, 28, 29. 3. *Given the angles* B, C, (fig. 26, 27, 28, 29), *and the opposite side* AC, *to find* BC, AB, *and the angle* A.

We have, by rule 1, as in case 1,

$$\text{tang } CP = \cos C \text{ tang } AC$$

Then, marking as in figure 27, we have, by rule 3,

$$\sin CP : \text{tang } (\text{co } C) :: \sin BP : \text{tang } (\text{co } B),$$

or $$\text{tang } (\text{co } C) : \sin CP :: \text{tang } (\text{co } B) : \sin BP,$$

then $$BC = CP \mp BP.$$

Again, marking as in figure 26, we have, by rule 4,

$$\sin (\text{co } AC) : \cos CP :: \sin (\text{co } AB) : \cos BP,$$

or $$\cos CP : \sin (\text{co } AC) :: \cos BP : \sin (\text{co } AB).$$

BC being found, we have, by rule 5,

$$\sin AC : \sin B :: \sin BC : \sin A.$$

Otherwise; by rule 1, we have, as in case 1,

$$\cot CAP = \cos AC \text{ tang } C:$$

and, marking as in figure 29, we have, by rule 4,

$$\sin (\text{co } C) : \cos (\text{co } CAP) :: \sin (\text{co } B) : \cos (\text{co } BAP),$$

or $$\cos C : \sin CAP :: \cos B : \sin BAP,$$

and $$A = CAP \mp BAP.$$

Then, marking as in figure 28, we have, by rule 3,

$$\sin (\text{co } CAP) : \text{tang } (\text{co } AC) :: \sin (\text{co } BAP) : \text{tang } (\text{co } AB)$$

A being found, we have, by rule 5,

$$\sin B : \sin AC :: \sin A : \sin BC.$$

Fig. 28, 29. 4. *Given the angles* A, C (fig. 28, 29), *and the included side* AC, *to find* AB, BC, *and the angle* B.

By rule 1, we have, as in case 1,

$$\cot CAP = \cos AC \text{ tang } C,$$

and $$BAP = A \mp CAP.$$

Marking as in figure 28, we have, by rule 3,

sin (co CAP) : tang (co AC) : : sin (co BAP) : tang (co AB).

Marking as in figure 29, we have, by rule 4,

sin (co C) : cos (co CAP) : : sin (co B) : cos (co BAP),

or cos (co CAP) : sin (co C) : : cos (co BAP) : sin (co B,)

or sin CAP : cos C : : sin BAP : cos B.

B being found, we have, by rule 5,

sin B : sin AC : : sin A : sin BC.

If the side BC had been required, and not AB, it would have been shorter to employ a perpendicular, let fall from the point C, by which means the required side BC would not be divided into segments. In this case, the side BC and the angle B might be found in a manner similar to that by which AB and B are found above.

Thus the rules of Lord Napier, together with the well known rule, that the sines of the sides are proportional to the sines of the opposite angles, furnish a complete solution of the various cases of spherical triangles, except where three sides are given to find an angle, or (which is nearly the same thing, by taking the supplemental triangle (*Geom.* 476)) three angles to find a side.

In what relates to oblique-angled spherical triangles of the above note, the writer has availed himself of the improved method of Bowditch, contained in a memoir on the *Application of Napier's rules*, and published in the third volume of the *Memoirs of the American Academy of Arts and Sciences.*

CHAPTER III.

APPLICATION OF ALGEBRA TO GEOMETRY.

Of the Geometrical construction of Algebraic Quantities.

63. As lines, surfaces, and solids are quantities, each admits of the operations which are performed upon numbers and algebraic quantities. But the results of such operations may be estimated in two principal ways, either by numbers or by lines. The first of these, as it supposes that each of the given quantities is expressed by numbers, is at present attended with no difficulty; it is only necessary to substitute in the place of the letters the numerical quantities which they represent, and to perform the operations indicated by the disposition of the signs and letters.

As to the manner of estimating by lines the results of solutions furnished by algebra, it is founded upon the import of certain fundamental expressions, to which all others are afterwards referred. We proceed to make known these expressions, and to explain how the others are referred to them. This is called *constructing* the algebraic quantities, or the problems which have led to these quantities.

64. Let it be proposed to construct such a quantity as $\frac{ab}{c}$, in which a, b, c, stand for known lines. We draw two indefinite Fig. 30. lines AZ, AX (*fig.* 30), making any angle with each other; upon one of these lines AX, we take a part AB, equal to the line represented by c, and a part AD, equal to one or the other of the two lines a and b, a, for example; then upon the second AZ, we take a part AC, equal to the line b. Having joined the extremities B, C, of the first and third by the line BC, we draw, through the extremity D of the second, the line DE parallel to BC; this will determine upon AZ the part AE as the value of $\frac{ab}{c}$. For the parallels DE, BC, give this proportion,

$$AB : AD :: AC : AE \quad (\textit{Geom.}\ 197),$$

or $$c : a :: b : AE;$$

therefore $$AE = \frac{ab}{c}.$$

In other words, it is necessary to find a fourth proportional to three given lines c, a, b; and, as we have given the method of finding this fourth proportional, we can employ it for the construction of the quantity $\frac{ab}{c}$. (*Geom.* 237).

It will be seen therefore, that, if it were proposed to construct $\frac{a^2}{c}$, it might be done in the same manner, since in this case the line b is equal to a.

If it were proposed to construct $\frac{ab + bd}{c + d}$; it is to be observed, that this quantity is the same as $\frac{(a + d)\,b}{c + d}$; regarding therefore $a + d$ as one line, represented by m, and $c + d$ also as one line, represented by n, we shall have $\frac{mb}{n}$ to be constructed, which refers itself to the preceding case.

Let the quantity to be constructed be $\frac{a^2 - b^2}{c}$; it will be recollected that $a^2 - b^2$ is equivalent to $(a + b)\,(a - b,)$ (*Alg.* 34), so that $\frac{a^2 - b^2}{c}$ may be represented under the form $\frac{(a+b)\,(a-b)}{c}$; and we have only to find a fourth proportional to c, $a + b$, $a - b$.

If the quantity to be constructed be $\frac{abc}{de}$, we put it under this form $\frac{ab}{d} \times \frac{c}{e}$; and, having constructed $\frac{ab}{d}$ in the manner just explained, we call m the line given by this construction; then $\frac{ab}{d} \times \frac{c}{e}$ becomes $\frac{mc}{e}$, which is constructed as above shown.

We see, therefore, that in order to construct $\frac{a^2 b}{c^2}$, we represent it under the form $\frac{a^2}{c} \times \frac{b}{c}$; we then construct $\frac{a^2}{c}$; and, having represented the value of this by m, we construct $\frac{mb}{c}$.

Thus the whole art consists in decomposing the quantity into portions, each of which returns to the form $\frac{ab}{c}$, or $\frac{a^2}{c}$; and, although this process may appear difficult in some cases, yet we easily arrive at the object proposed, by employing transformations.

If, for example, I had to construct $\frac{a^3 + b^3}{a^2 + c^2}$; I should put $b^3 = a^2 m$, and $c^2 = a n$; then $\frac{a^3 + b^3}{a^2 + c^2}$ becomes $\frac{a^3 + a^2 m}{a^2 + a n}$, which reduces itself to $\frac{a^2 + a m}{a + n}$, or $\frac{(a + m) a}{a + n}$, a quantity easy to be constructed, after what has been said, when m and n are known. Now to determine m and n, the equations $b^3 = a^2 m$, $c^2 = a n$, give $m = \frac{b^3}{a^2}$, and $n = \frac{c^2}{a}$, which are constructed according to the method already laid down.

Thus, while the quantity is rational, that is, without radical expressions, if the dimensions of the numerator do not exceed those of the denominator except by unity, we may always reduce the construction to the finding of a fourth proportional to three given lines.

It sometimes happens, that quantities present themselves under a form, that seems to render recourse to transformations of no use; it is when the quantity is not *homogeneous*, that is, when each of the terms of the numerator and denominator is not composed of the same number of factors; when the quantity, for example, is such as $\frac{a^3 + b}{c^2 + d}$.

But it should be observed, that we never arrive at a result of this kind, except when, in the course of an investigation, we suppose, with a view of simplifying the calculation, some one of the quantities equal to unity. If, for example, in $\frac{a^3 + b^2 c}{a^2 + c^2}$, I suppose b equal to 1, I shall have $\frac{a^3 + c}{a^2 + c^2}$. But, as we never undertake to construct a quantity without knowing the elements which we are to use for this construction, we always know in each case what is the quantity which is supposed equal to unity. We can always therefore restore it, and the above difficulty cannot occur; because, as the number of dimensions must be the same in each term of the numerator, and also of the denominator, although the number of terms may be different in the one from what it is in the other, we restore in each term a power of the line, which is taken for unity, sufficiently raised to complete the number of dimensions; thus, if I have to construct $\frac{a^3 + b + c^2}{a + b^2}$; d being supposed to be the line which is taken for unity, I write $\frac{a^3 + b\,d^2 + c^2\,d}{a\,d + b^2}$, which I should construct by making $b^2 = d\,m$, $c^2 = d\,n$, and $a^3 = d^2\,p$, which would change it into

$$\frac{d^2\,p + b\,d^2 + d^2\,n}{a\,d + d\,m},$$

or $\frac{d\,p + b\,d + d\,n}{a + m}$, or $\frac{(p + b + n)\,d}{a + m}$, a quantity easily constructed, when we have constructed the value of m, n, and p; namely, $m = \frac{b^2}{d}$, $n = \frac{c^2}{d}$, $p = \frac{a}{d^2}$, which is readily done after what has been said.

Hitherto we have supposed that the number of factors, or the dimensions of each term of the numerator exceeds the number of factors, or the dimensions of the denominator only by unity. It may exceed it by two or even three, but never by more than three, unless some line has been supposed equal to unity, or some of the factors do not represent numbers.

65. When the dimensions of the numerator of the proposed quantity exceed by two the dimensions of the denominator, the quantity expressed is a surface, the construction of which can always be reduced to that of a parallelogram, and consequently to that of a square. If, for example, the quantity to be constructed be

$$\frac{a^3 + a^2\,b}{a + c},$$

I should consider it as $a \times \frac{a^2 + a\,b}{a + c}$. Now $\frac{a + a\,b}{a + c}$ is easily constructed, after what has been laid down, by considering it as $a \times \frac{a + b}{a + c}$. Let us suppose therefore that m is the value of the line thus obtained; then $a \times \frac{a^2 + a\,b}{a + c}$ will become $a \times m$. Now if we make a the altitude and m the base of a parallelogram, we shall have $a \times m$ for the surface of this parallelogram (*Geom.* 174), therefore, reciprocally, this surface will represent $a \times m$, or $\frac{a^3 + a^2\,b}{a + c}$.

In like manner, the quantity $\frac{a^3+bc^2+d^3}{a+c}$ may be reduced to a similar construction by making $bc=am$, and $d^2=an$; for it will then become $\frac{a^3+amc+and}{a+c}$, or $a\left(\frac{a^2+mc+nd}{a+c}\right)$. Now the factor $\frac{a^2+mc+nd}{a+c}$ refers itself to the preceding constructions, as also the values of m, n. Having found the value of this factor, if I represent it by p, we have only to construct $a \times p$, that is, to make a parallelogram whose altitude is a and base p.

66. Lastly, if the dimensions of the numerator exceed the dimensions of the denominator by three, the quantity expresses a solid, the construction of which may always be reduced to a parallelopiped. If, for example, I had to construct $\frac{a^3 b + a^2 b^2}{a+c}$, I should consider this quantity as the same as $ab \times \frac{a^2+ab}{a+c}$; and, having constructed $\frac{a^2+ab}{a+c}$ in the manner already explained, if I represent by m, the line given by this construction, the question will be reduced to this, namely, to construct $ab \times m$. Now ab represents, as we have seen, a parallelogram; if, therefore, we conceive a parallelopiped, having for its base this parallelogram and for its altitude the line m, the solidity of this parallelopiped will represent $ab \times m$, that is, $\frac{a^3 b + a^2 b^2}{a+c}$.

67. What has been said will suffice for constructing any rational quantity; we proceed now to radical quantities of the second degree.

In order to construct $\sqrt{ab}$, it is necessary to draw an indefinite line AB (*fig.* 31), upon which we take the part CA, equal to the line a, and the part BC, equal to the line b; upon the whole AB as a diameter, we describe a semicircle, cutting in D, the perpendicular CD, raised upon AB at the point C; then CD will be the value of $\sqrt{ab}$; that is, the value of $\sqrt{ab}$ is obtained by finding a mean proportional between the two quantities represented by a, b. Indeed, we have

Fig. 31.

$$AC : CD :: CD : CB,$$

or
$$a : CD :: CD : b;$$

whence
$$\overline{CD}^2 = ab, \quad \text{or} \quad CD = \sqrt{ab}.$$

If we have to construct $\sqrt{3\,a\,b + b^2}$, or which is the same thing, $\sqrt{(3\,a + b)\,b}$, we should find a mean proportional between $3\,a + b$ and b.

In like manner, if the quantity to be constructed were

$$\sqrt{a^2 - b^2},$$

we should consider this the same as $\sqrt{(a + b)\,(a - b)}$ (*Alg.* 34); we then find a mean proportional between $a + b$ and $a - b$. If the quantity were $\sqrt{a^2 + b\,c}$, we make $b\,c = a\,m$, and then we shall have $\sqrt{a^2 + a\,m}$, or $\sqrt{(a + m)\,a}$, which is constructed by finding a mean proportional between $a + m$ and a after having constructed the value of $m = \frac{b\,c}{a}$ by the rules already given.

To construct $\sqrt{a^2 + b^2}$, we can in like manner make $b^2 = a\,m$, and construct $\sqrt{a^2 + a\,m}$, in the manner just explained. But the property of a right-angled triangle furnishes a more simple construction. If we draw the line AB (*fig.* 32), equal to a, and at its extremity A erect a perpendicular AC, equal to b, joining BC, we shall have $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 = a^2 + b^2$, and consequently $BC = \sqrt{a^2 + b^2}$. Fig. 32.

We can also, by means of a right-angled triangle, construct $\sqrt{a^2 - b^2}$ in a manner different from that above given; we draw a line AB (*fig.* 34), equal to a, and having described upon AB, as a diameter, the semicircle ACB, we draw from the point A a chord AC, equal to b; then, if we draw BC, this line will be the value of $\sqrt{a^2 - b^2}$; for the triangle ABC being right-angled, (*Geom.* 128), we shall have $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$; consequently, $\overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2 = a^2 - b^2$; therefore $BC = \sqrt{a^2 - b^2}$. Fig. 34.

Hence, also, $\sqrt{a^2 + b\,c}$ admits of a different construction from the above. We make $b\,c = m^2$, and construct $\sqrt{a^2 + m^2}$, as just shown, first finding for m a mean proportional between b and c, as indicated by the equation $b\,c = m^2$, which gives $m = \sqrt{b\,c}$.

If there are more than two terms under the radical sign, the construction is to be reduced to one of the preceding methods by means of transformations. If, for example, we have

$$\sqrt{a^2 + b\,c + e\,f},$$

we make $b\,c = a\,m$, $e\,f = a\,n$, and we have $\sqrt{a^2 + a\,m + a\,n}$,

or $$\sqrt{(a+m+n)a},$$
which may be constructed by finding a mean proportional between a and $a+m+n$, after having constructed the values of m and n, namely, $m=\frac{bc}{a}$, $n=\frac{ef}{a}$. We might, moreover, make
$$bc=m^2,\ ef=n^2,$$
and then we should have to construct $\sqrt{a^2+m^2+n^2}$. Now, when there are several positive squares contained under the radical sign, as $\sqrt{a^2+m^2+n^2+p^2+\&c.}$ we make
$$\sqrt{a^2+m^2}=h,$$
$\sqrt{h^2+n^2}=i$, $\sqrt{i^2+p^2}=k$, and so on; and, as each of the quantities is determined by the preceding, the last will give the value of $\sqrt{a^2+m^2+n^2+p^2+\&c.}$ In order to construct these quantities in the most simple manner, each hypothenuse is to be regarded successively as a side; having, for example, taken
Fig. 33. $AB=a$ (*fig.* 33), and raised the perpendicular $AC=c$, we join BC, which will be h; then at the point C we raise upon BC the perpendicular $CD=n$; and having drawn BD, which will be i, at the extremity D, we raise upon BD the perpendicular $DE=p$, and BE will be k, and equal to $\sqrt{a^2+m^2+n^2+p^2}$.

If some of the squares are negative, we combine the method just given with that for constructing $\sqrt{a^2-b^2}$.

Lastly, if the quantity to be constructed be of this form
$$a\frac{\sqrt{b+c}}{\sqrt{d+e}},$$
multiplying by $\sqrt{d+e}$, we change it into $a\frac{\sqrt{(b+c)(d+e)}}{d+e}$; then, finding a mean proportional between $b+c$ and $d+e$, and calling it m, we have $\frac{am}{d+e}$, which is easily constructed.

The construction often becomes much more simple by setting out always from the same principles; but these simplifications are derived from certain considerations which are peculiar to each question, and consequently can be made known only as the occasion presents itself. We will merely remark, in concluding, that although the construction of the radical quantities, which we have been considering, reduces itself to finding fourth proportionals, mean proportionals, and constructing right-angled triangles, still we can arrive at constructions more or less simple or elegant by the method employed for finding these mean proportionals; we shall now, therefore, make known two other methods of finding a mean proportional between two given lines.

The first consists in describing upon the greater AB (*fig.* 34) **Fig. 34.** of two given lines a semicircle ACB, and, having taken a part AD equal to the less, raising a perpendicular DC and drawing the chord AC, which will be a mean proportional between AB and AD; for, by drawing CB, the triangle ACB is right-angled (*Geom.* 128), and consequently AC is a mean proportional between the hypothenuse AB and the segment AD (*Geom.* 213).

The second method consists in drawing a line AB (*fig.* 35), **Fig. 35.** equal to the greater given line, and having taken a part AC equal to the less, describing upon the remainder BC a semicircle CDB, to which we draw the tangent AD; this tangent is a mean proportional between AB and AC (*Geom.* 228).

We see, therefore, that rational quantities may always be constructed by means of straight lines, and radical quantities of the second degree may be constructed by means of the circle and straight line united.

As to radical quantities of higher degrees, their construction depends upon the combination of different curved lines. We shall speak of these hereafter.

We proceed now to the consideration of questions, the solution of which depends either upon rational quantities or radical quantities of the second degree.

Geometrical questions, and remarks upon the manner of putting them into equations, and upon the different solutions derived from these equations.

68. The precept which we have given (*Alg.* 14) for putting a question into an equation is equally applicable to questions in geometry. Here, also, the thing sought is to be represented by a particular sign, and the reasoning is to be conducted by the aid of this sign, and of those which represent the other quantities, as if the whole were known, and we were undertaking to verify it. The method or manner of proceeding is called *Analysis*. In order to be able to carry on the investigation required for this verification, it is necessary to know at least some properties of the quantity sought. Moreover, it is not always necessary,

in order to verify a quantity, to see if it satisfies immediately the conditions of the question; this verification is often made more easily by inquiring whether this quantity has certain properties which are essentially connected with the conditions of the question. With this remark, which we shall have occasion to make use of, we pass to examples, which on subjects of this kind are always more easily understood than general directions.

69. We propose, for the first question, *to describe a square*
Fig. 36. ABCD (*fig.* 36), *in a given triangle* EHI.

By *a given triangle*, we understand a triangle, in which every thing is known; the sides, angles, altitude, &c.

With a little attention, we see that this question reduces itself to finding, in the altitude EF, a point G, through which AB being drawn parallel to HI, this line AB shall be equal to GF; thus the question presents itself very naturally, and we have only to determine the algebraic expression of AB and that of GF, and then to put them equal to each other.

We shall therefore designate the known altitude EF by a, the known base HI by b, and the unknown line GF by x; then EG will be equal to $a - x$.

Now, since AB is parallel to HI, we shall have

$$EF : EG :: FI : GB :: EI : EB :: HI : AB,$$

consequently $$EF : EG :: HI : AB,$$

or $$a : a - x :: b : AB;$$

whence $$AB = \frac{ab - bx}{a}.$$

But $$AB = GF = x,$$

therefore $$\frac{ab - bx}{a} = x,$$

and $$ab - bx = ax,\ ab = ax + bx = (a + b)x,$$

whence $$a = \frac{ab}{a + b}.$$

To construct this quantity, it is necessary to find a fourth proportional to $a + b$, b, and a (64), which is done in this manner. We apply from F to O a line FO equal to $a + b$, that is, equal to $EF + HI$, and join EO; then having taken $FM = HI = b$, we draw parallel to EO the line MG, which, by its meeting with EF, will give the determination of GF or the value of x; for the similar triangles EFO, GFM, give

$$FO : FM :: FE : FG,$$

or $$a+b:b::a:FG;$$

we have, therefore, $FG=\frac{ab}{a+b}$.

70. Let now the following question be proposed.

Given the length of the line BC (fig. 37), *and the angles* B, C, *contained by this line and the two lines* BA, CA, *to determine the altitude* AD *of the point of meeting of these two lines.* **Fig. 37.**

Angles are made to enter into an algebraic expression by the aid of the lines employed in trigonometry, namely, sines, tangents, &c. Thus, when it is said that an angle is given, the angle C, for example, it is to be understood that the value of its sine or tangent is given. This being premised, we designate BC by a, and AD by y. In the right-angled triangle ADC we have

$$CD:AD::\text{radius}:\text{tang}\ ACD\ (30),$$

or $$CD:y::r:m,$$

designating the radius by r, and the tangent of the angle ACD by m;

whence $$CD=\frac{ry}{m}.$$

In like manner, designating the tangent of ABD by n, we shall have $$BD:y::r:n,$$

whence $$BD=\frac{ry}{n}.$$

Now $$CD+BD=BC=a,$$

therefore $$\frac{ry}{m}+\frac{ry}{n}=a,$$

whence, making the denominators to disappear, transposing, and reducing, we have

$$y=\frac{amn}{rn+rm}.$$

This expression may be rendered more simple by introducing in the place of the tangents of the angles C, B, their cotangents, which we shall designate by p, q. Recollecting that the tangent is to radius, as radius to the cotangent (9), we shall have

$$m:r::r:p,\ \text{and}\ n:r::r:q;$$

whence $$m=\frac{r^2}{p},\ \text{and}\ n=\frac{r^2}{q}.$$

Substituting these values in the place of m and n, we shall have

$$y = \frac{\frac{a r^4}{p q}}{\frac{r^3}{q} + \frac{r^3}{p}} = \frac{\frac{a r^4}{p q}}{\frac{p r^3 + q r^3}{p q}} = \frac{a r^4}{p q} \times \frac{p q}{p r^3 + q r^3} = \frac{a r}{p + q}$$

71. Whence it will be perceived, that when, among quantities that may be regarded as given, those employed do not lead to a result so simple as may be desired, it is not necessary to commence the work anew, to assure one's self whether, by employing other given quantities, we cannot arrive at a more simple result; it is sufficient to express by equations the ratio of the given quantities first employed to those which we would introduce, as we have expressed m, n, by the equations $m = \frac{r^2}{p}$, $n = \frac{r^2}{p}$, which led by substitution merely to a solution dependent upon p and q.

72. We shall take, for a third example, a question which will show at once the manner of putting geometrical questions into equations, and how, by varying the preparation of these equations, new propositions may be discovered.

Fig. 38. *Given the three sides of a triangle* ABC (fig. 38), *to find the segments* AD, DC, *formed by the perpendicular* BD, *and the perpendicular itself* BD.

If these lines were all known, I should verify them in this manner. I should add the square of BD to the square of CD, and see if the sum were equal to the square of BC, to which it ought to be, since the triangle BDC is right-angled; I should also add the square of AD to the square BD, and see if the sum were equal to the square of AB.

Let us proceed then in this manner, designating the quantities to be employed, as follows,

$$BD = y, \qquad BC = a,$$
$$CD = x, \qquad AB = b, \qquad AC = c;$$

then AD, or $AC - CD = c - x$. Accordingly we have

$$x^2 + y^2 = a^2, \text{ and } c^2 - 2cx + x^2 + y^2 = b^2.$$

As x^2 and y^2 have in each equation only unity for a coefficient, I subtract the second equation from the first, which gives directly

$$2cx - c^2 = a^2 - b^2;$$

whence we have

$$x = \frac{a^2 - b^2 + c^2}{2c} = \frac{a^2 - b^2}{2c} + \tfrac{1}{2}c,$$

which may be written thus,

$$x = \tfrac{1}{2}\,\frac{(a+b)\,(a-b)}{c} + \tfrac{1}{2}\,c \ (Alg.\ 34).$$

Now under this form we see, after what has been said (64), that to obtain x, we have to find a fourth proportional to c, $a + b$, and $a - b$, to take one half of this and add it to $\frac{1}{2}$ c, or one half of the side AC.

Several other conclusions may be derived from these equations; I shall deduce some of them, that the learner may be able to read what is contained in an equation.

73. 1. The equation $2\,c\,x - c^2 = a^2 - b^2$ is the same as $c\,(2\,x - c) = (a+b)\,(a-b)$.

Now, since the product of the first two factors is equal to the product of the last two, we may consider the first two as the extremes, and the last two as the means of a proportion; we accordingly have

$$c : a + b :: a - b : 2\,x - c \quad \text{or} \quad x - (c - x);$$

$$\text{or} \quad AC : BC + AB :: BC - AB : CD - AD\dagger.$$

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† This proportion may be readily obtained, geometrically; thus, from the point B (*fig.* 38′), as a centre, and with the radius AB, describe the circle $AGFE$; then we have Fig. 38′.

$$CG = BC + BG = BC + AB,$$

$$CF = BC - BF = BC - AB,$$

and $CE = CD - DE = CD - AD$ (*Geom.* 105.)

But since AC, CG, are secants, drawn from a point without the circle to the concave part of the circumference,

$$AC : CG :: CF : CE \quad (Geom.\ 225),$$

that is, $AC : BC + AB :: BC - AB : CD - AD$, as above found.

Moreover, from this proportion we have

$$CD - AD = \frac{(BC + AB)\,(BC - AB)}{AC}.$$

Now, if to the sum of two quantities we add their difference, we shall have twice the greater; consequently, half of the sum added to half of the difference will give once the greater. Therefore,

$$CD = \tfrac{1}{2}\,\frac{(BC + AB)\,(BC - AB)}{AC} + \tfrac{1}{2}\,AC,$$

or

$$x = \tfrac{1}{2}\,\frac{(a+b)\,(a-b)}{c} + \tfrac{1}{2}\,c,$$

which agrees with the analytical determination of x (72).

74. 2. If from the point C as a centre, and with a radius equal to BC, we describe the arc BO, and draw the chord BO, we shall have

$$\overline{BD}^2 + \overline{DO}^2 = \overline{BO}^2;$$

now $\quad DO = CO - CD = BC - CD = a - x,$

therefore $\quad \overline{BO}^2 = y^2 + a^2 - 2\,a\,x + x^2;$

but we have found above $\quad y^2 + x^2 = a^2;$

consequently $\quad \overline{BO}^2 = 2\,a^2 - 2\,a\,x = 2\,a\,(a - x).$

Putting for x its value $\dfrac{a^2 - b^2 + c^2}{2\,c}$, (72),

since $2\,a\,c - a^2 - c^2 = -(a^2 - 2\,a\,c + c^2) = -(c - a)^2$, we shall have

$$\overline{BO}^2 = 2\,a\left(a + \frac{b^2 - a^2 - c^2}{2\,c}\right) = 2\,a\left(\frac{2\,a\,c - a^2 - c^2 + b^2}{2\,c}\right)$$

$$= \frac{a}{c}\left(b^2 - (c - a)^2\right).$$

Now, by considering $c - a$ as a single quantity, we find $b^2 - (c - a)^2 = (b + c - a)\,(b - c + a)$ (*Geom.* 184); hence

$$\overline{BO}^2 = \frac{a}{c}(b + c - a)\,(b - c + a),$$

which may be put under this form,

$$\overline{BO}^2 = \frac{a}{c}(a + b + c - 2\,a)\,(a + b + c - 2\,c).$$

If therefore we designate the sum of the three sides by $2\,s$, we shall have

$$\overline{BO}^2 = \frac{a}{c}(2\,s - 2\,a)\,(2\,s - 2\,c) = 4\,\frac{a}{c}(s - a)\,(s - c).$$

Letting fall from the point C upon OB the perpendicular CI, we obtain from the right-angled triangle CIO this proportion,

$$CO : OI :: R : \sin OCI, \quad (30),$$

that is, $\quad a : \tfrac{1}{2}\,BO :: R : \sin OCI,$

whence $\quad \tfrac{1}{2}\,BO = \dfrac{a \sin OCI}{R}$, or $BO = \dfrac{2\,a \sin OCI}{R};$

consequently $\quad \overline{BO}^2 = \dfrac{4\,a^2\,(\sin OCI)^2}{R^2}.$

Putting these two values of $\overline{BO}^2$ equal to each other, we have

$$\frac{4\,a^2\,(\sin OCI)^2}{R^2} = \frac{4\,a}{c}(s - a)\,(s - c),$$

or, dividing by $4\,a$, and making the denominators to disappear,

$$a\,c\,(\sin OCI)^2 = R^2\,(s - a)\,(s - c);$$

that is, dividing by $a\,c$, putting R equal to 1, and extracting the square root,

$$\sin OCI = \sqrt{\frac{(s-a)\,(s-c)}{a\,c}},$$

which agrees with the formula of art. 38.

75. 3. The equation $y^2 + x^2 = a^2$,

gives $$y^2 = a^2 - x^2 = (a+x)\,(a-x)\,;$$

putting for x its value, we have

$$y^2 = \left(a + \frac{a^2 - b^2 + c^2}{2\,c}\right)\left(a + \frac{b^2 - a^2 - c^2}{2\,c}\right)$$

$$= \left(\frac{2\,a\,c + a^2 + c^2 - b^2}{2\,c}\right)\left(\frac{2\,a\,c - a^2 - c^2 + b^2}{2\,c}\right)$$

$$= \left(\frac{(a+c)^2 - b^2}{2\,c}\right)\left(\frac{b^2 - (c-a)^2}{2\,c}\right)$$

$$= \left(\frac{(a+c+b)\,(a+c-b)}{2\,c}\right)\left(\frac{(b+c-a)\,(b-c+a)}{2\,c}\right);$$

consequently,

$$4\,c^2\,y^2 = (a+c+b)\,(a+c-b)\,(b+c-a)\,(b-c+a)$$
$$=(a+b+c)\,(a+b+c-2b)\,(a+b+c-2a)\,(a+b+c-2c)\,;$$

or, designating the sum of the three sides $a + b + c$ by $2\,s$,

$$4\,c^2\,y^2 = 2\,s\,(2\,s - 2\,b)\,(2\,s - 2\,a)\,(2\,s - 2\,c)$$
$$= 16\,s\,(s-b)\,(s-c)\,(s-a),$$

or, dividing by 16 and taking the square root,

$$\frac{c\,y}{2} = \sqrt{s\,(s-b)\,(s-c)\,(s-a)}.$$

But $\frac{c\,y}{2}$, or $\frac{AC \times BD}{2}$ is the surface of the triangle ABC. Hence, *to find the surface of a triangle by means of the three sides, we must subtract each side successively from the half sum, multiply the half sum and the three remainders continually together, and take the square root of this product.*

76. 4. The equation $2\,c\,x - c^2 = a^2 - b^2$,

gives $$b^2 = a^2 + c^2 - 2\,c\,x\,;$$

but, if the perpendicular fall without the triangle (*fig.* 39), since AD is now $c + x$ instead of $c - x$, designating the sides as before, we have Fig. 39.

$$y^2 + x^2 = a^2, \text{ and } y^2 + c^2 + 2\,c\,x + x^2 = b^2.$$

The first equation being subtracted from the second gives

$$c^2 + 2\,c\,x = b^2 - a^2, \text{ or } c\,(c + 2\,x) = (b+a)\,(b-a)\,;$$

whence $$c : b + a :: b - a : c + 2\,x.$$

Now $c + 2x$, or $x + c + x = CD + AD$;
consequently $AC : AB + BC :: AB - BC : CD + AD$†.

77. 5. The same equation $c^2 + 2cx = b^2 - a^2$,
gives $b^2 = a^2 + c^2 + 2cx$.
Comparing this with the equation

$$b^2 = a^2 + c^2 - 2cx,$$

which answers to figure 38; we see that b^2, the square of the side AB, opposite to the acute angle C, is less than the sum of the squares of the two other sides $a^2 + c^2$ by $2cx$; on the contrary, the square of the side AB opposite to the obtuse angle, figure 39, is equal to $a^2 + c^2 + 2cx$, that is, greater than the sum of the squares of the two other sides by $2cx$; which agrees with articles 191, 192, of the *Elements of Geometry*. By these propositions, we can determine, when the angles of a triangle are to be calculated by means of the sides, whether the angle sought be acute or obtuse.

78. 6. The two equations

$$b^2 = a^2 + c^2 - 2cx, \text{ and } b^2 = a^2 + c^2 + 2cx,$$

confirm what has been before said with regard to negative quantities; for we see that the segment CD (*fig.* 38, 39), takes different directions, according as the perpendicular BD falls within the triangle or without it. Now in these equations, the term $2cx$ has in fact contrary signs. Hence, whatever result we arrive at with regard to one of these triangles, we obtain that which belongs to the analogous case of the other, by giving the contrary sign to the parts which take different directions upon the same line. Now in the above theorem, respecting the surface of a triangle, the segment CD does not come into consideration; therefore the proposition is equally applicable to all kinds of plane triangles.

Fig. 38, 39.

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† This proportion may be obtained geometrically in a manner similar to that of art. 73; thus, from B (*fig.* 39′), as a centre, and with the radius BC, describe the circle $ECFG$; then we have

Fig. 39′.

$$AG = AB + BG = AB + BC,$$
$$AE = AB - BE = AB - BC,$$
$$AF = DF + AD = CD + AD;$$

but, since AG, AF, are secants drawn from a point without the circle to the concave part of the circumference,

$$AC : AG : AE :: AF,$$

that is, $AC : AB + BC :: AB - BC : CD + AD$.

We might deduce several other propositions from the same equations; but other objects claim our attention.

79. Although in putting geometrical questions into equations, we have more resources and more facilities, according as we are acquainted with a greater number of the properties of lines; still, as algebra itself furnishes the means of finding these properties, the number of propositions really necessary, is very limited. These two propositions, that *similar triangles have their homologous sides proportional*, and that *in a right-angled triangle the square of the hypothenuse is equal to the sum of the squares of the two other sides*, are the basis of the application of algebra to geometry. But there are many ways of making use of these propositions according to the nature of the question. Here, as in other things, there is a discretion to be exercised in the choice of means and manner of applying them. But, as this discretion is acquired in a great degree by practice, we proceed to illustrate these observations by several examples.

80. We propose, in the first place, this question. *From a point* A (fig. 40), *the situation of which is known, with respect to* Fig. 40.
two lines HD, DI, *that make with each other a known angle* HDI, *to draw a straight line* AEG, *in such a manner that the intercepted triangle* EDG *shall have a given surface, that is, a surface equal to a known square* c^2.

From the point A we draw the line AB parallel to DH, and the line AC perpendicular to DG produced. From the point E, where the line AEG must cut DH, let us suppose the perpendicular EF. If we knew EF and DG, by multiplying them together and taking half the product, we should have the surface of the triangle EDG, which was to be equal to c^2.

Let us then suppose $DG = x$; with regard to EF, let us see if we cannot determine the value of it, by means of x and what is known in the question.

Since the situation of the point A is supposed to be known, the distance BD of the parallel AB is to be regarded as known, as also the distance AC of the point A from the line DG produced. Designating therefore BD by a, and AC by b, we have from the similar triangles, ABG, EDG,

$$BG : DG :: AG : EG;$$

and from the similar triangles ACG, EFG,

$$AG : EG :: AC : EF;$$

whence $$BG : DG :: AC : EF;$$

that is, $a + x : x :: b : EF$;

therefore $EF = \frac{b\,x}{a + x}$;

and, since the surface of the triangle EDG must be equal to the square c^2,

we shall have $EF \times \frac{DG}{2}$, or $\frac{b\,x}{a + x} \times \frac{x}{2} = c^2$,

that is, $\frac{b\,x^2}{2\,a + 2\,x} = c^2$,

or making the denominators to disappear,

$$b\,x^2 = 2\,a\,c^2 + 2\,c^2\,x.$$

This equation, resolved according to the rules for equations of the second degree, gives these two values,

$$x = \frac{c^2}{b} \pm \sqrt{\frac{c^4}{c^2} + \frac{2\,a\,c^2}{b}}.$$

Of these values, that which has the sign minus is of no use as to the present question. In order to construct the first, I put it under the following form,

$$x = \frac{c^2}{b} + \sqrt{\left(\frac{c^2}{b} + 2\,a\right)\frac{c^2}{b}}.$$

Fig. 41. This being done, I draw the indefinite line PQ (*fig.* 41), and at some point C of this line, I raise the perpendicular $AC = b$, and upon CA, CP, I take the lines CO, CM, equal each to the side c of the given square; having joined AM, I draw parallel to it, through the point O, the line ON, by which we have CN for the value of $\frac{c^2}{b}$, since the similar triangles ACM, OCN, give

$$AC : OC :: CM : CN,$$

that is, $b : c :: c : CN$;

whence $CN = \frac{c^2}{b}$,

and the value of x becomes

$$x = CN + \sqrt{(CN + 2\,a) \times CN}.$$

Now $\sqrt{(CN + 2\,a) \times CN}$ expresses a mean proportional between CN and $CN + 2\,a$ (67). We have, therefore, only to determine this mean proportional, and to add it to CN. In order to do this, upon NC produced I take $CQ = 2\,a$; and upon the whole NQ, as a diameter, I describe the semicircumference NVQ, meeting AC produced in V; I apply the chord NV from N to

P, and I have *CP* for the value of x. For *NV* is a mean proportional between *CN* and *NQ* (*Geom.* 215), that is, between *CN* and $CN + 2\,a$;

therefore, NV, or $PN = \sqrt{(CN + 2\,a) \times CN}$,

and $CP = CN + PN = CN + \sqrt{(CN + 2\,a) \times CN} = x$,

If then we apply *CP* from *D* to *G* (*fig.* 40), we shall have the point *G*, through which and the point *A*, *AG* being drawn, we obtain the triangle *EDG* equal to the square c^2. **Fig. 40.**

81. If we would know what the second value of x signifies, namely,

$$x = \frac{c^2}{b} - \sqrt{\left(\frac{c^2}{b} + 2\,a\right)\frac{c^2}{b}},$$

it will be observed that, as there is nothing in the question to determine whether the inquiry relates rather to the angle *EDG*, than to its equal *E′DG′* formed by the lines *GD*, *ED*, produced; and the given quantities being the same for this case as for the other, this second solution must belong to the question in which the object is to do the same with respect to the angle *E′DG′*, which has been done in the angle *EDG*. Indeed, if we designate *DG′* by x, the other quantities being represented as before, the triangles *ABG′*, *E′DG′*, similar on account of the parallels *AB*, *DE′*, give $BG' : DG' :: AG' : G'E'$;

and, by letting fall the perpendicular *E′F′*, the similar triangles *ACG′*, *E′F′G′*, give

$$AG' : G'E' :: AC : F'E',$$

consequently $BG' : DG' :: AC : F'E'$;

that is, $a - x : x :: b : F'E'$;

whence $F'E' = \dfrac{b\,x}{a - x}$;

and, since the surface of triangle *G′E′D* must be equal to the square c^2,

we have $\dfrac{b\,x}{a - x} \times \dfrac{x}{2} = c^2$,

which gives $b\,x^2 = 2\,a\,c^2 - 2\,c^2\,x$,

consequently $x = \dfrac{-c^2}{b} \pm \sqrt{\dfrac{c^4}{b^2} + \dfrac{2\,a\,c^2}{b}}$,

values of x, which are precisely the same as those of the preceding case, only the signs are reversed, as they ought to be, since here the quantity x is taken in a direction opposite to that of the first supposition; a new confirmation of what we have already said more than once, that negative values are to be taken in a direction opposite to that of positive ones.

Fig. 41. The construction, which we have given for the preceding case, answers also for this, by applying NV (*fig.* 41), from N to K toward Q; then the value of x, which in the preceding case was CP, will be CK. Indeed, the value of x, which belongs to the present case, is

$$x = \frac{-c^2}{b} + \sqrt{\frac{c^4}{b^2} + \frac{2ac^2}{b}},$$

or

$$= -\frac{c^2}{b} + \sqrt{\left(\frac{c^2}{b} + 2a\right) \times \frac{c^2}{b}},$$

that is, $x = -CN + \sqrt{(CN + 2a) \times CN}$;

and, since $NV = \sqrt{(CN + 2a) \times CN}$,

we have $x = -CN + NV = -CN + NK = CK$;

Fig. 40. thus we apply CK from D to G' (*fig.* 40), and we have the point G', through which and the point A, $AG'E'$ being drawn, we shall have the triangle $G'DE'$ equal to the square c^2, or the second solution of the question.

Fig. 40. 82. We have supposed the point A (*fig.* 40), above the line
Fig. 42. BG; if it were below it (*fig.* 42), the quantity b, or line AC, would be negative, and the first two values of x would consequently be

$$x = -\frac{c^2}{b} \pm \sqrt{\frac{c^4}{b^2} - \frac{2ac^2}{b}},$$

or

$$x = -\frac{c^2}{b} \pm \sqrt{\left(\frac{c^2}{b} - 2a\right)\frac{c^2}{b}};$$

from which it is evident, that the problem is not possible, except when $2a$ is less than $\frac{c^2}{b}$; since, when it is greater, the quantity under the radical sign is negative, and consequently the values of x are imaginary or absurd (*Alg.* 107). When $2a$ is less than $\frac{c^2}{b}$, the two values of x are negative, that is, the problem is then impossible with regard to the angle HDI, while there are two solutions with regard to its equal $E'DG'$. In order to have these two solutions, it is necessary to construct the two values,

$$x = -\frac{c^2}{b} \pm \sqrt{\left(\frac{c^2}{b} - 2a\right)\frac{c^2}{b}},$$

which is done in the following manner. Having determined, as above, the value CN, of $\frac{c^2}{b}$ (*fig.* 43), we take $NQ = 2\,a$, and Fig. 43. having described upon NQ, as a diameter, the semicircle NVQ, we draw the tangent CV; we then apply CV from C to P toward N, and from C to K in the opposite direction; then NP, NK, will be the two values of x; we apply them from D to G (*fig.* 42), Fig. 42. and from D to G', and drawing through the point A and the points G and G' the straight lines EG, $E'G'$, each of the triangles EDG, $E'DG'$, will be equal to the square c^2. As to what we have said, that NP, NK (*fig.* 43), will be the two values of Fig. 43. x, we deduce it from this, that CV, being a mean proportional between CN and CQ, is equal to $\sqrt{CQ \times CN}$, or putting for these lines their values CV, or CP, or $CK = \sqrt{\left(\frac{c^2}{b} - 2\,a\right)\frac{c^2}{b}}$;

hence $$NP = CN - CP = \frac{c^2}{b} - \sqrt{\left(\frac{c^2}{b} - 2\,a\right)\frac{c^2}{b}},$$

and $$NK = CN + CK = \frac{c^2}{b} + \sqrt{\left(\frac{c^2}{b} - 2\,a\right)\frac{c^2}{b}}.$$

Now these two quantities are the same as the values of x, the signs being changed; therefore these same quantities applied from D toward G (*fig.* 42), will be the values of x. Fig. 42.

83. If the point A (*fig.* 44), were in the angle itself HDI, BD Fig. 44. falling in the direction opposite to that in which it first fell, a would be negative, and the first two values of x would become

$$x = \frac{c^2}{b} \pm \sqrt{\frac{c^4}{b^2} - \frac{2\,a\,c^2}{b}},$$

which are the same, the signs being changed, as those which we have just constructed. We ought, therefore, to construct them as we have done (*fig.* 43), but we should apply the values of x, Fig. 43. NP, NK, from D toward I, (*fig.* 44), and we shall have the Fig. 44. two triangles DEG, $DE'G'$, both of which satisfy the question.

84. Lastly, the point A (*fig.* 45), may be situated below BD, Fig. 45. but in the angle BDE'. In this case a, b, will be both negative, which would give

$$x = -\frac{c^2}{b} \pm \sqrt{\frac{c^4}{b^2} + \frac{2\,a\,c^2}{b}};$$

of which the signs are exactly the reverse of those belonging to the first values of x, found above. We should construct them,

Fig. 41. therefore, as we have done (*fig.* 41), and CK would be the positive value of x, and CP its negative value; and we should apply
Fig. 45. the former from D to G (*fig.* 45), toward B, and the latter in the opposite direction from D to G'.

We have insisted upon the different cases of this solution, in order to show, how they are all comprehended in a single equation, how they are deduced from this equation by merely changing the signs, and how the different positions of lines are denoted by difference of sines, and the reverse. It still remains to point out some of the uses of this solution.

85. If we had proposed this question; *from a given point* A
Fig. 46. (fig. 46), *without, or within, a given triangle* DHI, *to draw a line* AF *dividing this triangle into two parts* DEF, EFIH, *which shall be to each other in a given ratio, expressed by* m : n; this question would have its solution in that of the preceding. For, since the triangle DHI is given, and it is known what part of DHI, DEF is required to be, if we seek the fourth term of this proportion

$$m + n : m :: DHI : x,$$

this fourth term, or x, will be the triangle DEF. Now we can always find a square c^2 equal to this triangle (*Geom.* 243); the question then reduces itself to this, namely, to draw through the point A a line AEF, which shall form with the two sides DH, DI, a triangle DEF equal to the square c^2, which is the same as the preceding question.

86. It will be perceived also, that we can reduce to the same solution the following question, namely, to divide any rectilineal
Fig. 47. figure (*fig.* 47), by a line drawn from any point A, into two parts $BCFE$, $EFDHK$, which shall be to each other in any given ratio. Indeed, the figure $BCDHK$ being supposed to be known, all its angles and sides are known; we may regard as known, therefore, the triangle BLC formed by the two sides KB, DC, produced, since a side BC of this triangle, and the two adjacent angles LBC, LCB, the supplements of the angles CBK, BCD, are given; and, as $EBCF$ is a determinate portion of the whole surface, this is also known; the question then is reduced to this, namely, to draw a line AEF, which shall form in the angle KLD a triangle equal to a known square. It is moreover evident that this figure may be divided into a greater number of parts, of which the ratios are given.

87. A further remark might be made, that if some of the given quantities, belonging to an equation, which serve to resolve a question, are such that a change of the signs does not change the equation; or, if a change of the line or lines sought does not involve a change in the position or magnitude of the given lines; then among the different values of x, when there are several in the equation, there will always be found one which will be the proper solution for the case indicated by this change. For example, in the question which we have been considering, we have seen that one of the values of x gave directly the solution for the case in which the line AEG (*fig.* 40), was to traverse the angle HDI, as we had supposed in making the calculation; but we have seen at the same time, that the second value of x gave the solution of a case, not contemplated, which did not relate to the angle HDI, but to the angle opposite to it at the vertex. The reason of this is, that having in each case the same given quantities to employ, and the same reasoning to go through, we cannot but be conducted to the same equation; therefore the same equation ought to give the two solutions. This will be illustrated by other examples as we proceed. **Fig. 40.**

88. Let the following question be proposed. *From a given point* A, *without a circle* BDC (fig. 48), *to draw a straight line* AE *in such a manner that the part* DE, *intercepted in the circle, shall be equal to a given line.* **Fig. 48.**

Since the circle $BDEC$ is given, its diameter is supposed to be known; and, since the point A is given, if we draw through the centre O the straight line AOC, the line AB is to be considered as known, and consequently the line AC. In order to know how the line AE is to be drawn, we have only to determine what ought to be the magnitude of AD, that, when produced, the part DE should be equal to the given line. I designate AD by x, AB by a, AC by b, and the given line, to which DE is to be made equal, by c.

Since the figure $BDEC$ is a circle, the secants AC, AE, must be reciprocally proportional to the parts without the circle; that is, $AC : AE :: AD : AB$ (*Geom.* 225);

or $b : x + c :: x : a$;

whence $x^2 + c\,x = a\,b$,

an equation of the second degree, which, being resolved, gives

$$x = -\tfrac{1}{2}c \pm \sqrt{\tfrac{1}{4}c^2 + a\,b}.$$

of which the first value only, $-\frac{1}{2}c + \sqrt{\frac{1}{4}c^2 + ab}$, satisfies the question under consideration.

In order to finish the solution, it is necessary to construct this quantity, which can be done without employing the transformations made known, art. 64. For this purpose we draw from the point A the tangent AT, which, being a mean proportional between AB and AC, gives $\overline{AT}^2 = ab$; the value of x therefore becomes

$$x = -\tfrac{1}{2}c + \sqrt{\tfrac{1}{4}c^2 + \overline{AT}^2}.$$

The radius TO being drawn becomes a perpendicular to AT; if then we take TI equal to $\frac{1}{2}c$, by drawing AI, we shall have $AI = \sqrt{\frac{1}{4}c^2 + \overline{AT}^2}$; therefore, in order to obtain x, we have only to apply TI from I to R, and to describe from the point A, as a centre, and with the radius AR, the arc RD, which will determine the point sought D; for

$$AD, \text{or } AR = AI - IR = AI - TI = \sqrt{\tfrac{1}{4}c^2 + \overline{AT}^2} - \tfrac{1}{2}c = x.$$

In order now to know what the second value of x signifies, namely,

$$x = -\tfrac{1}{2}c - \sqrt{\tfrac{1}{4}c^2 + ab},$$

it must be observed that, as it is wholly negative, it can only fall in the direction opposite to that toward which AD tends. Let us see, then, if there be a question, depending upon the same quantities and the same reasoning, which fulfils this condition. If now we suppose a and b negative, the equation $x^2 + cx = ab$, undergoes no change; since, therefore, when the circle $BDEC$ becomes $B'D'E'C'$, situated toward the left in the same manner that $BDEC$ is toward the right, it follows that the solution of this case is contained in the same equation; the second value of x, or $-\frac{1}{2}c - \sqrt{\frac{1}{4}c^2 + ab}$, belongs to the same case, and satisfies the same conditions; if, therefore, in the preceding construction, we apply IT from I to R' on AI produced, and from the point A, as a centre, and with a radius equal to AR', we describe an arc cutting the circumference $B'D'E'C'$ in E', the point E' will be such that the part intercepted $E'D'$ will be equal to c. Indeed

$$AE' = AR' = AI + IR' = \sqrt{\tfrac{1}{4}c^2 + \overline{AT}^2} + \tfrac{1}{2}c,$$

that is, AE' is equal to the second value of x, the signs being changed. Now, since we apply this quantity in a direction opposite to that in which x tends, it follows that AE' is in reality the second value of x.

In fine, as the two circles are equal and situated in the same manner, the two solutions may both belong to the same circle, so that if we describe from the point A, as a centre, and with a radius AR', the arc $R'E$, the line AE will also resolve the question; indeed, it is evident that the point E, determined in this manner, is in the line AD, (obtained by the first construction), produced. But of the two solutions, furnished by algebra, the first falls on the right of the point A, and appertains to the point D of the convex circumference, while the second falls on the left, and appertains to the point E' of the concave part of the circumference.

89. Let us now suppose that it is proposed *to find in the direction of the given line* AB (fig. 49) *a point* C, *such, that its distance from the point* A *shall be a mean proportional between its distance from the point* B *and the whole line* AB. **Fig. 49.**

I shall designate the given line AB by a, and the distance sought AC by x; then BC will be $a - x$; and, as the proportion required is

$$AB : AC :: AC : CB,$$

or

$$a : x :: x : a - x,$$

we shall have

$$x^2 = a^2 - a\,x, \quad \text{or} \quad x^2 + a\,x = a^2,$$

an equation of the second degree, which, being resolved, gives

$$x = -\tfrac{1}{2}\,a \pm \sqrt{\tfrac{1}{4}\,a^2 + a^2}.$$

In order to construct the first value of x, we must, according to what has been said (67), raise at the point B, the perpendicular $BD = \frac{1}{2}\,a$; and, having drawn AD, we shall have

$$AD = \sqrt{\overline{BD}^2 + \overline{AB}^2} = \sqrt{\tfrac{1}{4}\,a^2 + a^2};$$

we have then only to subtract from this line the quantity $\frac{1}{2}\,a$, which is done by applying DB from D to O; then we shall have $AO = \sqrt{\frac{1}{4}\,a^2 + a^2} - \frac{1}{2}\,a$, that is, it will be equal to x. We then apply AO from A to C toward B, and C will be the point sought.

As to the second value of x, namely,

$$x = -\tfrac{1}{2}\,a - \sqrt{\tfrac{1}{4}\,a^2 + a^2},$$

if we apply BD from D to O' on AD produced, then we shall have

$$AO' = \tfrac{1}{2}\,a + \sqrt{\tfrac{1}{4}\,a^2 + a^2};$$

and, as the value of x is this quantity taken negatively, we apply AO' from A to C' on AB produced in a direction opposite to that toward which x is supposed in the solution to extend; and we shall have a second point C', which will also be such, that its distance from the point A will be a mean proportional between its distance from the point B and the whole line AB.

We remark in passing, that this question contains that of *dividing a line in extreme and mean ratio;* also the construction which we have obtained, is the same as that given in the *Elements of Geometry* (240). But it will be perceived, that we are made acquainted with this construction by algebra, whereas in the *Elements of Geometry* we supposed the construction and only demonstrated its truth.

90. With a little attention to the course pursued in the preceding questions, it will be evident that we have always taken for the unknown quantity a line, which being once known serves, by observing the conditions of the question, to determine all the others. This is the course to be pursued in all cases, but there is a choice with regard to the line to be used; there are often several, each of which has the property of determining all the others, if once known. Among these, some would lead to more simple equations than others. The following rule is given to aid in such cases.

91. *If among the lines or quantities, which would, when taken each for the unknown quantity, serve to determine all the other quantities, there are two which would in the same way answer this purpose, and it could be foreseen that each would lead to the same equation (the signs + and — excepted); then we ought to employ neither of these, but take for the unknown quantity one which depends equally upon both;* that is, their half sum, or their half difference, or a mean proportional between them, or &c., and we shall always arrive at an equation more simple than by employing either the one or the other.

The question we have resolved art. 88, may be used to illustrate what is here said. In this question there is no reason for
Fig. 48. taking AD (*fig.* 48), rather than AE, for the unknown quantity; by taking AD for the unknown quantity x, we have $x + c$ for AE; and, by taking AE for the unknown quantity x, we should have $x - c$ for AD; and, as to the rest the mode of proceeding is the same for each case; so that the equations differ only in the

signs. If, therefore, instead of taking either for the unknown quantity, I take their half sum, and designate it by x, since their half difference $DE = c$ is given, we shall have

$AE = x + \frac{1}{2} c$, and $AD = x - \frac{1}{2} c$, (*Note, page* 81);

whence, according to the proposition adopted in the first solution,

$$(x + \tfrac{1}{2} c)(x - \tfrac{1}{2} c) = a\,b,$$

or

$$x^2 - \tfrac{1}{4} c^2 = a\,b,$$

a more simple equation than the former, and which gives

$$x = \sqrt{\tfrac{1}{4} c^2 + a\,b};$$

and, since $AE = x + \frac{1}{2} c$, we have immediately

$$AE = \tfrac{1}{2} c + \sqrt{\tfrac{1}{4} c^2 + a\,b},$$

and

$$AD = -\tfrac{1}{2} c + \sqrt{\tfrac{1}{4} c^2 + a\,b},$$

as before found (88).

The following question will furnish several examples of the application of the same principle.

92. *From a point* D (fig. 50), *situated in the right angle* IAE, *and equally distant from the two sides* IA, AE, *to draw a straight line* DB *in such a manner that the part* CB, *comprehended in the right angle* EAB *shall be equal to a given line.* Fig. 50.

Having let fall the perpendiculars DE, DI, I can take indifferently for the unknown quantity CE or AB, AC or IB, CD or DB. If I take, for example, CE for the unknown quantity; designating CE by x, and each of the two equal lines DE, DI, supposed to be known, by a; calling at the same time, the given line to which BC is to be made equal, c, I shall have

$$AC = AE - CE = a - x;$$

and the similar triangles DEC, CAB, give this proportion,

$$CE : DE :: AC : AB,$$

that is,

$$x : a :: a - x : AB,$$

whence

$$AB = \frac{a^2 - a\,x}{x}.$$

Now, by the property of right-angled triangles,

$$\overline{AC}^2 + \overline{AB}^2 = \overline{BC}^2;$$

substituting for these lines their algebraic values, we shall have

$$(a - x)^2 + \left(\frac{a^2 - a\,x}{x}\right)^2 = c^2,$$

or

$$a^2 - 2\,a\,x + x^2 + \frac{a^4 - 2\,a^3\,x + a^2\,x^2}{x^2} = c^2,$$

or, making the denominators to disappear, transposing and reducing,

$$x^4 - 2\,a\,x^3 + 2\,a^2\,x^2 - c^2\,x^2 - 2\,a^3\,x + a^4 = 0,$$

an equation of the fourth degree, but which is much less simple than others which may be employed for the solution of this question.

If, instead of taking *CE* for the unknown quantity, we take *IB*; and, designating *IB* by x, proceed as above, we should have an equation which does not differ from that just found, except that, instead of $a - x$, we should have $x - a$, which would lead to precisely the same thing, since the squares of these quantities are employed. If we should take *AB* for the unknown quantity, the result would differ only in its signs from that in which *AC* is taken for the unknown quantity. With regard to *DB* and *DC*, the equation in which one is taken for the unknown quantity would not differ, except in its signs, from that in which the other is taken for the unknown quantity; therefore neither of these lines should be used for this purpose. But, if we take for the unknown quantity the sum of the two lines *DB*, *DC*, and represent this sum by $2\,x$, we shall have

$$DB = x + \tfrac{1}{2}\,c, \quad \text{and} \quad DC = x - \tfrac{1}{2}\,c.$$

Now, in order to find *AB*, *AC*, on account of the parallels *DI*, *CA*, we have the following proportions,

$$DC : CB :: IA \text{ or } DE : AB,$$
$$DB : CB :: DI : AC,$$

that is,

$$x - \tfrac{1}{2}\,c : c :: a : AB,$$
$$x + \tfrac{1}{2}\,c : c :: a : AC,$$

whence

$$AB = \frac{a\,c}{x - \frac{1}{2}\,c}, \text{ and } AC = \frac{a\,c}{x + \frac{1}{2}\,c}$$

consequently, since the right-angled triangle *CAB* gives

$$\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2,$$

we shall have

$$\frac{a^2\,c^2}{(x - \frac{1}{2}\,c)} + \frac{a^2\,c^2}{(x + \frac{1}{2}\,c)^2} = c^2,$$

or, making the denominators to disappear and dividing by c^2,

$$a^2\,(x + \tfrac{1}{2}\,c)^2 + a^2\,(x - \tfrac{1}{2}\,c)^2 = (x - \tfrac{1}{2}\,c)^2\,(x + \tfrac{1}{2}\,c)^2;$$

which, by performing the operations indicated, becomes, after reduction,

$$x^4 - (\tfrac{1}{2}\,c^2 + 2\,a^2)\,x^2 = \tfrac{1}{2}\,a^2\,c^2 - \tfrac{1}{16}\,c^4,$$

to an equation of the fourth degree, indeed, but one which may be more easily resolved than the preceding, since the resolution may be effected after the manner of an equation of the second degree (*Alg.* 160).

We can moreover arrive at equations sufficiently simple, by employing two unknown quantities, one being the sum and the other the difference of the two lines AB, AC; that is, if we make

$$AB + AC = 2\,x, \quad \text{and} \quad AB - AC = 2\,y,$$

which would give

$$AB = x + y, \quad \text{and} \quad AC = x - y;$$

the right-angled triangle ABC gives

$$\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2,$$

and the similar triangles ABC, IBD, give

$$AB : AC :: IB : ID,$$

from which we obtain the two equations necessary for determining x and y. The value of x^2, being deduced from one of these and substituted in the other, will give for y an equation of the second degree. But, leaving what remains of the calculation as an exercise for the learner, we return to our equation.

According to the rule for completing the square we have

$$x^4 - (\tfrac{1}{2}c^2 + 2a^2)x^2 + (\tfrac{1}{4}c^2 + a^2)^2 = (\tfrac{1}{4}c^2 + a^2)^2 + \tfrac{1}{2}a^2c^2 - \tfrac{1}{16}c^4$$
$$= a^2c^2 + a^4,$$

or, extracting the square root,

$$x^2 - (\tfrac{1}{4}c^2 + a^2) = \pm\sqrt{a^2c^2 + a^4},$$

and consequently

$$x^2 = \tfrac{1}{4}c^2 + a^2 \pm \sqrt{a^2c^2 + a^4},$$

extracting the square root of this, we have

$$x = \pm\sqrt{\tfrac{1}{4}c^2 + a^2 \pm \sqrt{a^2c^2 + a^4}},$$

or

$$x = \pm\sqrt{\tfrac{1}{4}c^2 + a^2 \pm a\sqrt{c^2 + a^2}}.$$

Of the four values of x resulting from the double combination of the two signs $\pm$, there is only one which relates to the question, as it has been proposed, and this is

$$x = +\sqrt{\tfrac{1}{4}c^2 + a^2 + a\sqrt{c^2 + a^2}}.$$

The value, $x = +\sqrt{\tfrac{1}{4}c^2 + a^2 - a\sqrt{c^2 + a^2}}$, resolves the question for the case in which it is required that the line CB should be in the same angle with the point D (*fig.* 51), when x **Fig. 51.**

represents not the half sum, but the half difference of the two lines BD, DC; of this we may be easily satisfied by calling this difference $2x$, and resolving the problem in the manner above given. For we shall have

$$DB = \tfrac{1}{2}c + x, \quad CD = \tfrac{1}{2}c - x,$$

and the parallels DI, CA, give

$$DB : CB :: DI : CA, \quad \text{and} \quad CD : CB :: AI : AB,$$

or $\tfrac{1}{2}c + x : c :: a : CA$, and $\tfrac{1}{2}c - x : c :: a : AB$;

whence $$CA = \frac{ac}{\tfrac{1}{2}c + x}, \quad \text{and } AB = \frac{ac}{\tfrac{1}{2}c - x},$$

consequently, the triangle CAB being right-angled, we have

$$\frac{a^2c^2}{(\tfrac{1}{2}c + x)^2} + \frac{a^2c^2}{(\tfrac{1}{2}c - x)^2} = c^2,$$

or $$x^4 - (\tfrac{1}{2}c^2 + 2a^2)x^2 = \tfrac{1}{2}a^2c^2 - \tfrac{1}{16}c^4,$$

an equation precisely the same with that which we found for the
Fig. 50. sum of the two lines BB, CD (*fig.* 50). Therefore, as the same equation satisfies the two cases, one of the roots must be the sum and the other the difference of the two lines. Now it is evident, that the two roots that must be taken, are those we have been considering, since the two others, being wholly negative, can only relate to cases entirely the reverse of those contemplated in each resolution.

As to these two other roots, in order to find to what cases they belong, it is to be observed, that there is nothing in the present question, or at least in the equation, to determine whether
Fig. 50. the point D (*fig.* 50), is, as we have first supposed, below AI and on the left of AE, or whether it is above the first and on the right of the second, as it is represented relatively to $A'I'$, $A'E'$. Now, in this case the quantity a, falling in a direction opposite to that in which it first occurred, is negative; consequently, we shall have the solution adapted to this case, if we put $-a$ in the place of $+a$ in the equation

$$x^4 - (\tfrac{1}{2}c^2 + 2a^2)x^2 = \tfrac{1}{2}a^2c^2 - \tfrac{1}{16}c^4,$$

found above; but, as this substitution produces no change, it follows that this same equation ought also to resolve the two new cases; therefore, these two other values of x are, the one the
Fig. 50. sum of the two lines DB', DC', (*fig.* 50), and the other their
Fig. 51. difference (*fig.* 51); and we see indeed that in this new position, the points B, C, fall in directions opposite to those in which they

first occurred, and that, consequently, the sum as well as the difference of the two lines DB', DC', ought to be negative, as in fact they are, according to the equation.

In order to construct the solution just found, we take upon EA produced (*fig.* 50, 51), the part $AN = c$, and having drawn IN, we apply this last to DI produced, from I to K; upon DK, as a diameter, we describe the semicircle KLD meeting AI produced in L. From the middle H of AN, we draw IH, and apply it from I to M (*fig.* 50), and we have LM for the first value of x. But in figure 51 we describe from the point L as a centre, and with a radius equal to IH, a small arc cutting IK in M, and IM will be the second value of x; and, since BD is equal to $x + \frac{1}{2}c$, we have $BD = LM + AH$ (*fig.* 50), and Fig. 50, 51. Fig. 50. Fig. 50.

$$BD = IM + AH \ (\textit{fig. } 51);$$ Fig. 51.

thus we have only to describe from the point D, as a centre, and with the radius BD just found, an arc cutting IA produced in some point B, the straight line DB will be the line required. Indeed, the right-angled triangle IAN (*fig.* 50, 51), gives Fig. 50, 51.

$$IN, \text{ or } IK = \sqrt{\overline{IA}^2 + \overline{AN}^2} = \sqrt{a^2 + c^2};$$

and, since LI is a mean proportional between DI and IK, we have

$$\overline{IL}^2 = DI \times IK = a\sqrt{a^2 + c^2}.$$

Now the right-angled triangle IAH gives

$$IH, \text{or } (\textit{fig. } 50)\ IM, \text{or } (\textit{fig. } 51)\ LM = \sqrt{\overline{IA}^2 + \overline{AH}^2} = \sqrt{a^2 + \tfrac{1}{4}c^2};$$ Fig. 50. Fig. 51.

and the right-angled triangle LIM (*fig.* 50), gives Fig. 50.

$$LM = \sqrt{\overline{IM}^2 + \overline{IL}^2} = \sqrt{a^2 + \tfrac{1}{4}c^2 + a\sqrt{a^2 + c^2}} = x;$$

and (*fig.* 51), $IM = \sqrt{\overline{LM}^2 - \overline{IL}^2} = \sqrt{a^2 + \tfrac{1}{4}c^2 - a\sqrt{a^2 + c^2}} = x$. Fig. 51.

It should be observed, with respect to this last value, that the construction which we have given supposes that IH (*fig.* 51) is greater than IL, or at least equal to it. If it were less, the question would be impossible with respect to the last case. This also will be made evident by the aid of algebra; for, in the value Fig. 51.

$$x = \sqrt{a^2 + \tfrac{1}{4}c^2 - a\sqrt{a^2 + c^2}},$$

if $a^2 + \frac{1}{4}c^2$, which is $\overline{IH}^2$, is less than $a\sqrt{a^2 + c^2}$, which is $\overline{IL}^2$, the whole expression taken together is negative, and consequently the value of x becomes imaginary.

By taking for the unknown quantity the sum of the two lines
Fig. 50. *DB*, *DC*, (*fig.* 50), or the difference (*fig.* 51), we arrive at an
Fig. 51. equation more simple than by taking *CE*, or *AC*, or *AB*, or *IB*; because the relation of the lines *DB*, *DC*, to the lines *IB*, *AB*, is similar to that of these same lines *DB*, *DC*, to the lines *AC*, *CE*; that is, we can determine them by similar operations, whether we employ *IB*, *AB*, or *AC*, *CE*. As, in general, the equation ought to contain all the different relations which the quantity sought can have with those on which it depends, this equation will be the more simple, according as the quantity, selected for the unknown quantity, shall have a smaller number of different relations with the others. See a striking example of this in the following solution of the same question.

Fig. 52. 93. Since *CAB* (*fig.* 52) is a right-angle, if we imagine a circle described upon *CB* as a diameter, the circumference would pass through the point *A*. We draw the line *DA*, which, produced, would meet the circumference in *M*; then it is evident, that since the lines *DI*, *DE*, are equal, the angle *DAI*, or its equal *BAM*, will contain 45°; and since this last has for measure half of the arc *MB* (*Geom.* 126), this arc will contain 90°; if then we draw the radius *LM*, the triangle *DLM* will be right-angled, and, consequently, by letting fall upon *DM* the perpendicular *LN*, the side *LM* will be a mean proportional between *DM* and *MN* (*Geom.* 213), or since *AN* is equal to *NM* (*Geom.* 105), between *DM* and *AN*. Hence we shall arrive at a very simple solution by taking *AN* for the unknown quantity.

We represent this line *AN* by x, and the line *DA*, supposed to be known, by d; then *DM* will be $d + 2x$; and, since, according to what has been said,

$$DM : LM :: LM : MN,$$

we have $$d + 2x : \tfrac{1}{2}c :: \tfrac{1}{2}c : x;$$
whence $$dx + 2x^2 = \tfrac{1}{4}c^2, \quad \text{or} \quad x^2 + \tfrac{1}{2}dx = \tfrac{1}{8}c^2,$$
resolving this equation, we have

$$x = -\tfrac{1}{4}d \pm \sqrt{\tfrac{1}{16}d^2 + \tfrac{1}{8}c^2}.$$

In order to construct this quantity, I write it thus,

$$x = -\tfrac{1}{4}d \pm \sqrt{\tfrac{1}{16}d^2 + \tfrac{1}{16}c^2 + \tfrac{1}{16}c^2}.$$

I then take, upon the sides *A o*, *AI*, of the right-angle *IA o*, the parts *A m*, *A n*, equal each to $\tfrac{1}{4}c$, and finishing the square *A m p n*, I draw the diagonal *A p*, which will be perpendicular to *DA*, and equal to $\sqrt{\frac{1}{16}c^2 + \frac{1}{16}c^2}$; I take, upon AD, a part Ar equal to $\frac{1}{4}d$, or $\frac{1}{4}AD$, and, drawing pr, I have

$$pr = \sqrt{\overline{Ar}^2 + \overline{Ap}^2} = \sqrt{\tfrac{1}{16}d^2 + \tfrac{1}{16}c^2 + \tfrac{1}{16}c^2}.$$

In order, therefore, to obtain the first value of x I have only to subtract from pr the quantity $\frac{1}{4}d$, which is done by describing from the point r, as a centre, and with the radius rp, an arc cutting DM in N, which gives AN for the first value of x; so that by raising at the point N the perpendicular NL, and cutting it in L by an arc described from the point A, as a centre, and with the radius $\frac{1}{2}c$, we shall have the point L, through which, and the point D, DCB being drawn, the solution is completed.

As to the second value of x, namely,

$$x = -\tfrac{1}{4}d - \sqrt{\tfrac{1}{16}d^2 + \tfrac{1}{16}c^2 + \tfrac{1}{16}c^2},$$

it is obtained by applying rp from r to N', for then AN', being equal to $Ar + rN'$, is equal to $\frac{1}{4}d + \sqrt{\frac{1}{16}d^2 + \frac{1}{16}c^2 + \frac{1}{16}c^2}$, that is, it is equal to the second value of x, the signs being changed; and as it falls in a direction opposite to that of the first value, it will be the true value of x for the second case. We raise, therefore, at the point N' the perpendicular $N'L'$, and cutting it in L' by an arc described likewise from the point A, as a centre, and with the radius $\frac{1}{2}c$, we draw through the points L' and D the straight line $B'L'D$, and we have the second solution of which the question is susceptible. To be convinced of this we have only to apply to figure 53 what we have said of figure **Fig. 53.**
52 at the beginning of this solution; we shall see, that calling AN or MN, x, and retaining the other denominations, we shall have

$$DM : ML :: ML : MN,$$

that is, $\quad 2x - d : \frac{1}{2}c :: \frac{1}{2}c : x;$

whence $\quad 2x^2 - dx = \frac{1}{4}c^2,$

and $\quad x = \frac{1}{4}d \pm \sqrt{\frac{1}{16}d^2 + \frac{1}{16}c^2 + \frac{1}{16}c^2},$

of which one of the values is precisely the same as that under consideration, the signs only being different, as they ought to be.

But it is important to remark in this place, that it may happen that the arc which we would describe from the point A (*fig.* 52), **Fig. 52.**
as a centre, and with the radius $\frac{1}{2}c$, will not meet the perpendicular $N'L'$, because the quantity $\frac{1}{2}c$ may be less than AN'. Now we have said that, when questions of the second degree are impossible, algebra makes it known; still in the equation

$$x = -\tfrac{1}{4}d - \sqrt{\tfrac{1}{16}d^2 + \tfrac{1}{16}c^2 + \tfrac{1}{16}c^2}$$

there is nothing to show in what cases this impossibility exists; for the whole under the radical sign is necessarily positive.

See the solution of this difficulty. It is not to be denied, that when a question expressed algebraically is impossible, algebra manifests this impossibility; but it is to be carefully observed, that this is the case, when we have expressed by algebra every thing which the question supposes, either expressly or by implication; now this is not done in the case before us. The question implies that the three points *D*, *A*, *L*, are not in the same straight line; and this is what we have not expressed algebraically; we have expressed that *LM* is a mean proportional between *DM* and *NM*, a property which belongs indeed to a right-angled triangle, but which may also take place, when the three points *D*, *A*, *L*, are supposed to be in a straight line. Indeed, it is evident that this question may be proposed. *To find in the*
Fig. 54. *direction* DL (fig. 54), *what distance must be left between the two straight lines* DA, ML, *of known magnitude, in order that* ML *may be a mean proportional between* DM *and* MN, *the point* N *being the middle of* AM. Now this question leads, as one may easily assure himself, to precisely the same equation as the above, and this equation gives two solutions, one for the case where the two points *A* and *M* are between *D* and *L*, and the other for the contrary case. It is not surprising then, that, when the first question becomes impossible, at least in one of its cases, algebra should give no intimation of it, since it gives the solution of this second question which is always possible.

94. We are led by what is here said to remark, that there are two kinds of questions, namely, *concrete*, and *abstract*. By the first, we are to understand such as are of a nature similar to the one before the last, in which the thing sought is specified or pointed out by some condition, property, or peculiar construction, which the equation does not express; abstract questions, on the contrary, are those in which the quantities are considered simply as quantities, and in which the equation expresses every thing contained in the question, as in the one last solved. These may always have as many answers, either positive or negative, as there are real solutions to the equations, whereas the number of answers to a concrete question is often less than the number of solutions, and less even than the number of positive solutions

of the equation. The following question is of this kind and will illustrate what is here said.

95. Let $ABED$ (*fig.* 55), represent a sphere generated by the revolution of a semicircle ABE about its diameter AE. The sector ABC, by this revolution, generates a spherical sector, which is composed of a spherical segment generated by the revolution of the semisegment ABP, and of a cone generated by the revolution of the right-angled triangle BPC. It is required to find when the spherical segment and cone are equal to each other. **Fig. 55.**

In order to resolve this question, it must be recollected that the spherical sector is equal to the product of the surface of the zone BAD by a third of the radius AC (*Geom.* 546). Now the surface of the zone is found by multiplying the circumference $ABED$ by the altitude AP of this zone (*Geom.* 538); consequently, if we represent by $r : c$ the ratio of the radius of a circle to its circumference, and if we designate AC by a, and AP by x, we shall have the circumference $ABDE$ by this proportion

$$r : c :: a : ABDE, \quad \text{and} \quad ABDE = \frac{a\,c}{r};$$

therefore the surface of the zone will be $\frac{a\,c\,x}{r}$; and, consequently the solidity of the sector $\frac{a\,c\,x}{r} \times \frac{1}{3}\,a$, or, $\frac{a^2\,c\,x}{3\,r}$.

To obtain the solidity of the cone, we must multiply the surface of the circle which serves as a base, that is, the surface of the circle whose radius is BP, by a third of the altitude CP (*Geom.* 524). Now, since $CP = CA - AP = a - x$, and $CB = a$, we have in the right-angled triangle BPC,

$$BP = \sqrt{\overline{CB}^2 - \overline{PC}^2} = \sqrt{a^2 - a^2 + 2\,a\,x - x^2} = \sqrt{2\,a\,x - x^2}.$$

But the surface of the circle whose radius is BP, is found by multiplying its circumference by half of the radius; and this circumference is the fourth term of the proportion

$$r : c :: \sqrt{2\,a\,x - x^2} \quad (Geom.\ 287),$$

or

$$c\,\frac{\sqrt{2\,a\,x - x^2}}{r};$$

multiplying this by half of $\sqrt{2\,a\,x - x^2}$, which is the radius, we have

$$\frac{c\,(2\,a\,x - x^2)}{2\,r}$$

for the base of the cone. If we multiply this by a third of the altitude CP, that is, by $\frac{a-x}{3}$, the result

$$\frac{c\,(2\,a\,x - x^2)}{2\,r} \times \frac{a-x}{3}$$

is the solidity of the cone.

Now, in order that the cone may be equal to the segment, the sector, which is the sum of both, must be double of each; hence

$$\frac{a^2\,c\,x}{3\,r} = 2\,c \times \frac{2\,a\,x - x^2}{2\,r} \times \frac{a-x}{3},$$

or, by suppressing the factor 2, common to the numerator and denominator of the second member,

$$\frac{a^2\,c\,x}{3\,r} = \frac{c\,(2\,a\,x - x^2)\,(a-x)}{3\,r},$$

the equation which resolves the question.

And this equation may be simplified by suppressing the common divisor $3\,r$, and the common multiplier $c\,x$, which will leave

$$a^2 = (2\,a - x)\,(a - x),$$

or

$$x^2 - 3\,a\,x = -\,a^2,$$

from which we obtain

$$x = \tfrac{3}{2}\,a \pm \sqrt{\tfrac{5}{4}\,a^2}.$$

Of these two solutions, only $x = \frac{3}{2}\,a - \sqrt{\frac{5}{4}\,a^2}$ can satisfy the question, since it is evident that $x = \frac{3}{2}\,a + \sqrt{\frac{5}{4}\,a^2}$, being more than $2\,a$, that is, more than the diameter, cannot belong to the sphere.

To construct the solution $x = \frac{3}{2}\,a - \sqrt{\frac{5}{4}\,a^2}$, we put it under this form, $x = \frac{3}{2}\,a - \sqrt{\frac{9}{4}\,a^2 - a^2}$; and having taken AM equal to $\frac{3}{2}\,a$, we describe upon AM, as a diameter, the semicircle AOM, then inscribing AO equal to a, we draw OM, and apply it from M to P toward A; the point P, thus found, will determine the altitude AP or x. Indeed, on account of the right-angled triangle AOM, we have

$$OM, \text{ or } PM = \sqrt{\overline{AM}^2 - \overline{AO}^2} = \sqrt{\tfrac{9}{4}\,a^2 - a^2};$$

therefore $AP = AM - PM = \frac{3}{2}\,a - \sqrt{\frac{9}{4}\,a^2 - a^2} = x.$

As to the second solution, $x = \frac{3}{2}a + \sqrt{\frac{5}{4}a^2}$, it is inapplicable to the present question; we shall see that it belongs, as well as the first, to this other question, of an abstract nature, furnished by the equation

$$x^2 - 3ax = -a^2, \quad \text{or} \quad 3ax - x^2 = a^2.$$

The known line AN (fig. 56), *being divided into three equal parts at the points* B *and* D, *to find in the direction of this line a point* P, *such that the part* AD *shall be a mean proportional between the distances of the point* P *from the extremities* A *and* N. Indeed, if we designate the third AD of the line AN by a, and AP by x, we shall have $PN = 3a - x$; and the conditions of the question give this proportion Fig. 56.

$$x : a :: a : 3a - x;$$

whence

$$3ax - x^2 = a^2,$$

of which the two roots are

$$x = \frac{3}{2}a \pm \sqrt{\frac{5}{4}a^2},$$

as above found. We shall have both also by the same construction, except that for the second root, or $\frac{3}{2}a + \sqrt{\frac{5}{4}a^2}$, we apply MO from M to P toward N, and then AP and AP' will be the two values of x.

Other Applications of Algebra.

96. In order to resolve the last question, we were obliged to calculate the algebraic expression of a spherical sector, and of the cone, which makes a part of it. The bodies, which are the subject of consideration in geometry, being the elements of all others, often present themselves in our inquiries, and especially in physico-mathematical questions.

If we represent by $r : c$ the ratio of the radius to the circumference of a circle, a ratio which is known with a degree of exactness sufficient for practical purposes (*Geom.* 292), the circumference of any other circle whose radius is a will be $\frac{ac}{r}$, (*Geom.* 287), and its surface $\frac{ac}{r} \times \frac{1}{2}a$, or $\frac{a^2c}{2r}$ (*Geom.* 289). From this it will be evident, that the surfaces of circles increase as the squares of their radii; for $\frac{c}{2r}$ being always of the same value, the quantity $\frac{a^2 c}{2 r}$ increases only in proportion to the increase of a^2.

If h be the altitude of a cylinder, the radius of whose base is a, we shall have $\frac{a^2 c}{2 r} \times h$ for the solidity of this cylinder (*Geom.* 516); for the same reason we shall have $\frac{a'^2 c}{2 r} \times h'$ for the solidity of another cylinder, whose altitude is h' and the radius of whose base is a'; consequently the solidity of the two cylinders will be to each other

$$:: \frac{a^2 c}{2 r} \times h : \frac{a'^2 c}{2 r} \times h', \text{ or } :: a^2 h : a'^2 h';$$

that is, the solidities of cylinders are to each other as the altitudes multiplied by the squares of the radii of the bases. If the altitudes are proportional to the radii of the bases, we shall have

$$h : h' :: a : a',$$

and consequently $$h' = \frac{h a'}{a},$$

and the ratio $$a^2 h : a'^2 h'$$

becomes $$a^2 h : \frac{a'^3 h}{a},$$

or, multiplying by a and dividing by h,

$$a^3 : a'^3;$$

that is, the solidities are as the cubes of the radii of the bases (*Geom.* 518).

We have seen in the *Elements of Geometry*, that surfaces depend upon the product of two dimensions, and solids upon the product of three dimensions; so that, if the several dimensions of one of two solids, or two surfaces, which we would compare, have to the several dimensions of the other, each the same ratio, the two surfaces will be to each other as the squares, and the two solids as the cubes, of the homologous dimensions; and more generally still, if any two quantities of the same nature are expressed each by the same number of factors, and if the several factors of the one have to the several factors of the other, each the same ratio, the two quantities will be to each other as their homologous factors, raised to a power whose exponent is equal to the number of factors. If, for example, the two quantities were $a\, b\, c\, d$, $a'\, b'\, c'\, d'$, and we had

$$a : a' : : b : b' : : c : : c' : d : d',$$

then we should have

$$b' = \frac{a'\,b}{a},\ c' = \frac{a'\,c}{a},\ d' = \frac{a'\,d}{a},$$

and consequently,

$$a\,b\,c\,d : a'\,b'\,c'\,d' : : a\,b\,c\,d : \frac{a'^4\,b\,c\,d}{a^3},$$

$$: : \quad a \quad : \frac{a'^4}{a^3},$$

$$: : \quad a^4 \quad : a'^4.$$

What is here said is true not only of simple quantities; the same may be shown with respect to compound quantities. Let the quantities whose dimensions are proportional be

$$a\,b + c\,d, \quad a'\,b' + c'\,d';$$

since, by supposition,

$$a : a' : : b : b' : : c : c' : : d : d',$$

we shall have

$$b' = \frac{a'\,b}{a},\ c' = \frac{a'\,c}{a},\ d' = \frac{a'\,d}{a},$$

and consequently

$$a\,b + c\,d : a'\,b' + c'\,d' : : a\,b + c\,d : \frac{a'^2\,b}{a} + \frac{a'^2\,c\,d}{a^2},$$

$$: : a\,b + c\,d : \frac{a'^2\,a\,b + a'^2\,c\,d}{a^2},$$

$$: : a^2\,(a\,b + c\,d) : a'^2\,(a\,b + c\,d),$$

$$: : \quad a^2 \quad : \quad a'^2.$$

It follows, from what is here demonstrated, that the surfaces of similar figures are as the squares of their homologous dimensions, and that the solidities of similar solids are as the cubes of their homologous dimensions; for, whatever these figures and these solids may be, the former may always be considered as composed of similar triangles, having their altitudes and bases proportional (*Geom.* 219), and the latter as composed of similar pyramids, having their three dimensions also proportional (*Geom.* 433).

It will hence be perceived, that quantities may be readily compared, when they are expressed algebraically; and this may be done, whether the quantities be of the same or of a different species, as a cone and a sphere, a prism and a cylinder, provided only that they are of the same nature, that is, both solids, or both surfaces, or both &c.

97. We are taught in the *Elements of Geometry*, how to find the solidity of the frustum of a pyramid, and the frustum of a cone (*Geom.* 422, 527). If now we designate by h the altitude of the entire pyramid, and by h' the altitude of the pyramid cut off, by s the surface of the inferior base, and by s' the surface of the superior base, we shall have

$$s : s' :: h^2 : h'^2 \quad (\textit{Geom. } 409),$$

and consequently $$h'^2 = \frac{h^2 s'}{s}, \text{ or } h' = h\sqrt{\frac{s'}{s}}.$$

But, if we designate by k the altitude of the frustum, we shall have $$k = h - h',$$

or, substituting for h' its value

$$k = h - h\sqrt{\frac{s'}{s}} = \frac{h\sqrt{s} - h\sqrt{s'}}{\sqrt{s}} \quad (\textit{Alg. } 165);$$

whence we deduce, by the common rules of algebra,

$$h = \frac{k\sqrt{s}}{\sqrt{s} - \sqrt{s'}}.$$

Now the solidity of the entire pyramid is $s \times \frac{h}{3} = \frac{h\,s}{3}$,

and the solidity of the pyramid cut off is $s' \times \frac{h'}{3} = \frac{h'\,s'}{3}$,

or, putting for h' its value,

$$s' \times \frac{h}{3}\sqrt{\frac{s'}{s}};$$

hence the solidity of the frustum will be

$$\frac{h\,s}{3} - \frac{h\,s'\sqrt{s'}}{3\sqrt{s}}, \text{ or } \frac{h}{3}\left(s - \frac{s'\sqrt{s'}}{\sqrt{s}}\right), \text{ or } \frac{h}{3}\left(\frac{s\sqrt{s} - s'\sqrt{s'}}{\sqrt{s}}\right);$$

putting for h its value, found above, we shall have

$$\frac{k\sqrt{s}}{3(\sqrt{s} - \sqrt{s'})} \times \frac{(s\sqrt{s} - s'\sqrt{s'})}{\sqrt{s}},$$

which is reduced to

$$\frac{k}{3}\left(\frac{s\sqrt{s} - s'\sqrt{s'}}{\sqrt{s} - \sqrt{s'}}\right),$$

or, the whole being divided by $\sqrt{s} - \sqrt{s'}$,

$$\frac{k}{3}(s + \sqrt{ss'} + s').$$

We see, therefore, that the frustum of a pyramid, or of a cone, is composed of three pyramids of the same altitude, of which one has for its base the inferior base s of the frustum, another the superior base s', and the third the mean proportional between these $\sqrt{s s'}$, which agrees with the propositions above referred to.

98. If a represent the radius of a sphere, $\frac{a^2 c}{2 r}$ will be the surface of a great circle of this sphere, and $\frac{4 a^2 c}{2 r}$ or $\frac{2 a^2 c}{r}$ will be the entire surface (*Geom.* 536); consequently, $\frac{2 a^2 c}{r} \times \frac{1}{3} a$, or $\frac{a^2 c}{2 r} \times \frac{4}{3} a$, or $\frac{c}{2 r} \times \frac{4 a^3}{3}$ will be the solidity of this sphere (*Geom.* 546).

If we designate by x the altitude of any segment, we shall have, as in art. 95, $\frac{a^2 c x}{3 r}$ for the solidity of the sector, and $\frac{c}{2 r}$ $(2 a x - x^2) \frac{a - x}{3}$ for the solidity of the cone, which makes a part of it; hence the solidity of the segment will be

$$\frac{a^2 c x}{3 r} - \frac{c}{2 r} (2 a x - x^2) \frac{a - x}{3}$$

$$= \frac{c}{3 r}\left(a^2 x - \frac{2 a x - x^2}{2} (a - x)\right)$$

$$= \frac{c}{3 r}\left(\frac{2 a^2 x - 2 a^2 x + a x^2 + 2 a x^2 - x^3}{2}\right)$$

$$= \frac{c}{3 r}\left(\frac{3 a x^2 - x^3}{2}\right) = \frac{c x^2}{2 r} (a - \tfrac{1}{3} x);$$

from which it will be seen, that the solidity of the segment is equal to the product of the circle, whose radius is the altitude of this segment, multiplied by the radius of the sphere minus the third of this altitude.

When we have the algebraic expression of quantities, it is easy to resolve several questions that may be raised respecting these quantities. If, for example, it were asked, what must be the altitude of a cone which shall be equal in solidity to a given sphere, and which shall have for the radius of its base, the radius of the sphere; designating this altitude by h, and the radius of the base by a, we shall have for the solidity of the cone

$$\frac{c}{2 r} \times \frac{a^2 h}{3};$$

and, since it must be equal to the sphere, which has also a for its radius, we have

$$\frac{c}{2r} \times \frac{a^2 h}{3} = \frac{c}{2r} \times \frac{4a^3}{3},$$

whence $$h = 4a.$$

The altitude of the cone, therefore, must be double the diameter of the sphere, a result, the truth of which is evident from the consideration, that the sphere, being $\frac{2}{3}$ of the circumscribed cylinder (*Geom.* 549), is double a cone of the same base and same altitude, that is, equal to a cone of the same base and double the altitude (*Geom.* 525).

Of Curved Lines and particularly of Conic Sections.

99. Curved lines are not merely a subject of speculation. So long as the questions, which we have to resolve do not exceed the second degree, we have no occasion for these lines; but in questions of higher degrees, they become necessary. We proceed now to give a general idea of curved lines, and of the use that may be made of them in the construction of equations, to which we arrive in the solution of questions.

Among the curved lines, which are considered in geometry, some are such that every point may be determined by the same law, that is, by like calculations and operations; with respect to others, the several points are determined by different laws, that is, by different calculations and operations; but this difference is subject to a law.

As to lines drawn at random, such, for example, as are traced by the pen in writing, they are incapable of being made the subject of a strict geometry. Still the researches with which we are occupied lead, by direct and certain processes, to the construction of figures that seem to be regulated by no law.

100. To be able to describe the curved lines, which make the subject of geometry, it is necessary to know the law to which the different points of a curve are subjected. Now this law may be given in several ways; one is by indicating a process, according to which the curve may be described by a continued motion; of this nature is the circle, which is described by making a given line to revolve in a plane about a given point. Another way is by making known some property that belongs to every point of

the curve; thus, knowing that every angle which has its vertex in the circumference of a circle, and the sides of which are drawn to the extremities of the diameter, is a right angle, we can find successively each of the points of a circle, whose diameter is known, by drawing from one of the extremities *A* (*fig.* 57) of this diameter, a sufficient number of lines *AC*, *AD*, *AE*, *AF*, &c., and letting fall from the other extremity the perpendiculars *BC*, *BD*, *BE*, *BF*, &c.; then *C*, *D*, *E*, *F*, &c., determined in this manner, belong each to the circumference of a circle, whose diameter is *AB*. Fig. 57.

Lastly, this law may be given by an equation, and it may always be supposed to be thus given, because the two other ways, of which we have spoken, serve to find the equation which expresses this law. It is principally under this last point of view, that we shall consider curves. It is at the same time the most simple and the most comprehensive, for recognizing the properties, peculiarities, and uses of curves. Let us then see how an equation may be made to express the nature of a curve; and since, as yet we are acquainted with the circle only, we begin with this.

101. Let us accordingly suppose that *AMB* (*fig.* 58) is a curve, of which no other property is known except this, that the perpendicular *PM* let fall from any point *M* of this curve upon the line *AB* is a mean proportional between the two parts *AP*, *PB*. Let us see how, by the aid of algebra, we can find each of the points of this curve, and its different properties. If I designate the line *AB* by a, the part *AP* by x, and the perpendicular *PM* by y; then *PB* will be $a - x$; and, since we have supposed *PM* a mean proportional between *AP* and *PB*, we shall have Fig. 58.

$$x : y :: y : a - x;$$

and consequently $$y^2 = a x - x^2.$$

Let us imagine *AB* to be divided into a certain number of equal parts, ten, for example, and that through the several points of division, perpendiculars $p\,m$, $p\,m$, &c., are raised; it is evident that if, in the equation just found, we suppose x successively equal to each of the lines $A\,p$, $A\,p$, &c., y will be equal to the corresponding lines $p\,m$, $p\,m$, &c., since, according to the equation y^2, $= a x - x^2$, y is always a mean proportional between x and $a - x$, whatever x may be, which is the property belonging by supposition, to each perpendicular $p\,m$. We can, therefore, find successively each of the points of this curve, by giving successively to x different values, and calculating the corresponding values of y. If, as we have supposed, a be divided into ten parts, we shall have $a = 10$, and consequently

$$y^2 = 10\,x - x^2.$$

If then we suppose successively

$$x = 1,\ x = 2,\ x = 3,\ x = 4,\ x = 5,$$
$$x = 6,\ x = 7,\ x = 8,\ x = 9,\ x = 10;$$

we shall find successively

$$y = \sqrt{9},\ \ y = \sqrt{16},\ y = \sqrt{21},\ y = \sqrt{24},\ y = \sqrt{25},$$
$$y = \sqrt{24},\ y = \sqrt{21},\ y = \sqrt{16},\ y = \sqrt{9},\ \ y = \sqrt{0},$$

or

$$y = 3,\quad y = 4,\quad y = 4{,}5,\quad y = 4{,}9,\quad y = 5,$$
$$y = 4{,}9,\quad y = 4{,}5,\quad y = 4,\quad y = 3,\quad y = 0.$$

Thus, if we apply these values of y successively to the perpendiculars corresponding to the values 1, 2, &c., of x, the points m, m, &c., determined in this manner, will belong to the curve, the property of which is, that each perpendicular $p\,m$ is a mean proportional between the two parts $A\,p$ and $p\,B$ of the straight line AB, a curve which we shall soon see is the circumference of a circle.

It will be recollected that each of the expressions $\sqrt{9}$, $\sqrt{16}$, &c., has two values (*Alg.* 106), the one positive and the other negative. Thus, besides the values of y, given above, we have also these others,

$$y = -3,\quad y = -4,\quad y = -4{,}5,\ y = -4{,}9,\ y = -5,$$
$$y = -4{,}9,\ y = -4{,}5,\ y = -4,\quad y = -3,\quad y = 0.$$

To obtain the points of the curve, according to these new values of y, it is necessary, conformably to the doctrine of negative quantities, to produce the perpendiculars $p\,m$, $p\,m$, &c., and to apply in the opposite direction, that is, from p to m' the quantities $p\,m'$, $p\,m'$, &c., equal respectively to $p\,m$, $p\,m$, &c.

In order to have a greater number of points belonging to the curve, it is only necessary to suppose AB divided into a greater number of parts, a hundred, for example; or, in other words, preserving a of the same value, and giving to x intermediate values between those above assumed, we might find intermediate values of y, and consequently new points of the curve.

From the value $y = 0$, found above, it is evident that the curve meets the line AB at the point B, where $x = a = 10$; since the perpendicular $p\,m$ in this case, having zero for its value, the distance of the point m from the straight line AB is 0. It is manifest, that it ought to meet the line AB at the point A also. Indeed, since at the places where the curve meets this line, the value of y must be 0, to find these places, it is only necessary to suppose y equal to zero, in the equation $y^2 = a\,x - x^2$, which then becomes $0 = a\,x - x^2 = x\,(a - x)$; and this is zero in two cases, when $x = 0$, and when $x = a$; accordingly, y will be equal to zero in these two cases. Now x is evidently equal to 0 at the point A, and it is equal to a at the point B; therefore the curve does in fact meet the line AB at the points A and B.

From this example we shall begin to perceive how an equation serves to determine different points of a curve; before proceeding to others, we must explain certain words that we shall have occasion to use hereafter.

102. When we would express, by an equation, the nature of a curve, we refer, or are supposed to refer, each of the points m, m, &c., to two fixed lines AB and OAO, which make with each other a determinate angle, either acute, right, or obtuse; and, by imagining that from each point m, lines $m\,p$, and $m\,p'$, &c., are drawn parallel to OAO and AB, it is evident that the situation of this point will be known, when the lines $m\,p'$ or $A\,p$, and $p\,m$, are known; or, which amounts to the same thing, if we know one of these lines and its ratio to the other. Now, when it is said that an equation expresses the nature of a curve, it is to be understood that this equation gives, for each point m, the ratio of $A\,p$ to $p\,m$, so that one being known, the other is found by this means, and, according as this ratio is more or less compounded, the curve itself is of a higher or lower order.

The lines $A\,p$, or $m\,p'$, which measure the distance of each point m from OAO, one of the lines of comparison, are called *abscissas;* and the lines $m\,p$, or $p'\,A$, which measure the distance AB of the other line of comparison, are called *ordinates;* the line AB is called the *axis of the abscissas*, and the line OAO the *axis of the ordinates;* the point A, from which we begin to count the abscissas, is called the *origin of the abscissas*, and that from which we begin to count the ordinates, the *origin of the ordinates*. In figure 58 these two points are the same, namely, A. There is nothing which requires us to count the abscissas from the same point to which the ordinates are referred; but, when there is no circumstance to determine us otherwise, it is always most simple to count them from the same point.

The lines Ap, pm, have the common name of *coordinates of the curve;* and, considered as belonging indifferently to any point of the curve, they are called *indeterminate.* The same name is given to the letters or algebraic characters, by which these lines Ap, pm, &c., are designated.

103. Let us now return to our equation, and see if we can deduce from it the properties of the curve.

1. From the middle C of AB, we draw to any point M of the curve, the straight line CM; wherever this may fall, the triangle MPC will be right-angled, and we shall accordingly have

$$\overline{MP}^2 + \overline{PC}^2 = \overline{MC}^2,$$

that is, since $PC = AC - AP = \frac{1}{2}a - x$,

$$y^2 + \tfrac{1}{4}a^2 - ax + x^2 = \overline{MC}^2.$$

Now, since MP, or y, is a mean proportional between AP and PB, we have $y^2 = ax - x^2$; whence, by substituting for y^2 this value, we have

$$ax - x^2 + \tfrac{1}{4}a^2 - ax + x^2 = \overline{MC}^2,$$

or

$$\tfrac{1}{4}a^2 = \overline{MC}^2;$$

which gives $MC = \frac{1}{2}a$; and this is the case in whatever part of the curve M falls; therefore, every point M is equally distant from the point C, that is, the curve is the circumference of a circle.

2. From any point M of the curve, the straight lines MA, MB, being drawn to the extremities A, B, the right-angled triangles MPA, MPB, give

$$\overline{AP}^2 + \overline{PM}^2 = \overline{AM}^2, \quad \text{and} \quad \overline{PM}^2 + \overline{PB}^2 = \overline{MB}^2,$$

or, substituting the algebraic values,

$$x^2 + y^2 = \overline{AM}^2, \quad \text{and} \quad y^2 + a^2 - 2ax + x^2 = \overline{MB}^2;$$

adding these two equations, and putting for y^2 its value $ax - x^2$, we have

$$a^2 - 2ax + 2x^2 + 2ax - 2x^2 = \overline{AM}^2 + \overline{MB}^2;$$

that is,

$$\overline{AM}^2 + \overline{MB}^2 = a^2 = \overline{AB}^2,$$

the property of a right-angled triangle, from which we learn that the angle AMB is always a right angle, in whatever part of the curve M falls (*Geom.* 128).

3. If, in the equation

$$x^2 + y^2 = \overline{AM}^2,$$

we put for y^2 its value $a\,x - x^2$, we shall have

$$x^2 + a\,x - x^2 = a\,x = \overline{AM}^2;$$

from which we obtain this proportion

$$a : AM :: AM : x,$$

or
$$AB : AM :: AM : AP;$$

that is, the chord AM is a mean proportional between the diameter AB and the segment, or abscissa, AP (*Geom.* 213).

We might thus find all the other properties of the circle made known in the *Elements of Geometry*, by setting out always with this supposition, that the ordinates $p\,m$ are respectively mean proportionals between $A\,p$ and $p\,B$.

We have counted the abscissas from the point A, the origin of the diameter, and we have had the equation $y^2 = a\,x - x^2$. If we would count the abscissas from the centre, or in other words would take CP, $C\,p$, &c., for the abscissas; designating these lines by z, we should have

$$CP = AC - AP,$$

or
$$z = \tfrac{1}{2}\,a - x,$$

and consequently,
$$x = \tfrac{1}{2}\,a - z.$$

Putting, therefore, for x this value in the equation

$$y^2 = a\,x - x^2,$$

we shall have

$$y^2 = a\,(\tfrac{1}{2}\,a - z) - (\tfrac{1}{2}\,a - z)^2,$$

or
$$y^2 = \tfrac{1}{4}\,a^2 - z^2,$$

for the equation of the circle, the coordinates being supposed to be perpendicular, and to have their origin at the centre.

In fine, any property, which belongs essentially to every point of the curve, will give, by being translated into algebra, the same equation for the curve, at least so long as we take the same abscissas and the same ordinates; but when we change the origin or direction of the coordinates, or both, we may have a different equation; still it will always be of the same degree. We have just seen the truth of the last part of this proposition in the change of the abscissas, which, instead of $y^2 = a\,x - x^2$, led to the equation $y^2 = \frac{1}{4}a^2 - z^2$; and this, being deduced from the first, has for its basis the same property. But if we were to set out with this property, namely, that the several distances MC are all the same, and each $= \frac{1}{2}a$; then designating CP by z, and PM by y, we should have on account of the right-angled triangle MPC

$$y^2 + z^2 = \tfrac{1}{4}a^2,$$

which gives
$$y^2 = \tfrac{1}{4}a^2 - z^2;$$

the same equation as that just obtained, although deduced from a different property.

Of the Ellipse.

104. LET us now inquire what would be the curve in which Fig. 59. the sum of the distances $MF + Mf$ (*fig.* 59), of each point from two fixed points F, f, is equal to a given line a.

To find the properties of this curve, which is called an *ellipse*, an equation is to be sought which shall express the relation, arising from this known property, between the perpendiculars PM, drawn from each point M to a determinate line, as Ff, and their distances FP, or AP, from some point F, or A, taken arbitrarily.

For this purpose, I take for the origin of the abscissas the point A, determined by applying from the middle C of Ff the line $CA = \frac{1}{2}a$; and having made $CB = CA$, I designate the lines to be used, as follows, namely,

$$AP = x, \qquad PM = y,$$
$$FM = z, \; AF, \text{ supposed to be known}, = c,$$

then
$$FP = AP - AF = x - c(^*),$$
$$Mf = FMf - FM = a - z,$$
$$fP = PB - Bf = AB - AP - Bf = a - x - c.$$

This being supposed, the right-angled triangles FPM, fPM, give

$$\overline{FM}^2 = \overline{PM}^2 + \overline{FP}^2, \quad \text{and} \quad \overline{Mf}^2 = \overline{PM}^2 + \overline{fP}^2,$$

(*) If the point M had been taken in such a manner, that the perpendicular MP would fall between A and F, then FP would be $c - x$; but this would produce no change in the final equation, because, in the formation of this equation, we employ only the square of FP, which is always $x^2 - 2cx + c^2$.

or $$z^2 = y^2 + x^2 - 2\,c\,x + c^2,$$
and $$a^2 - 2a\,z + z^2 = y^2 + a^2 - 2\,a\,x + x^2 - 2ac + 2cx + c^2.$$
Subtracting the latter of these equations from the former and suppressing a^2, which is found in both members of the result, we have
$$2\,a\,z = 2\,a\,x + 2\,a\,c - 4\,c\,x,$$
and consequently
$$z = \frac{a\,x + a\,c - 2\,c\,x}{a}.$$
Putting for z this value in the equation
$$z^2 = y^2 + x^2 - 2\,c\,x + c^2,$$
we have
$$\frac{a^2\,x^2 + 2\,a^2\,c\,x + a^2\,c^2 - 4\,a\,c\,x^2 - 4\,a\,c^2\,x + 4\,c^2\,x^2}{a^2}$$
$$= y^2 + x^2 - 2\,c\,x + c^2;$$
or, making the denominator to disappear, transposing and reducing,
$$a^2\,y^2 = 4\,a^2\,c\,x - 4\,a\,c^2\,x - 4\,a\,c\,x^2 + 4\,c^2\,x^2,$$
or $$a^2\,y^2 = (4\,a\,c - 4\,c^2)\,a\,x + (4\,c^2 - 4\,a\,c)\,x^2,$$
or, since $4\,c^2 - 4\,a\,c$ is the same as $-(4\,a\,c - 4\,c^2)$,
$$a^2\,y^2 = (4\,a\,c - 4\,c^2)\,a\,x - (4\,a\,c - 4\,c^2)\,x^2,$$
or, lastly, $$a^2\,y^2 = (4\,a\,c - 4\,c^2)\,(a\,x - x^2),$$
whence
$$y^2 = \frac{4\,a\,c - 4\,c^2}{a^2}\,(a\,x - x^2).$$
Such is the equation of the curve, in which, any point M being taken, the sum of the distances MF, Mf, from two fixed points F, f, is equal to a given line a.

105. This equation would enable us to describe the curve by points, if we were to give successively to x different values, as we have done above, with respect to the circle. But, the mode of proceeding being the same, we shall not repeat the calculation.

106. We can also describe the ellipse by points, in this manner; having made CB (*fig.* 60) $= CA = \frac{1}{2}\,a$, we take $B\,r$, equal to any part of AB less than Af, and from the point f, as a centre, and with a radius equal to $B\,r$, we describe arcs above and below AB, which we cut by arcs described from the point F, as a centre, and with the radius $A\,r$; all the points M, M', M'', M''', found in this manner, belong to the ellipse. Fig. 60.

107. The fundamental property, from which we have derived the equation, furnishes also a very simple method of describing

this curve by a continued motion. Having taken the two points
Fig. 60. F, f (*fig.* 60), at pleasure, and having fixed at these points, by means of pins, the extremities of a thread of a greater length than the distance Ff, if we stretch this thread by a style M, carrying it round at the same time, the style will trace the curve in question, since the sum of the two distances of the style from the two points F, f, will always be equal to the whole length of the thread.

108. It will be perceived then, that, if the length of the thread is taken equal to AB, the curve will pass through the two points A, B; for, since $Cf = CF$, we have $AF = Bf$, and consequently

$$AF + Af = Af + Bf = a,$$

and

$$BF + Bf = BF + AF = a.$$

This is made evident also by the equation; for, in order to know where the curve would meet Ff produced, we must make $y = 0$; now this supposition gives

$$\frac{4\,a\,c - 4\,c^2}{a^2}(a\,x - x^2) = 0;$$

and, as $\frac{4\,a\,c - 4\,c^2}{a^2}$ cannot become zero, the equation requires that

$$a\,x - x^2, \quad \text{or} \quad x\,(a - x) = 0,$$

which can take place in two cases, namely, when $x = 0$, that is, at the point A, and when $x = a$, that is, at the point B.

109. It is also evident from the equation, that the curve ex-
Fig. 59. tends above, as well as below, the line AB (*fig.* 59), and that it is the same on each side of the axis. Indeed, the equation gives

$$y = \pm\sqrt{\frac{4\,a\,c - 4\,c^2}{a^2}(a\,x - x^2)},$$

from which we learn, that for each value of x, or AP, there are two values of y, or PM perfectly equal; but, having contrary signs, they must be applied in contrary directions.

It is moreover evident, that if from the middle C of AB, we raise the perpendicular DD', the curve will be divided into two parts perfectly equal and similar; this is a consequence of the manner in which it is described; it is also a consequence of the equation. But this will be more easily perceived, after we have further considered this equation.

110. The line AB is called the *transverse axis* of the ellipse, and the line DD' the *conjugate axis*. The two points F, f, are called *foci*; and the points A, B, D, D', the vertices of the axes, and the point C the *centre*.

111. If we would obtain the value of the ordinate $F m''$, which passes through the focus, we must suppose AP, or $x = AF = c$. In this case we shall have

$$y^2 = \frac{4\,a\,c - 4c^2}{a^2}\,(a\,c - c^2) = \frac{4\,(a\,c - c^2)^2}{a^2};$$

and, extracting the square root,

$$y = \pm \frac{2\,(a\,c - c^2)}{a};$$

therefore $$m''\,m''' = \frac{4\,(a\,c - c^2)}{a}.$$

This line $m''\,m'''$ is called the *parameter* of the ellipse. *The parameter, then, is less than the quadruple of the distance* c *of the vertex from the focus,* since its value $\frac{4\,(a\,c - c^2)}{a}$, which is the same as $4\,c - \frac{4\,c^2}{a}$, is obviously less than $4\,c$.

If we designate this value of the parameter by p, we shall have $p = \frac{4\,a\,c - 4\,c^2}{a}$, and consequently $\frac{p}{a} = \frac{4\,a\,c - 4\,c^2}{a^2}$; the equation of the ellipse, therefore may be changed by substitution, into one of a more simple form, namely, $y^2 = \frac{p}{a}\,(a\,x - x^2)$.

112. If we would know what is the value of the line CD, we have only to suppose, in the equation

$$y^2 = \frac{4\,a\,c - 4\,c^2}{a^2}\,(a\,x - x^2),$$

that AP, or x, is AC, or $\frac{1}{2}\,a$; we shall then have

$$y^2 = \frac{4\,a\,c - 4\,c^2}{a^2}\,(\tfrac{1}{2}\,a^2 - \tfrac{1}{4}\,a^2),$$

which is reduced to $$y^2 = a\,c - c^2;$$

that is, $$\overline{CD}^2 = a\,c - c^2 = c\,(a - c) = AF \times BF,$$

whence $$AF : CD : : CD : BF.$$

We see, therefore, that *CD, or the semiconjugate axis, is a mean proportional between the two distances of the same focus from the vertices* A, B.

As the line DD' is one of the most remarkable lines in the ellipse, we shall introduce it into the equation in preference to the line AF, or c. For this purpose we designate DD' by b; and we shall have $CD = \frac{b}{2}$; and, since we have found $\overline{CD}^2 = a\,c - c^2$, we shall have $\frac{b^2}{4} = a\,c - c^2$, or $b^2 = 4\,a\,c - 4\,c^2$. By making this substitution, the equation of the ellipse becomes

$$y^2 = \frac{b^2}{a^2}(a\,x - x^2).$$

We have found $p = \frac{4\,a\,c - 4\,c^2}{a}$, or $p\,a = 4\,a\,c - 4\,c^2$, and $b^2 = 4\,a\,c - 4\,c^2$; therefore

$$p\,a = b^2,$$

or, reducing this equation to a proportion,

$$a : b :: b : p;$$

in other words, *the parameter is a third proportional to the transverse and conjugate axes.*

113. If, in the equation $y^2 = \frac{b^2}{a^2}(a\,x - x^2)$, we make the denominator to disappear, we shall have

$$a^2\,y^2 = b^2\,(a\,x - x^2),$$

and consequently, $y^2 : a\,x - x^2 :: b^2 : a^2$,

or, since $a\,x - x^2 = x\,(a - x) = AP \times PB$,

$$\overline{PM}^2 : AP \times PB :: \overline{DD'}^2 : \overline{AB}^2;$$

that is, *the square of any ordinate is to the rectangle of the corresponding abscissas, as the square of the conjugate axis is to the square of the transverse;* and since this property belongs to every point of the ellipse, it follows that *the squares of the ordinates are to each other as the rectangles of the corresponding abscissas.*

114. The equation $y^2 = \frac{b^2}{a^2}(a\,x - x^2)$ does not differ from the equation of the circle (101), described upon AB, as a diameter (*fig.* 61), except that the quantity $a\,x - x^2$ is multiplied by $\frac{b^2}{a^2}$ that is, by the ratio of the square of the conjugate axis to the square of the transverse; so that if we designate by u any ordinate PN of the circle, we shall have

Fig. 61.

$$u^2 = a\,x - x^2;$$

putting for $a\,x - x^2$ its value u^2 in the equation of the ellipse, we obtain

$$y^2 = \frac{b^2}{a^2}\,u^2,$$

or, extracting the square root,

$$y = \frac{b}{a}\,u, \quad \text{or} \quad a\,y = b\,u,$$

which gives $y : u :: b : a$,

or $PM : PN :: DD' : AB,$

$:: CD : AC$ or CE.

It will hence be perceived, that *the ordinates of the ellipse are merely the ordinates of the circle described upon the transverse axis, diminished in the ratio of the transverse axis to the conjugate.*

The ellipse, therefore, is easily described by means of the circle, which may be regarded as an ellipse, of which the two axes a, b, are equal, or in which the distance from the vertex to the focus is equal to the semitransverse axis, or in which the parameter is equal to a diameter; for, by supposing, in the above equations, $b = a$, or $c = \frac{1}{2}\,a$, or $p = a$, we have $y^2 = a\,x - x^2$, the equation of the circle.

115. It is clear, therefore, from the equations already obtained, that it is not with the ellipse as with the circle; for a single line, namely, the diameter, determines the circle, whereas the transverse axis is not sufficient for determining the ellipse; it is necessary to know further, either the conjugate axis, or its parameter, or the distance from the vertex to the focus. When we know the transverse axis and the distance c, the ellipse is easily described, as above shown. But if the transverse and conjugate axes are given, it is necessary, in order to describe the ellipse by a continued motion, to determine the foci, which is easily done by describing with the semitransverse, as a radius, and the extremity of the conjugate, as a centre, two arcs cutting the transverse at the two points F and f (*fig.* 59), which will be Fig. 59.
the foci; for the sum of the distances $FD + Df$, being equal to a, when these two lines are equal, each must be equal to $\frac{1}{2}\,a$.

If the transverse axis and the parameter are given, the conjugate axis is determined by finding a mean proportional between these lines, as is manifest from the proportion $a : b :: b : p$ (112). The conjugate axis being found, we can proceed in the manner already explained.

Fig. 59. 116. *If, through any point* M (fig. 59) *of the ellipse, we draw the line* fMG *from one of the foci, making the part* MG *equal to the distance* MF; *and, having drawn* GF, *we let fall upon this line the perpendicular* MOT, *this last will be a tangent to the ellipse, that is, it will meet it in only one point* M.

Indeed, on account of the equal lines MF, MG, the perpendicular MT bisects FG; consequently, if from any other point N of the line MT, we draw the two straight lines NG, NF, these will be equal to each other. Let us then suppose MT to meet the ellipse in some other point N; drawing Nf, we should have $FN + Nf = MF + Mf = GM + Mf = Gf$; but Gf is less than $GN + Nf$ (*Geom.* 40), and consequently less than $FN + Nf$; therefore the point N is without the ellipse (*Geom.* 97).

117. The angles FMO, OMG, are equal, according to the construction just given; but OMG is equal to the opposite angle fMN; consequently, FMO is equal to fMN; therefore *two lines drawn from the two foci to the same point of the ellipse, make equal angles with the tangent to this point.*

We learn by experiment that, when a ray of light falls upon a plane surface, the angle of reflection is equal to the angle of incidence; therefore, all the rays proceeding from one of the foci F, and falling upon the curve MAM', supposed to be capable of reflection, would be concentrated in the other focus.

If from the point M we raise upon MT the perpendicular MI, which will at the same time be perpendicular to the curve, this line will divide the angle FMf into two equal parts; for, if from the right angles IMT, IMN, we subtract the equal angles FMT, fMN, the remaining angles FMI, IMf, will be equal.

118. We can therefore determine the distance PI, from the ordinate to the point where the perpendicular MI meets the axis. This line PI is called the *sub-normal*, and the line MI the *normal.*

To obtain the value of PI, we first find that of FI. Since the angle FMf is divided into two equal parts by MI, we have

$$Mf : MF :: fT : FI \quad (Geom. 201),$$

and hence $Mf + MF : Mf - MF :: fI + FI : fI - FI.$

But $$Mf + MF = a,$$

and, since $$MF = z,$$

we have $$Mf = a - z,$$

and consequently $Mf - MF = a - 2z;$

also $\quad fI + FI = Ff = AB - 2\,AF = a - 2\,c,$
and $\quad fI - FI = Ff - 2\,FI = a - 2\,c - 2\,FI;$
therefore, putting for the above lines their algebraic values, we have

$$a : a - 2\,z :: a - 2\,c : a - 2\,c - 2\,FI,$$

whence

$$a^2 - 2\,a\,c - 2\,a \times FI = a^2 - 2\,a\,c - 2\,a\,z + 4\,c\,z,$$

and from this we deduce

$$FI = \frac{a\,z - 2\,c\,z}{a};$$

or, putting for z its value $\frac{a\,x + a\,c - 2\,c\,x}{a}$ (104), we have

$$FI = \frac{a^2\,c - 2\,a\,c^2 + a^2\,x - 4\,a\,c\,x + 4\,c^2\,x}{a^2};$$

but $FI = FP + PI = AP - AF + PI = x - c + PI,$
therefore $\quad FI - x + c = PI,$
or substituting for FI the above value,

$$PI = \frac{a^2\,c - 2\,a\,c^2 + a^2\,x - 4\,a\,c\,x + 4\,c^2\,x}{a^2} - x + c,$$

$$= \frac{a^2\,c - 2\,a\,c^2 + a^2\,x - 4\,a\,c\,x + 4\,c^2\,x - a^2\,x + a^2\,c}{a^2},$$

$$= \frac{2\,a^2\,c - 2\,a\,c^2 - 4\,a\,c\,x + 4\,c^2\,x}{a^2},$$

$$= \frac{2\,a\,(a\,c - c^2) - 4\,x\,(a\,c - c^2)}{a^2} = \frac{2\,a - 4\,x}{a^2}\,(a\,c - c^2),$$

or, putting for $a\,c - c^2$ its value $\frac{b^2}{4}$ (112),

$$PI = b^2\,\frac{a - 2\,x}{2\,a^2} = \frac{b^2}{a^2}\,(\tfrac{1}{2}\,a - x).$$

119. PI being found, it is easy to determine the distance PT from the ordinate to the point of meeting of the tangent, which is called the *subtangent*; for, the triangle IMT being right-angled, and PM a perpendicular let fall from the right angle,

$$PI : PM :: PM : PT, \quad (\textit{Geom.}\ 213),$$

that is $\quad \frac{b^2}{a^2}\,(\tfrac{1}{2}\,a - x) : y :: y : PT,$

whence

$$PT = \frac{a^2\,y^2}{b^2\,(\tfrac{1}{2}\,a - x)},$$

or, putting for y^2 its value, $\frac{b^2}{a^2}(a\,x - x^2)$,

$$PT = \frac{a\,x - x^2}{\frac{1}{2}\,a - x}.$$

By means of the algebraic expression of the lines PI and PT, we can draw a perpendicular and a tangent to the ellipse at any point M; for, when the point M is given, the perpendicular MP being drawn, we have the value of AP, or x; and, as a and b are supposed to be known, every thing is known which enters into the value of PI and that of PT.

120. From the expression for PT, we shall perceive, that if we draw a tangent to the circle described upon the transverse Fig. 61. axis AB (*fig.* 61) at the point N, where the circumference of the circle is met by the ordinate PM of the ellipse, the tangents NT, MT, will meet the transverse axis in the same point T. For, since the conjugate b does not enter into the expression for PT, this line PT will always remain the same, while a and x remain the same. Thus the tangents to the corresponding points of all the ellipses described upon AB as the transverse axis meet in the same point T.

Fig. 59. If to PT (*fig.* 59) we add CP, which is $\frac{1}{2}\,a - x$, we shall have

$$CT = \frac{(a\,x - x^2)}{\frac{1}{2}\,a - x} + \frac{1}{2}\,a - x = \frac{\frac{1}{4}\,a^2}{\frac{1}{2}\,a - x} = \frac{\overline{AC}^2}{CP},$$

whence $$CP : AC : : AC : CT.$$

121. If we would have the expression for TM, it is easily obtained by means of the right-angled triangle TPM, which gives

$$\overline{TM}^2 = \overline{TP}^2 + \overline{PM}^2 = \frac{(a\,x - x^2)^2}{(\frac{1}{2}\,a - x)^2} + \frac{b^2}{a^2}(a\,x - x^2)$$

$$= \left(ax + x^2 + \frac{b^2}{a^2}(\tfrac{1}{2}\,a - x)^2\right)\frac{a\,x - x^2}{(\frac{1}{2}\,a - x)^2}.$$

Fig. 59. 122. If from any point M (*fig.* 59) of the ellipse, we draw to the conjugate axis DD' the perpendicular or ordinate MP', designating DP' by x', and MP' by y', we shall have

$$DP' = CD - CP' = CD - PM,$$

that is, $$x' = \tfrac{1}{2}\,b - y,$$

and consequently $$y = \tfrac{1}{2}\,b - x'.$$

In like manner, we shall have

$$MP' = CP = CA - AP$$

that is, $y' = \frac{1}{2} a - x,$

and consequently $x = \frac{1}{2} a - y'.$

If we substitute for x and y these values in the equation

$$y^2 = \frac{b^2}{a^2} (a x - x^2), \quad \text{or} \quad a^2 y^2 = b^2 (a x - x^2),$$

we shall have

$$a^2 (\tfrac{1}{2} b - x')^2 = b^2 (a (\tfrac{1}{2} a - y') - (\tfrac{1}{2} a - y')^2)$$
$$a^2 (\tfrac{1}{4} b^2 - b x' + x'^2) = b^2 (\tfrac{1}{2} a^2 - a y' - \tfrac{1}{4} a^2 + a y' - y'^2)$$
$$\tfrac{1}{4} a^2 b^2 - a^2 b x' + a^2 x'^2 = \tfrac{1}{4} a^2 b^2 - b^2 y'^2$$
$$b^2 y'^2 = a^2 b x' - a^2 x'^2,$$

whence

$$y'^2 = \frac{a^2}{b^2} (b x' - x'^2),$$

an equation similar to that obtained for the transverse axis, and from which similar results may be derived; thus, *the square of the ordinate* P′M *to the conjugate axis is to the rectangle of the abscissas* DP′ × P′D′, *as the square of the transverse axis is to the square of the conjugate;* for, from the above equation, we have the proportion,

$$y'^2 : b x' - x'^2 : : a^2 : b^2;$$

in which $b x' - x'^2 = x' (b - x') = DP' \times P'D'.$

We infer also from the same equation, that *the squares of the ordinates to the conjugate axis are to each other as the rectangles of the corresponding abscissas*; and that *the ellipse may be constructed by means of the circle described upon the conjugate axis, by producing the ordinates of the circle in the ratio of the conjugate axis to the transverse.* See fig. 61.

123. We see, therefore, that the properties relating to the conjugate axis are similar to those we have found, with respect to the transverse, at least, so far as the foci are not concerned.

If we would determine on the conjugate axis the lines $P'I'$, $P'T'$, CT', and MT' (*fig.* 62), we can readily obtain them by Fig. 62.
means of the analogous ones, found on the transverse, and the similar triangles, which will be immediately seen by recurring to the figure. If we would obtain the value of these lines by means of the abscissas DP', or x', we shall always find the expressions similar to those we have found, by means of x, for the analogous lines on the transverse axis.

We give a *parameter* also to the conjugate axis; but we understand by this line, not one which passes through a focus, for the conjugate axis has no focus, but a third proportional to the conjugate axis and the transverse.

124. Hitherto we have counted the abscissas only from the vertex; if we would count them from the centre C, designating the abscissa CP by z, we shall have

$$AP, \text{ or } x = \tfrac{1}{2} a - z,$$

hence, in the following expressions, namely,

$$y^2 = \frac{b^2}{a^2}(a x - x^2),$$

$$PI = \frac{b^2}{a^2}(\tfrac{1}{2} a - x) \ (118),$$

$$PT = \frac{a x - x^2}{\frac{1}{2} a - x} \quad (119),$$

$$CT = \frac{\frac{1}{4} a^2}{\frac{1}{2} a - x} \quad (120),$$

$$\overline{TM}^2 = \left(a x - x^2 + \frac{b^2}{a^2}(\tfrac{1}{2} a - x)^2\right)\frac{a x - x^2}{(\frac{1}{2} a - x)^2} \quad (121),$$

if we substitute for x the above value, $\frac{1}{2} a - z$, we shall have successively

$$y^2 = \frac{b^2}{a^2}\left(a(\tfrac{1}{2} a - z) - (\tfrac{1}{2} a - z)^2\right)$$

$$= \frac{b^2}{a^2}(\tfrac{1}{2} a^2 - a z - \tfrac{1}{4} a^2 + a z - z^2) = \frac{b^2}{a^2}(\tfrac{1}{4} a^2 - z^2),$$

$$PI = \frac{b^2}{a^2}(\tfrac{1}{2} a - (\tfrac{1}{2} a - z) = \frac{b^2}{a^2}(\tfrac{1}{2} a - \tfrac{1}{2} a + z) = \frac{b^2}{a^2} z,$$

$$PT = \frac{a(\frac{1}{2} a - z) - (\frac{1}{2} a - z)^2}{\frac{1}{2} a - (\frac{1}{2} a - z)} = \frac{\frac{1}{4} a^2 - z^2}{z},$$

$$CT = \frac{\frac{1}{4} a^2}{\frac{1}{2} a - (\frac{1}{2} a - z)} = \frac{\frac{1}{4} a^2}{z},$$

$$\overline{TM}^2 = \left(a(\tfrac{1}{2}a - z) - (\tfrac{1}{2}a - z)^2 + \frac{b^2}{a^2}\overline{\tfrac{1}{2}a - (\tfrac{1}{2}a - z)}^2\right)\frac{a\frac{1}{2}(a - z) - (\frac{1}{2}a - z)^2}{(\frac{1}{2}a - (\frac{1}{2}a - z))^2},$$

$$= \left(\tfrac{1}{2}a^2 - az - \tfrac{1}{4}a^2 + az - z^2 + \frac{b^2}{a^2}z^2\right)\frac{\frac{1}{2}a^2 - az - \frac{1}{4}a^2 + az - z^2}{z^2},$$

$$= \left(\tfrac{1}{4} a^2 - z^2 + \frac{b^2 z^2}{a^2}\right)\frac{\frac{1}{4} a^2 - z^2}{z^2}.$$

The equation $y^2 = \frac{b^2}{a^2}(\frac{1}{4} a^2 - z^2)$ gives

$$y = \pm \frac{b}{a}\sqrt{\tfrac{1}{4} a^2 - z^2},$$

from which it is evident, that for each value of CP or z we have two ordinates PM and PM'. As the values of z begin at C and end at A, this equation would at first seem to give only half of the ellipse DAD'; but there is no reason for giving to z positive rather than negative values; by giving the latter, we have the ordinates $p\,m$, which determine the second half of the ellipse; and, since putting $-z$ for $+z$ in the expression $\pm \frac{b}{a}\sqrt{\frac{1}{4}a^2 - z^2}$ does not alter its value, it follows that the half DBD' is perfectly equal and similar to the half DAD'.

125. If, from any point M of the ellipse (*fig.* 63), we draw through the middle C of the axis AB, that is, through the centre, the straight line MCM', terminated by the other part of the ellipse, this line is called a *diameter;* and, if through the vertex M, we draw the tangent MT, and through the centre C the diameter NN' parallel to MT, NN' is the *conjugate diameter* of MM'. A line $m\,O$ drawn from a point m of the ellipse parallel to MT, and terminated by the diameter MM', is called an ordinate to this diameter, and MO is called an abscissa. The parameter to the diameter MM' is a third proportional to MM' and NN'. **Fig. 63.**

126. We proceed now to show that the ordinates $m\,O$, to any diameter have properties similar to those of the ordinates to the axes.

In order to this, I let fall from the points m and O the perpendiculars $m\,p$, OQ, upon the axis AB; I draw the line $m\,S$ parallel to the same axis, and I designate the lines to be used, as follows, namely,

$$AB = a,\ PM = y,\ CP = z,\ Qp = g,\ CQ = k;$$

and I have

$$AP = \tfrac{1}{2}a - z,\ PB = \tfrac{1}{2}a + z,$$
$$Ap = CA - Cp = CA - CQ - Qp = \tfrac{1}{2}a - k - g,$$
$$pB = CB + Cp = \tfrac{1}{2}a + k + g.$$

The similar triangles TPM, $m\,S\,O$, give

$$TP : PM :: m\,S \text{ or } Qp : SO;$$

that is,

$$\frac{\frac{1}{4}a^2 - z^2}{z} : y :: g : SO;$$

whence
$$SO = \frac{g\,y\,z}{\frac{1}{4}a^2 - z^2}.$$

The similar triangles CMP, COQ, give

$$CP : PM :: CQ : QO,$$

that is,
$$z : y :: k : QO;$$

whence
$$QO = \frac{k\,y}{z};$$

therefore $p\,m = QS = QO - SO = \frac{k\,y}{z} - \frac{g\,y\,z}{\frac{1}{4}\,a^2 - z^2}$.

Now, since the point m is a point in the ellipse,

$$\overline{p\,m}^2 : \overline{PM}^2 :: A\,p \times p\,B : AP \times PB \ (113),$$

that is,

$$\left(\frac{ky}{z} - \frac{gyz}{\frac{1}{4}a^2 - z^2}\right) : y^2 :: (\tfrac{1}{2}a - k - g)(\tfrac{1}{2}a + k + g) : (\tfrac{1}{2}a - z)(\tfrac{1}{2}a + z),$$

or

$$\frac{k^2\,y^2}{z^2} - \frac{2\,g\,k\,y^2\,z}{z\,(\frac{1}{4}\,a^2 - z^2)} + \frac{g^2\,y^2\,z^2}{(\frac{1}{4}\,a^2 - z^2)^2} : y^2$$
$$:: \tfrac{1}{4}\,a^2 - k^2 - 2\,g\,k - g^2 : \tfrac{1}{4}\,a^2 - z^2;$$

multiplying the extremes and means, and observing what quantities are multiplied and divided at the same time by $\frac{1}{4}\ a^2 - z^2$, or by z, we shall have

$$\frac{k^2\,y^2}{z^2}(\tfrac{1}{4}\,a^2 - z^2) - 2\,g\,k\,y^2 + \frac{g^2\,y^2\,z^2}{\frac{1}{4}\,a^2 - z^2}$$
$$= \tfrac{1}{4}\,a^2\,y^2 - k^2\,y^2 - 2\,g\,k\,y^2 - g^2\,y^2,$$

or, by developing the term $\frac{k^2\,y^2}{z^2}\ (\frac{1}{4}\,a^2 - z^2)$, suppressing the common terms, $-\,k^2\,y^2$, and $-\,2\,g\,k\,y^2$, and dividing by y^2, we obtain

$$\frac{\frac{1}{4}\,a^2\,k^2}{z^2} + \frac{g^2\,z^2}{\frac{1}{4}\,a^2 - z^2} = \tfrac{1}{4}\,a^2 - g^2,$$

the equation required for our purpose; but before employing it in the way intended, let us deduce from it a proposition which we shall have occasion for.

If we suppose the point O, here considered as any point whatever, to be the point C, that is, if the line $O\,m$ pass through the centre, or become CN, then CQ or k becomes zero, and the line $Q\,p$ or g becomes CR. Now, if, in the equation just found, we make $k = 0$, we shall have, after making the denominator to disappear, transposing and reducing, and dividing by $\frac{1}{4}\,a^2$,

$$g^2 = \tfrac{1}{4}\,a^2 - z^2,$$

that is,

$$\overline{CR}^2 = \tfrac{1}{4}\,a^2 - z^2 = (\tfrac{1}{2}\,a - z)\,(\tfrac{1}{2}\,a + z) = AP \times PB.$$

Returning to our inquiry, we designate the lines CM, CN, &c., as follows, namely,

$$CM = \tfrac{1}{2}\, a', \quad CN = \tfrac{1}{2}\, b', \quad m\,O = y', \quad CO = z'.$$

The similar triangles CPM, CQO, give

$$CM : CO :: CP : CQ,$$

or

$$\tfrac{1}{2}\, a' : z' :: z : k;$$

whence

$$k = \frac{z\,z'}{\tfrac{1}{2}\, a'}, \quad \text{and} \quad k^2 = \frac{z^2\, z'^{\,2}}{\tfrac{1}{4}\, a'^{\,2}}.$$

The triangles CNR, $m\,S\,O$, similar on account of the sides being parallel, give

$$m\,O : m\,S :: CN : CR,$$

or

$$y' : g :: \tfrac{1}{2}\, b' : CR,$$

whence

$$CR = \frac{\tfrac{1}{2}\, b'\, g}{y'}, \quad \text{and} \quad \overline{CR}^2 = \frac{\tfrac{1}{4}\, b'^{\,2}\, g^2}{y'^{\,2}};$$

but we have seen that $\overline{CR}^2 = \tfrac{1}{4}\, a^2 - z^2$,

therefore

$$\frac{\tfrac{1}{4}\, b'^{\,2}\, g^2}{y'^{\,2}} = \tfrac{1}{4}\, a^2 - z^2,$$

from which we deduce

$$g^2 = \frac{y'^{\,2}\,(\tfrac{1}{4}\, a^2 - z^2)}{\tfrac{1}{4}\, b'^{\,2}}.$$

Resuming now the equation

$$\frac{\tfrac{1}{4}\, a^2\, k^2}{z^2} + \frac{g^2\, z^2}{\tfrac{1}{4}\, a^2 - z^2} = \tfrac{1}{4}\, a^2 - g^2,$$

and substituting for g^2 and k^2 the values above found, we shall have

$$\tfrac{1}{4}\, a^2\, \frac{z^2\, z'^2}{\tfrac{1}{4}\, a'^2\, z^2} + \frac{y'^2\, z^2(\tfrac{1}{4}\, a^2 - z^2)}{\tfrac{1}{4}\, b'^2(\tfrac{1}{4}\, a^2 - z^2)} = \tfrac{1}{4}\, a^2 - \frac{\tfrac{1}{4}\, a^2\, y'^2}{\tfrac{1}{4}\, b'^2} + \frac{y'^1\, z^2}{\tfrac{1}{4}\, b'^2},$$

or, reducing, and dividing by $\tfrac{1}{4}\, a^2$,

$$\frac{z'^{\,2}}{\tfrac{1}{4}\, a'^2} = 1 - \frac{y'^{\,2}}{\tfrac{1}{4}\, b'^{\,2}},$$

or, making the denominators to disappear,

$$\tfrac{1}{4}\, b'^{\,2}\, z'^{\,2} = \tfrac{1}{16}\, a'^{\,2}\, b'^{\,2} - \tfrac{1}{4}\, a'^{\,2}\, y'^{\,2},$$

from which we obtain

$$y'^{\,2} = \frac{b'^2}{a'^2}\,(\tfrac{1}{4}\, a'^{\,2} - z'^{\,2}),$$

and thence the proportion

$$y'^{\,2} : \tfrac{1}{4}\, a'^{\,2} - z'^{\,2} :: b'^{\,2} : a'^{\,2},$$

or

$$\overline{m\,O}^2 : MO \times OM' :: \overline{NN'}^2 : \overline{MM'}^2.$$

Thus the equation, with respect to any two conjugate diameters, is similar to that before obtained respecting the two axes.

127. If, in the equation $y'^2 = \frac{b'^2}{a'^2}(\frac{1}{4}a'^2 - z'^2)$, we make $y' = 0$, we shall have

$$\tfrac{1}{4}a'^2 - z'^2 = 0,$$

and consequently $z' = \pm \frac{1}{2}a'$.

The curve, therefore, meets the line MM' in two points M and M', equally distant from the centre C; thus, *in the ellipse, all the diameters bisect each other at the centre.*

128. As the equation $y'^2 = \frac{b'^2}{a'^2}(\frac{1}{4}a'^2 - z'^2)$ gives

$$y' = \pm \frac{b'}{a'}\sqrt{\tfrac{1}{4}a'^2 - z'^2},$$

it is evident, that if $m\,O$ be produced, so as to make $O\,m' = O\,m$, the point m' will belong to the curve; therefore, *each diameter of the ellipse bisects the lines drawn parallel to the tangent which passes through the origin* M.

129. We may hence infer; 1. that the tangent, at the extremity N of the diameter NN', is parallel to the diameter MM';

2. since $y' = \pm \frac{b'}{a'}\sqrt{\frac{1}{4}a'^2 - z'^2} = \pm \frac{b'}{a'}\sqrt{(\frac{1}{2}a' + z')(\frac{1}{2}a' - z')}$,

we may infer, that the ordinates $O\,m$ to the diameter MM', are those of the circle, having MM' for its diameter, but diminished or augmented in the ratio of a' to b', and inclined at an angle equal to that of the conjugate diameters. If $a' = b'$, these ordinates are precisely equal to those of this same circle.

If we would know in what part of the ellipse the two conjugate diameters are equal, we have only to find in what part $CP = CR$, or $\overline{CP}^2 = \overline{CR}^2$, or $z^2 = \frac{1}{4}a^2 - z^2$. Now this equation gives

$$z^2 + z^2 = \tfrac{1}{4}a^2,$$

whence $z^2 = \frac{1}{8}a^2$, and $z = \sqrt{\frac{1}{8}a^2} = \frac{1}{2}a\sqrt{\frac{1}{2}}$,

which may be thus constructed; having described upon the
Fig. 61. transverse axis AB (*fig.* 61), as a diameter, the semicircle $ANEB$, cut in E by the conjugate axis CD, we bisect the arc AE in N'', and letting fall the perpendicular $N''\,P$, which cuts the ellipse in M'' and M', CM'', CM', equal to each other, will be the semiconjugate diameters. For, if we designate CP by z,

since the triangle CPN'' is right-angled and isosceles, the angle ACN'' being 45°, we have

$$z^2 + z^2 = \overline{CN''}^2 = \tfrac{1}{4}\,a^2,$$

hence
$$z^2 = \tfrac{1}{8}\,a^2,$$

and
$$z = \tfrac{1}{2}\,a\,\sqrt{\tfrac{1}{2}}.$$

130. If, from the centre C (*fig.* 63), we draw to TM the perpendicular CF, the similar triangles TPM, TCF, give Fig. 63.

$$TM : PM :: CT : CF;$$

whence

$$CF = \frac{PM \times CT}{TM}.$$

In like manner, the triangles TPM, CNR, similar on account of the parallel sides, give

$$TM : TP :: CN : CR;$$

whence

$$CN = \frac{TM \times CR}{TP}.$$

consequently, multiplying the above equations, member by member, we have

$$CF \times CN = \frac{PM \times CT \times TM \times CR}{TM \times TP} = \frac{PM \times CT \times CR}{TP},$$

or, taking the squares,

$$\overline{CF}^2 \times \overline{CN}^2 = \frac{\overline{PM}^2 \times \overline{CT}^2 \times \overline{CR}^2}{\overline{TP}^2}.$$

Now, we have already found

$$\overline{PM}^2 \text{ or } y^2 = \frac{b^2}{a^2}\left(\tfrac{1}{4}\,a^2 - z^2\right),$$

$$\overline{CT}^2 = \frac{\frac{1}{16}\,a^4}{z^2} \qquad (124),$$

$$\overline{TP}^2 = \frac{\left(\frac{1}{4}\,a^2 - z^2\right)^2}{z^2} \qquad (124),$$

$$\overline{CR}^2 = \tfrac{1}{4}\,a^2 - z^2 \qquad (126);$$

substituting these quantities for their equals in the above result, we have

$$\overline{CF}^2 \times \overline{CN}^2 = \frac{\frac{b^4}{a^2}\left(\frac{1}{4}\,a^2 - z^2\right)\frac{\frac{1}{16}\,a}{z^2}\left(\frac{1}{4}\,a^2 - z^2\right)}{\frac{\left(\frac{1}{4}\,a^2 - z^2\right)^2}{z^2}}$$

$$\overline{CF}^2 \times \overline{CN}^2 = \frac{\frac{\frac{1}{16} a^4 b^2}{a^2} \frac{(\frac{1}{4} a^2 - z^2)(\frac{1}{4} a^2 - z^2)}{z^2}}{\frac{(\frac{1}{4} a^2 - z^2)^2}{z^2}}$$

$$= \tfrac{1}{16} a^2 b^2,$$

and $$CF \times CN = \tfrac{1}{4} a\, b.$$

If now we draw the tangent NT'', which produced meets TM in I, $CN \times CF$ expresses the surface of the parallelogram $CMIN$; and $\frac{1}{4}\ a\, b$, or $\frac{1}{2} a \times \frac{1}{2}\ b$, expresses that of the rectangle of the two semiaxes; therefore, *the parallelograms, formed by tangents drawn through the vertices of conjugate diameters, are equal to each other, and equal to the rectangle of the two axes.*

131. The similar triangles TPM, CRN, give, also,

$$PT : PM :: CR : RN,$$

whence

$$RN = \frac{CR \times PM}{PT},$$

and $$\overline{RN}^2 = \frac{\overline{CR}^2 \times \overline{PM}^2}{\overline{PT}^2},$$

$$= \frac{(\frac{1}{4} a^2 - z^2) \frac{b^2}{a^2} (\frac{1}{4} a^2 - z^2)}{\frac{(\frac{1}{4} a^2 - z^2)^2}{z^2}},$$

$$= \frac{(\frac{1}{4} a \ - z^2) \frac{b^2}{a^2} (\frac{1}{4} a^2 - z^2)\, z^2}{(\frac{1}{4} a^2 - z^2)^2},$$

$$= \frac{b^2 z^2}{a^2};$$

but the right-angled triangles CRN, CPM, give

$$\overline{CR}^2 + \overline{RN}^2 = \overline{CN}^2, \quad \overline{CP}^2 + \overline{PM}^2 = \overline{CM}^2,$$

whence

$$\overline{CR}^2 + \overline{RN}^2 + \overline{CP}^2 + \overline{PM}^2 = \overline{CN}^2 + \overline{CM}^2;$$

substituting for the lines in the first member their algebraic values, we have

$$\tfrac{1}{4} a^2 - z^2 + \frac{b^2 z^2}{a^2} + z^2 + \frac{b^2}{a^2} (\tfrac{1}{4} a^2 - z^2) = \overline{CN}^2 + \overline{CM}^2,$$

or, the whole being reduced,

$$\tfrac{1}{4} a^2 + \tfrac{1}{4} b^2 = \overline{CN}^2 + \overline{CM}^2;$$

therefore, *the sum of the squares of any two semiconjugate diameters is equal to the sum of the squares of the semiaxes.*

132. If, in the equation

$$\overline{CN}^2 = \overline{CR}^2 + \overline{RN}^2,$$

we substitute for $\overline{CR}^2$ and $\overline{RN}^2$ their algebraic values, we shall have

$$\overline{CN}^2 = \tfrac{1}{4}a^2 - z^2 + \frac{b^2 z^2}{a^2},$$

but $$\overline{TM}^2 = \left(\tfrac{1}{4}a^2 - z^2 + \frac{b^2 z^2}{a^2}\right)\frac{\tfrac{1}{4}a^2 - z^2}{z^2} \quad (121),$$

consequently $$\overline{TM}^2 = \overline{CN}^2 \times \frac{\tfrac{1}{4}a^2 - z^2}{z^2}.$$

Now the similar triangles TPM, $MP'T'$, give, by taking the squares of the homologous sides,

$$\overline{PT}^2 : \overline{TM}^2 :: \overline{P'M}^2 : \overline{MT'}^2,$$

or $$\frac{(\tfrac{1}{4}a^2 - z^2)^2}{z^2} : \overline{CN}^2 \times \frac{\tfrac{1}{4}a^2 - z^2}{z^2} :: z^2 : \overline{MT'}^2,$$

whence $$\overline{MT'}^2 = \overline{CN}^2 \times \frac{z^2}{\tfrac{1}{4}a^2 - z^2},$$

and, multiplying together the members of this equation, and that above, we have

$$\overline{TM}^2 \times \overline{MT'}^2 = \left(\overline{CN}^2 \times \frac{\tfrac{1}{4}a^2 - z^2}{z^2}\right)\left(\overline{CN}^2 \times \frac{z^2}{\tfrac{1}{4}a^2 - z^2}\right),$$

$$= \overline{CN}^4,$$

or $TM \times MT' = \overline{CN}^2$.

If now we designate the parameter to the diameter MM' by p', we shall have

$$2\,CM : 2\,CN :: 2\,CN : p' \quad (125),$$

whence $$2p' \times CM = 4\,\overline{CN}^2,$$

or $$\tfrac{1}{2}p' \times CM = \overline{CN}^2;$$

therefore, comparing this value of $\overline{CN}^2$ with the one above, we have

$$TM \times MT' = \tfrac{1}{2}p' \times CM,$$

or, in other words,

$$CM : TM :: MT' : \tfrac{1}{2}p'.$$

Fig. 64. If upon TT' (*fig.* 64), as a diameter, we describe a circle, the circumference will pass through the point C, since the angle TCT' is a right angle. If now we produce CM till it meets the circumference in V, we shall have, by the nature of the circle (*Geom.* 223),

$$CM : TM :: MT' : MV;$$

therefore, comparing this proportion with the one above, we obtain

$$MV = \tfrac{1}{2} p'.$$

133. From what is here shown we may easily find the axes of an ellipse, and consequently derive a simple method of describing it, when we know only two conjugate diameters MM', NN', and the angle contained by them.

We produce CM, making MV equal to the semiparameter, and from the middle X of CV, we raise a perpendicular XZ, meeting in Z the indefinite line TT', drawn through the point M parallel to NN'; from the point Z, as a centre, and with the distance ZC, as a radius, we describe a circle meeting TT' in two points T, T', through which and the point C, TC, $T'C$, being drawn, these are the directions of the two axes. We then determine the magnitude of the two axes by letting fall the perpendiculars MP, MP', and taking CA a mean proportional between CT and CP; and CD, a mean proportional between CT' and CP'; for it has been shown (120), that $CP : CA :: CA : CT$; and it is easy to prove, by means of the similar triangles TPM, TCT', and the known values of TP, PM, and CT, that $CT' = \frac{\overline{CD}^2}{CP'}$, that is, that $CP' : CD :: CD : CT'$.

Of the Hyperbola.

134. Let us now consider the curve, the property of which is Fig. 65. that the difference of the distances from any point M (*fig.* 65) to two fixed points F, f, is the same throughout, and equal to a given line a.

We proceed to find, as in the case of the ellipse, an equation, which shall express the relation between the lines PM perpendicular to Ff, and their distances FP, or AP, from some fixed point F, or A, taken arbitrarily upon the line Ff.

Taking for the origin of the abscissas the point A, determined by applying from the middle C of Ff the line CA equal to $\frac{1}{2}$ a, I make CB equal to CA. This being done, I designate the lines to be used as follows, namely,

$AP = x$, $PM = y$,

$FM = z$, AF, supposed to be known, $= c$;

then $FP = AF - AP = c - x$ (*),

$fP = fA + AP = fB + AB + AP = c + a + x$;

and, since $Mf - MF = a$,

we have $Mf = a + FM = a + z$.

The right-angled triangles FPM, fMP, give

$$\overline{FP}^2 + \overline{PM}^2 = \overline{FM}^2, \text{ and } \overline{fP}^2 + \overline{PM}^2 = \overline{fM}^2,$$

that is,

$$c^2 - 2cx + x^2 + y^2 = z^2,$$

and

$$c^2 + 2ac + a^2 + 2cx + 2ax + x^2 + y^2 = a^2 + 2az + z^2,$$

Subtracting the first of these equations from the second, and suppressing a^2 common to both members of the result, we have

$$4cx + 2ac + 2ax = 2az,$$

whence $$z = \frac{2cx + ac + ax}{a};$$

putting therefore for z this value in the first of the above equations, we obtain

$$c^2 - 2cx + x^2 + y^2 = \frac{4c^2x^2 + 4ac^2x + a^2c^2 + 4acx^2 + 2a^2cx + a^2x^2}{a^2},$$

or, making the denominator to disappear, transposing and reducing,

$$a^2y^2 = 4a^2cx + 4ac^2x + 4acx^2 + 4c^2x^2,$$
$$= (4ac + 4c^2)(ax + x^2);$$

whence $$y^2 = \frac{4ac + 4c^2}{a^2}(ax + x^2).$$

135. This equation will enable us, by giving to x successive values, to describe the curve by points.

We can moreover trace the curve by means of points, if we take an extent Br (*fig.* 66) greater than BF, and describe from **Fig. 66.** the point f, as a centre, and with the radius Br, an arc to be cut by another arc described from the point F as a centre and with a radius equal to Ar.

(*) If the point P were on the other side of F, with respect to A, FP would be $x - c$; but this would not change the final equation. See note to page 116.

Lastly, this curve may be described by a continued motion, in the following manner. We fix at the point f a rule which admits of being turned about this point. At the point F and some point Q of this rule, we attach the extremities of a thread FMQ, less than fQ, the difference between which and fQ being equal to AB; then, by means of a style M, we apply a part MQ of the thread against the rule, moving the style toward A, turning the rule at the same time, and keeping the thread stretched, the part FM will thus become less and less, and the style M will describe MA, a part of the curve in question, and which is called a *hyperbola*. Indeed, it isevident, that as the whole extent fQ, or $fM + MQ$, is always of the same magnitude, and $FM + MQ$, also of the same magnitude, the difference between these, or

$$fM + MQ - FM - MQ,$$

or

$$fM - FM,$$

will in like manner be throughout of the same magnitude.

136. Since the equation $y^2 = \frac{4\,a\,c + 4\,c^2}{a^2}(a\,x + x^2)$ gives

$$y = \pm\sqrt{\frac{4\,a\,c + 4\,c^2}{a^2}(a\,x + x^2)},$$

it is manifest, that for every abscissa AP or x, we have two equal ordinates PM, PM', which fall upon opposite sides of AB produced; thus the curve has a second branch AM' perfectly equal to the first, and each is of unlimited extent, since it is evident that the more we increase x, the more we increase the values of

$$y, \text{ or } \pm\sqrt{\frac{4\,a\,c + 4\,c^2}{a^2}(a\,x + x^2)}.$$

137. If, in this same quantity, we make x negative, that is, if we suppose the point P to fall above A, it will become

$$\pm\sqrt{\frac{4\,a\,c + 4\,c^2}{a^2}(x^2 - a\,x)};$$

now $x^2 - a\,x$, or $x\,(x - a)$, being negative so long as x is smaller than a, the quantity

$$\pm\sqrt{\frac{4\,a\,c + 4\,c^2}{a^2}(x^2 - a\,x)}$$

is imaginary; consequently y has no real value from A to B; but as soon as x is greater than a, $x^2 - a\,x$ becoming positive, the values of y become real; there commences, therefore, at B

a new portion of the curve $m\,B\,m'$, which, like the first, extends without limit on each side of AB produced. This is moreover perfectly equal to MAM', for, if we take $B\,p$ equal to AP, $x^2 - a\,x$, or $A\,p \times p\,B$, becomes equal to $AP \times PB$; consequently, $p\,m$ is also equal to PM.

138. If in the equation $y^2 = \frac{4\,a\,c + 4\,c^2}{a^2}\,(a\,x + x^2)$ we make $y = 0$, we shall find $a\,x + x^2$, or $x\,(a + x) = 0$, which gives $x = 0$, and $x + a = 0$, or $x = -a$; therefore the curve meets AB at the two points A and B. This line AB is called the *transverse axis.*

139. If we suppose $AP = AF$, that is, $x = c$, for the purpose of obtaining the value of the ordinate $F\,m''$, passing through the point F, (which is called the *focus*, as also the point f), we shall have

$$y = \pm \sqrt{\frac{4\,a\,c + 4\,c^2}{a^2}\,(a\,c + c^2)},$$

$$= \pm \sqrt{\frac{4\,(a\,c + c^2)^2}{a^2}} = \pm\,\frac{2\,(a\,c + c^2)}{a};$$

whence the double ordinate $m''\,m''' = \frac{4\,(a\,c + c^2)}{a}$. This line is called the *parameter* of the hyperbola; if we represent it by p, we shall have

$$p = \frac{4\,(a\,c + c^2)}{a},$$

and consequently,

$$\frac{p}{a} = \frac{4\,(a\,c + c^2)}{a^2},$$

Substituting $\frac{p}{a}$ for its equal, in the equation of the curve, we have

$$y^2 = \frac{p}{a}\,(a\,x + x^2).$$

From the value of p we may draw the conclusion, that *the parameter of the transverse axis of the hyperbola is more than quadruple the distance of the vertex* A *from the focus* F; for this value, $p = \frac{4\,a\,c + 4\,c^2}{a}$, reduces itself to $p = 4\,c + \frac{4\,c^2}{a}$, which is evidently greater than $4\,c$.

140. If upon the middle C of AB we raise a perpendicular DD', making the half CD a mean proportional between c and $a + c$, that is, between AF and fA, this perpendicular is called the *conjugate axis* of the hyperbola. Designating DD' by b, we shall have

$$c : \tfrac{1}{2} b : : \tfrac{1}{2} b : a + c,$$

whence $\tfrac{1}{4} b^2 = c\,(a + c)$, or $b^2 = 4\,a\,c + 4\,c^2$.

Putting b^2 for its equal in the equation

$$y^2 = \frac{4\,a\,c + 4\,c^2}{a^2}\,(a\,x + x^2),$$

the equation of the curve becomes

$$y^2 = \frac{b^2}{a^2}\,(a\,x + x^2).$$

The three equations above obtained, it will be seen, do not differ from the corresponding ones, respecting the ellipse, except in the sign of c^2 and x^2.

The equation $y^2 = \frac{b^2}{a^2}\,(a\,x + x^2)$ makes known also a property analogous to what we have remarked in the ellipse; indeed, if we make the denominator to disappear, we shall have

$$a^2\,y^2 = b^2\,(a\,x + x^2),$$

which gives this proportion

$$y^2 : a\,x + x^2 : : b^2 : a^2,$$

or $$\overline{PM}^2 : AP \times PB : : \overline{DD'}^2 : \overline{AB}^2,$$

$$: : \overline{CD}^2 : \overline{AC}^2;$$

therefore, *the square of an ordinate to the transverse axis of a hyperbola is to the rectangle of the corresponding abscissas, as the square of the conjugate axis is to the square of the transverse*, and consequently, *the squares of the ordinates are to each other, as the rectangles of the corresponding abscissas.*

When the two axes a, b, are equal, the equation becomes

$$y^2 = a\,x + x^2,$$

which does not differ from the equation of the circle, except in the sign of the square x^2. The curve in this case is called an *equilateral hyperbola.*

From the equation $p = \frac{4\,a\,c + 4\,c^2}{a}$, we deduce

$$a\,p = 4\,a\,c + 4\,c^2;$$

but $$b^2 = 4\,a\,c + 4\,c^2,$$

consequently $a\,p = b^2$,
which gives $a : b :: b : p$;
therefore *the parameter to the transverse axis is a third proportional to this axis and the conjugate.*

141. If from the point D to the point A we draw the straight line DA, the right-angled triangle DCA will give

$$DA = \sqrt{\overline{CD}^2 + \overline{AC}^2} = \sqrt{\tfrac{1}{4}\,b^2 + \tfrac{1}{4}\,a^2},$$

or, substituting for b^2 its value $4\,a\,c + 4\,c^2$,

$$DA = \sqrt{c^2 + a\,c + \tfrac{1}{4}\,a^2},$$
$$= c + \tfrac{1}{2}\,a = AF + CA = CF.$$

Therefore to find the foci, when the axes are given, we have only to apply DA from C to F and f; and reciprocally, to find the conjugate axis, when the transverse and the foci are given, we describe from the point A, as a centre, and with a radius equal to CF, an arc cutting the perpendicular DD' in the points D, D'.

142. We see, therefore, that the description of the hyperbola depends upon two quantities, namely, the transverse and conjugate axes, or the transverse axis and the foci, or the transverse axis and the parameter. After what has been said, the description is easily reduced to one of the methods just explained. If, for example, we have given the transverse axis and the parameter, taking a mean proportional between these two lines, we have the conjugate axis, by which we are enabled to find the foci.

143. If we take upon Mf (*fig.* 67) the part MG, equal to MF, **Fig. 67.**
and joining FG, draw through the point M perpendicularly to FG the line MOT, this line will be a tangent to the hyperbola, that is, it will meet the curve only in one point M.

From some other point N, taken in TM, let the two straight lines Nf, NF, be drawn to the foci, and join NG; it is evident by the construction, that NF, NG, are equal (*Geom.* 52); now Nf is less than $NG + Gf$, and consequently less than $NF + Gf$; therefore $Nf - NF$ is less than Gf, that is, less than $Mf - MF$; accordingly the point N is without the hyperbola. The same may be shown with respect to any other point in TM, except M.

The angles FMO, OMG, are equal by the preceding construction; but OMG is equal to its opposite NMQ, consequently FMO is equal to NMQ; therefore FM, drawn from the focus F, and MQ, ($f\,M$ produced,) drawn from the other focus, make equal angles with the tangent at the point M. Accordingly, if F were a luminous point, all the rays proceeding from this point and falling upon MAM', would, after being reflected, take the direction of rays coming from the point f.

144. Let us now determine the *subtangent* PT. Since the angle FMf is bisected by the tangent MT, we have

$$fM : MF :: fT : FT \quad (\textit{Geom.}\ 201);$$

now, MF being z, fM is equal to $z + a$; moreover

$$Ff, \text{ or } Bf + AB + AF = a + 2c,$$
$$fT, \text{ or } Ff - FT = a + 2c - FT;$$

consequently, substituting these values in the above proportion, we have

$$z + a : z :: a + 2c - FT : FT,$$

whence $\quad z \times FT + a \times FT = az + 2cz - z \times FT,$

or $\quad (2z + a)\, FT = az + 2cz,$

which gives

$$FT = \frac{az + 2cz}{2z + a}$$
$$= \frac{(2c + a)\, z}{(2z + a)};$$

but we have found $\quad z = \dfrac{2cx + ac + ax}{a} \quad (134),$

and accordingly $2z + a = \dfrac{4cx + 2ac + 2ax + a^2}{a}$

$$= \frac{(2c + a)\, 2x + (2c + a)\, a}{a}$$
$$= \frac{(2c + a)(2x + a)}{a};$$

substituting these values in the place of z and $2z + a$ in the above expression for FT, we shall have

$$FT = \frac{(2c + a)\,\dfrac{2cx + ac + ax}{a}}{(2c + a)\,\dfrac{2x + a}{a}},$$

or, suppressing the common factor, $\dfrac{2c + a}{a}$,

$$FT = \frac{2cx + ac + ax}{2x + a}.$$

Having thus found FT, it is easy to obtain the value of the subtangent PT; for

$$PT = FT - FP = FT - AF + AP = FT - c + x,$$

or, putting for FT its value,

$$PT = \frac{2\,c\,x + a\,c + a\,x}{2\,x + a} - c + x$$

$$= \frac{2\,c\,x + a\,c + a\,x - 2\,c\,x + 2\,x^2 - a\,c + a\,x}{2\,x + a}$$

$$= \frac{2\,a\,x + 2\,x^2}{2\,x + a}$$

$$= \frac{a\,x + x^2}{x + \frac{1}{2}\,a}.$$

We see, therefore, that the expression for the subtangent in the hyperbola does not differ, except in the signs, from that found for the subtangent in the ellipse (119).

145. If from PT we subtract AP, we shall have AT, or the distance from the vertex to the point where the tangent meets the axis; that is,

$$AT = \frac{a\,x + x^2}{\frac{1}{2}\,a + x} - x$$

$$\frac{\frac{1}{2}\,a\,x}{\frac{1}{2}\,a + x}.$$

146. From this expression of AT, we shall take occasion to make some remarks upon the curvature of the hyperbola. We have seen that each of the two branches AM, AM', is infinite. Still the curvature is such that all the tangents, that can be drawn to the several points of these infinite branches, meet the axis in the space comprehended between A and C. Indeed, if in the value of AT we substitute for x all imaginable quantities from 0 to infinite, the value of AT increases only from 0 to $\frac{1}{2}\,a$; for, when x is infinite, the denominator $\frac{1}{2}\,a + x$ must be regarded as the same as x; since, if $\frac{1}{2}\,a$ is to be retained, it would imply that x could be augmented, and consequently would destroy the supposition we have made, that x is infinite. Now in this case the quantity AT reduces itself to $\frac{\frac{1}{2}\,a\,x}{x}$, that is, to $\frac{1}{2}\,a$; therefore, the tangent at the extremity of each of the branches AM, AM', would pass through the centre C; and, since the opposite branches $B\,m$, $B\,m'$, are equal to AM, AM', and the points A and B are equally distant from C, it follows that these same tangents are also tangents at the extremities of the branches $B\,m$, $B\,m'$; thus CX, CY (*fig.* 66) produced, would represent the lines in question. Fig. 66.

147. These tangents are called the *asymtotes* of the hyperbola; they are, as we have seen, lines which, proceeding from the centre, approach continually the hyperbola without meeting it, except at an infinite distance.

Fig. 65. If through the vertex A (*fig.* 65) we draw the straight line $A\,t$ parallel to PM, the similar triangles $TA\,t$, TPM, give

$$TP : PM :: TA : A\,t,$$

that is,

$$\frac{a\,x + x^2}{\frac{1}{2}\,a + x} : y :: \frac{\frac{1}{2}\,a\,x}{\frac{1}{2}\,a + x} : A\,t\,;$$

whence

$$A\,t = \frac{\frac{1}{2}\,a\,x\,y}{\frac{1}{2}\,a + x} \times \frac{\frac{1}{2}\,a + x}{a\,x + x^2} = \frac{\frac{1}{2}\,a\,y}{a + x},$$

or, putting for y its value, $\frac{b}{a}\sqrt{a\,x + x^2}$,

$$A\,t = \frac{\frac{1}{2}\,b\sqrt{a\,x + x^2}}{a + x},$$

which, when x is infinite, becomes $\frac{\frac{1}{2}\,b\sqrt{x^2}}{x}$, that is, $\frac{1}{2}\,b$ or CD, since $a\,x$ must be suppressed against x^2 and a against x. We determine the asymtotes in this manner. We raise at the point

Fig. 66. A (*fig.* 66) a perpendicular AL, and produce it so as to make AL, AL', each equal to CD; then through the centre C and the points L, L', two straight lines CL, CL', being drawn, these will be the asymtotes.

Fig. 65. 148. In order to find the expression for CT (*fig.* 65), AT is to be subtracted from CA, which gives

$$CT = \tfrac{1}{2}\,a - \frac{\frac{1}{2}\,a\,x}{\frac{1}{2}\,a + x} = \frac{\frac{1}{4}\,a^2}{\frac{1}{2}\,a + x} = \frac{\overline{CA}^2}{CP} \quad (145),$$

from which we have this proportion

$$CP : CA :: CA : CT.$$

149. If we would have the expression for TM, the right-angled triangle TPM gives

$$\overline{TM}^2 = \overline{PM}^2 + \overline{PT}^2,$$

or, taking the algebraic values of the terms of the second member, (140, 141), we have

$$\overline{TM}^2 = \frac{b^2}{a^2}(a\,x + x^2) + \frac{(a\,x + x^2)^2}{(\frac{1}{2}\,a + x)^2}$$

$$= \left(\frac{b^2}{a^2}(\tfrac{1}{2}\,a + x)^2 + a\,x + x^2\right)\frac{(a\,x + x^2)}{(\frac{1}{2}\,a + x)^2}.$$

150. To obtain the expression for PI, or the *subnormal*, the triangles TPM, MPI, similar on account of the perpendicular PM being let fall from the right angle TMI, give

$$TP : PM :: PM : PI,$$

or

$$\frac{a\,x + x^2}{\frac{1}{2}\,a + x} : y :: y : PI,$$

whence

$$PI = \frac{y^2\,(\frac{1}{2}\,a + x)}{a\,x + x^2},$$

or, since

$$y^2 = \frac{b^2}{a^2}\,(a\,x + x^2),$$

$$PI = \frac{b^2}{a^2}\,(\tfrac{1}{2}\,a + x).$$

151. We proceed now to find the equation with respect to the conjugate axis DD'. In order to this we let fall upon the conjugate axis the perpendicular MP', and designating MP' by y', and DP' by x', we shall have

$$CP' = PM = y \qquad = \tfrac{1}{2}\,b - x',$$

$$P'M = CP = \tfrac{1}{2}\,a + x = y', \text{ and hence } x = y' - \tfrac{1}{2}\,a.$$

Substituting, therefore, for x and y, these values in the equation

$$y^2 = \frac{b^2}{a^2}\,(a\,x + x^2), \quad \text{or} \quad a^2\,y^2 = b^2\,(a\,x + x^2),$$

we shall have

$$a^2\,(\tfrac{1}{2}\,b - x')^2 = b^2\,(a(y' - \tfrac{1}{2}\,a) + (y' - \tfrac{1}{2}\,a)^2)$$

$$a^2\,(\tfrac{1}{4}\,b^2 - b\,x' + x'^2) = b^2\,(a\,y' - \tfrac{1}{2}\,a^2 + y'^2 - a\,y' + \tfrac{1}{4}\,a^2)$$

$$= b^2\,(-\tfrac{1}{4}\,a^2 + y'^2)$$

$$y'^2 = \frac{a^2}{b^2}\,(\tfrac{1}{2}\,b^2 - b\,x' + x'^2).$$

We see, therefore, that it is not with the hyperbola as with the ellipse, since the equation with regard to the conjugate axis is not similar to that respecting the transverse.

152. If we would have the equation with respect to the axis AB by counting the abscissas from the centre C; designating CP by z, we shall have

$$z = CA + AP = \tfrac{1}{2}\,a + x,$$

and $$x = z - \tfrac{1}{2}a.$$

Substituting this value for x in the equation

$$y^2 = \frac{b^2}{a^2}(a\,x + x^2),$$

we shall have

$$y^2 = \frac{b^2}{a^2}(z^2 - \tfrac{1}{4}a^2),$$

for the equation, with respect to the transverse axis, the abscissas being counted from the centre.

With regard to the conjugate axis DD', if we designate CP', by z', we shall have

$$z' = CD - DP' = \tfrac{1}{2}b - x',$$

and hence

$$x' = \tfrac{1}{2}b - z'.$$

Substituting this value for x' in the equation

$$y'^2 = \frac{a^2}{b^2}(\tfrac{1}{2}b^2 - b\,x' + x'^2),$$

already found for the conjugate axis (151), we shall have

$$y'^2 = \frac{b^2}{a^2}\left(\tfrac{1}{2}b^2 - b\,(\tfrac{1}{2}b - z') + (\tfrac{1}{2}b - z')^2\right),$$

$$= \frac{a^2}{b^2}(z'^2 + \tfrac{1}{4}b^2).$$

153. If we would refer to the centre, the expressions for PT, CT, PI, and TM, above found (144, 148, 150, 149), we have only to substitute, as in the first part of the last article, $z - \frac{1}{2}a$ for x; thus,

$$PT = \frac{a\,x + x^2}{x + \frac{1}{2}a} = \frac{a\,(z - \frac{1}{2}a) + (z - \frac{1}{2}a)^2}{z - \frac{1}{2}a + \frac{1}{2}a} = \frac{z^2 - \frac{1}{4}a^2}{z};$$

$$CT = \frac{\frac{1}{4}a^2}{\frac{1}{2}a + x} = \frac{\frac{1}{4}a^2}{\frac{1}{2}a + z - \frac{1}{2}a} = \frac{\frac{1}{4}a^2}{z};$$

$$PI = \frac{b^2}{a^2}(\tfrac{1}{2}a + x) = \frac{b^2}{a^2}.(\tfrac{1}{2}a + z - \tfrac{1}{2}a) = \frac{b^2\,z}{a^2};$$

$$\overline{TM}^2 = \left(\frac{b^2}{a^2}(\tfrac{1}{2}a + x)^2 + a\,x + x^2\right)\frac{a\,x + x^2}{(\frac{1}{2}a + x)^2};$$

$$= \left(\frac{b^2\,z^2}{a^2} + z^2 - \tfrac{1}{4}a^2\right)\frac{z^2 - \frac{1}{4}a^2}{z^2}.$$

If MT be produced till it meets the conjugate axis in T', the similar triangles TPM, TCT', give

$$TP : PM :: CT : CT',$$

or

$$\frac{z^2 - \frac{1}{4}a^2}{z} : y :: \frac{\frac{1}{4}a^2}{z} : CT';$$

whence

$$CT' = \frac{\frac{1}{4}a^2 y}{z^2 - \frac{1}{4}a^2},$$

but, from the equation $y^2 = \frac{b^2}{a^2}(z^2 - \frac{1}{4}a^2)$ (152), we have

$$z^2 - \frac{1}{4}a^2 = \frac{a^2 y^2}{b^2},$$

consequently,

$$CT' = \frac{\frac{1}{4}a\ y}{\frac{a^2 y^2}{b^2}} = \frac{\frac{1}{4}b^2}{y} = \frac{\overline{CD}^2}{PM} = \frac{\overline{CD}^2}{CP'},$$

therefore,

$$CP' : CD :: CD : CT'.$$

154. If, through the centre C of the hyperbola (*fig.* 68), we draw any straight line MCM' terminated in each direction by the hyperbola, this line is called a *diameter*. Any straight line $m\,O$, drawn from a point m of the curve parallel to the tangent at M and terminated by the diameter MM' produced, is called an *ordinate* to this diameter. MO, OM' are the corresponding *abscissas*. We proceed to show, that the properties of the ordinates $m\,O$, with respect to the diameters terminated by the curve, are the same as those of the ordinates MP to the transverse axis. Fig. 68.

We let fall from the points m, O, the perpendiculars $m\,p$, OQ, upon the transverse axis AB; and from the point m we draw $m\,S$ parallel to AP. Designating PM by y, CP by z, $Q\,p$ by g, and CQ by k, we have

$$AP = CP - CA = z - \tfrac{1}{2}a,$$
$$BP = CP + BC = z + \tfrac{1}{2}a,$$
$$A\,p = C\,p - CA = CQ - Q\,p - CA = k - g - \tfrac{1}{2}a,$$
$$B\,p = Cp + BC = k - g + \tfrac{1}{2}a.$$

The similar triangles CPM, CQO, give

$$CP : PM :: CQ : QO,$$

that is,

$$z : y :: k : QO;$$

whence

$$QO = \frac{k\,y}{z}.$$

The similar triangles TPM, $m\,SO$, give

$$PT : PM :: m\,S \text{ or } Q\,p : SO,$$

that is, (153),

$$\frac{z^2 - \frac{1}{4}a^2}{z} : y :: g : SO;$$

whence

$$SO = \frac{g\,y\,z}{z^2 - \frac{1}{4}a^2};$$

therefore

$$m\,p = SQ = QO - SO = \frac{k\,y}{z} - \frac{g\,y\,z}{z^2 - \frac{1}{4}a^2}.$$

Now, since the point m belongs to the hyperbola,

$$\overline{p\,m}^2 : \overline{PM}^2 :: A\,p \times p\,B : AP \times PB \text{ (140)},$$

that is,

$$\left(\frac{ky}{z} - \frac{g\,y\,z}{z^2 - \frac{1}{4}a^2}\right)^2 : y^2 :: (k - g - \tfrac{1}{2}a)(k - g + \tfrac{1}{2}a) : (z - \tfrac{1}{2}a)(z + \tfrac{1}{2}a),$$

or $$\frac{k^2\,y^2}{z^2} - \frac{2\,g\,k\,y^2\,z}{z\,(z^2 - \frac{1}{4}a^2)} + \frac{g^2\,y^2\,z^2}{(z^2 - \frac{1}{4}a^2)^2} : y^2$$
$$:: k^2 - 2\,g\,k + g^2 - \tfrac{1}{4}a^2 : z^2 - \tfrac{1}{4}a^2;$$

taking the product of the extremes and means, and observing what quantities are at the same time multiplied and divided by $z^2 - \frac{1}{4}a^2$, and what by z, we shall have

$$\frac{k^2\,y^2}{z^2}(z^2 - \tfrac{1}{4}a^2) - 2\,g\,k\,y^2 + \frac{g^2\,y\;\;z^2}{z^2 - \frac{1}{4}a^2}$$
$$= k^2\,y^2 - 2\,g\,k\,y^2 + g^2\,y^2, - \tfrac{1}{4}a^2\,y^2,$$

or, developing the term $\frac{k^2\,y^2}{z^2}$ $(z^2 - \frac{1}{4}a^2)$, suppressing $k^2\,y^2$, and $-2\,g\,k\,y^2$, common to each member, and dividing by y^2, we obtain

$$-\frac{\frac{1}{4}a^2\,k^2}{z^2} + \frac{g^2\,z^2}{z^2 - \frac{1}{4}a^2} = g^2 - \tfrac{1}{4}a^2,$$

an equation which will serve to demonstrate the property in question. But, before making use of it, we shall observe that, if on each side of the centre C, we take upon the axis AB the part CR a mean proportional between BP and AP, that is, such as will give

$$\overline{CR}^2 = AP \times PB = z^2 - \tfrac{1}{4}a^2;$$

and if, having raised the perpendicular RN', terminated in N' by the line NN', passing through the centre C parallel to TM, we make $CN = CN'$, the line NN', is called a diameter, and the *conjugate* of MM'; also the third proportional to MM' and NN' is called the *parameter* to the diameter MM'.

Returning to our inquiry, we designate CM, CN, &c., as follows, namely,

$$CM = \tfrac{1}{2}a',\ CN \text{ or } CN' = \tfrac{1}{2}b',\ CO = z',\ O\,m = y'.$$

The similar triangles CPM, CQO, give

$$CM : CP :: CO : CQ,$$

that is, $\tfrac{1}{2}a' : z :: z' : k$;

whence

$$k = \frac{z\,z'}{\frac{1}{2}a'}, \quad \text{and} \quad k^2 = \frac{z^2\,z'^2}{\frac{1}{4}a'^2}.$$

The triangles $m\,SO$, $CN'R$, similar on account of the sides being parallel, give

$$CN' : CR :: m\,O : m\,S,$$

that is, $\tfrac{1}{2}b' : CR :: y' : g$;

whence $g = \dfrac{CR \times y'}{\frac{1}{2}b'}$ and $g^2 = \dfrac{\overline{CR}^2 \times y'^2}{\frac{1}{4}b'^2}$,

or, since $\overline{CR}^2 = z^2 - \tfrac{1}{4}a^2$, by construction,

$$g^2 = \frac{y'^2\,(z^2 - \frac{1}{4}a^2)}{\frac{1}{4}b'^2}.$$

Resuming now the equation

$$-\frac{\frac{1}{4}a^2\,k^2}{z^2} + \frac{g^2\,z^2}{z^2 - \frac{1}{4}a^2} = g^2 - \tfrac{1}{4}a^2,$$

and, substituting for k^2 and g^2 the values above found, we shall have

$$-\frac{\frac{1}{4}a^2\,z^2\,z'^2}{\frac{1}{4}a'^2\,z^2} + \frac{y'^2\,z^2(z^2 - \frac{1}{4}a^2)}{\frac{1}{4}b'^2(z^2 - \frac{1}{4}a^2)} = \frac{y'^2\,(z^2 - \frac{1}{4}a^2)}{\frac{1}{4}b'^2} - \tfrac{1}{4}a^2,$$

or, reducing and dividing by $\tfrac{1}{4}a^2$,

$$-\frac{z'^2}{\frac{1}{4}a'^2} = -\frac{y'^2}{\frac{1}{4}b'^2} - 1;$$

or, making the denominators to disappear,

$$-\tfrac{1}{4}b'^2\,z'^2 = -\tfrac{1}{4}a'^2\,y'^2 - \tfrac{1}{16}a'^2\,b'^2,$$

from which we obtain

$$y'^2 = \frac{b'^2}{a'^2}\,(z'^2 - \tfrac{1}{4}a'^2),$$

an equation similar to that respecting the transverse axis.

155. If we make $y' = 0$, we find $z'^2 - \tfrac{1}{4}a'^2 = 0$, which gives $z' = \pm\tfrac{1}{2}a'$; the curve, therefore, meets the line MM' in two opposite points M and M', distant from the centre each $\tfrac{1}{2}a'$, or CM; thus all the diameters bisect each other at the centre.

156. Since the equation $y'^2 = \frac{b'^2}{a'^2}(z'^2 - \frac{1}{4}a'^2)$ gives

$$y' = \pm \frac{b'}{a'}\sqrt{z'^2 - \frac{1}{4}a'^2},$$

that is, two equal values having contrary signs, it is evident that if $m\,O$ be produced, so as to make $O\,m' = O\,m$, the point m' will belong to the curve; therefore, *each diameter*, MM', *produced, bisects the lines drawn parallel to the tangent that passes through the origin*, M.

157. From the same equation, $y'^2 = \frac{b'^2}{a'^2}(z'^2 - \frac{1}{4}a'^2)$, we obtain

$$a'^2\,y'^2 = b'^2\,(z'^2 - \frac{1}{4}a'^2),$$

and thence the proportion

$$y'^2 : z'^2 - \frac{1}{4}a'^2 :: b'^2 : a'^2,$$

or $$\overline{m\,O}^2 : MO \times OM' :: \overline{NN'}^2 : \overline{MM'}^2;$$

that is, *the square of any ordinate* m O, *to a diameter terminated by the curve, is to the rectangle* MO × OM' *of the corresponding abscissas, as the square of the conjugate diameter is to the square of the first diameter.*

158. If from the centre C we let fall upon TM the perpendicular CF, the similar triangles CFT, TPM, give

$$TM : PM :: CT : CF;$$

whence

$$CF = \frac{PM \times CT}{TM}.$$

The similar triangles CRN', TPM, give

$$PT : TM :: CR : CN' \text{ or } CN;$$

whence

$$CN = \frac{TM \times CR}{PT}.$$

Multiplying the above equations, member by member, we have

$$CF \times CN = \frac{PM \times CT \times TM \times CR}{TM \times PT},$$

$$= \frac{PM \times CT \times CR}{PT};$$

or, taking the squares of both members,

$$\overline{CF}^2 \times \overline{CN}^2 = \frac{\overline{PM}^2 \times \overline{CT}^2 \times \overline{CR}^2}{\overline{PT}^2}.$$

But $\overline{PM}^2 = y^2 = \frac{b^2}{a^2}(z^2 - \frac{1}{4}a^2)$ (152),

$$\overline{PT}^2 = \frac{(z^2 - \frac{1}{4}a^2)^2}{z^2} \quad (153),$$

$$\overline{CR}^2 = z^2 - \frac{1}{4}a^2 \quad (154),$$

$$\overline{CT}^2 = \frac{\frac{1}{16}a^4}{z^2} \quad (153);$$

substituting these values for the quantities which they represent in the above equation, we shall have

$$\overline{CF}^2 \times \overline{CN}^2 = \frac{\frac{b^2}{a^2}(z^2 - \frac{1}{4}a^2) \times \frac{\frac{1}{16}a^4}{z^2} \times (z^2 - \frac{1}{4}a^2)}{\frac{(z^2 - \frac{1}{4}a^2)^2}{z^2}},$$

$$= \tfrac{1}{16}a^2 b^2,$$

or $CF \times CN = \frac{1}{4}ab$.

Now, if we produce MT (*fig.* 68) to the asymptote in I, MI will be equal to CN, as we shall see below, and $CIMN$ will be a parallelogram, the surface of which will be Fig. 68.

$$CF \times MI = CF \times CN;$$

therefore, wherever the point M shall fall, the parallelogram $CIMN$ will always be equal in surface to the rectangle of the two semiaxes, that is, to $\frac{1}{2}a \times \frac{1}{2}b$, or $\frac{1}{4}ab$.

159. The similar triangles TPM, CRN' (*fig.* 68), give Fig. 68.

$$TP : PM :: CR : RN';$$

whence

$$RN' = \frac{PM \times CR}{TP},$$

and

$$\overline{RN'}^2 = \frac{\overline{PM}^2 \times \overline{CR}^2}{\overline{TP}^2},$$

or, the algebraic values of the terms of the second member being substituted,

$$\overline{RN'}^2 = \frac{\frac{b^2}{a^2}(z^2 - \frac{1}{4}a^2)(z^2 - \frac{1}{4}a^2)}{\frac{(z^2 - \frac{1}{4}a^2)}{z^2}}$$

$$= \frac{b^2 z^2}{a^2}.$$

But the right-angled triangles CPM, CRN', give

$$CM = \overline{CP}^2 + \overline{PM}^2, \text{ and } \overline{CN'}^2 \text{ or } \overline{CN}^2 = \overline{CR}^2 + \overline{RN'}^2,$$

consequently, subtracting the second equation from the first, we have

$$\overline{CM}^2 - \overline{CN}^2 = \overline{CP}^2 + \overline{PM}^2 - \overline{CR}^2 - \overline{RN'}^2,$$

or, substituting the algebraic values,

$$\overline{CM}^2 - \overline{CN}^2 = z^2 + \frac{b^2}{a^2}(z^2 - \tfrac{1}{4}a^2) - (z^2 - \tfrac{1}{4}a^2) - \frac{b^2 z^2}{a^2}$$
$$= \tfrac{1}{4}a^2 - \tfrac{1}{4}b^2;$$

that is, *the difference of the squares of two semiconjugate diameters is always the same, and equal to the difference of the squares of the semiaxes.*

It follows from this, that in the equilateral hyperbola each diameter is equal to its conjugate; for, if $a = b$, we shall have $\overline{CM}^2 - \overline{CN}^2 = 0$, and consequently $CM = CN$.

160. If, in the equation $\overline{CN}^2 = \overline{CR}^2 + \overline{RN'}^2$, we substitute for CR and RN' their algebraic values, we shall have

$$\overline{CN}^2 = z^2 - \tfrac{1}{4}a^2 + \frac{b^2 z^2}{a^2};$$

but, according to what has already been shown (153),

$$\overline{TM}^2 = \left(\frac{b^2 z^2}{a^2} + z^2 - \tfrac{1}{4}a^2\right)\frac{z^2 - \tfrac{1}{4}a^2}{z^2}.$$

therefore

$$\overline{TM}^2 = \overline{CN}^2 \times \frac{z^2 - \tfrac{1}{4}a^2}{z^2}.$$

Now the similar triangles MPT, $MP'T'$, give, by taking the squares of the homologous sides,

$$\overline{PT}^2 : \overline{TM}^2 :: \overline{P'M}^2 : \overline{T'M}^2,$$

or

$$\frac{(z^2 - \tfrac{1}{4}a^2)^2}{z^2} : \frac{\overline{CN}^2 \times (z^2 - \tfrac{1}{4}a^2)}{z^2} :: z^2 : \overline{T'M}^2,$$

whence

$$\overline{T'M}^2 = \frac{\overline{CN}^2 \times z^2}{z^2 - \tfrac{1}{4}a^2};$$

Multiplying therefore the two last equations, member by member, we shall have

$$\overline{TM}^2 \times \overline{T'M}^2 = \overline{CN}^4,$$

or

$$TM \times T'M = \overline{CN}^2.$$

If now we designate by p' the parameter to the diameter MM' we shall have

$$2\,CM : 2\,CN :: 2\,CN : p';$$

whence $\qquad 2\,p' \times CM = 4\,\overline{CN}^2,$

or $\qquad \frac{1}{2}\,p' \times CM = \overline{CN}^2,$

therefore, $\qquad TM \times T'M = \frac{1}{2}\,p' \times CM,$

which gives the proportion

$$CM : TM :: T'M : \tfrac{1}{2}\,p'.$$

161. From what is here said we may readily find the axes of the hyperbola, and thence derive a simple method of describing the curve, when we have given only two conjugate diameters and the angle contained by them. We take upon MC (*fig.* 69), Fig. 69. a line MH equal to $\frac{1}{2}\,p'$, and from the middle I of CH we raise a perpendicular IK cutting in some point K the line MT' drawn through the point M parallel to the conjugate NN'. From this point K, as a centre, and with a radius equal to the distance from K to C, we describe a circle meeting MT' in two points T, T', through which and the centre TC, CT' being drawn, these will be the directions of the two axes; for it is evident, 1. that the angle TCT' will be a right angle, since the circumference passes through the point C, and has TT' for its diameter; 2. by the nature of the circle, we have

$$CM : TM :: T'M : MH,$$

therefore, since by construction,

$$MH = \tfrac{1}{2}\,p',$$

we have

$$CM : TM :: T'M : \tfrac{1}{2}\,p'.$$

Having thus found the direction of the two axes, we determine their magnitude by letting fall from the point M the perpendiculars MP, MP', and taking CA a mean proportional between CP and CT, and CD a mean proportional between CP' and CT', agreeably to what has already been made known (148, 153).

When the two conjugate diameters are equal, the parameter is of the same magnitude; accordingly $MH = MC$, the two points of section H and C coinciding, and MC is a tangent to the circle; so that to find the centre K, it is only necessary to raise upon CM a perpendicular at the point C.

Of the Hyperbola with reference to its Asymptotes.

162. The hyperbola, considered with reference to its asymptotes, has certain properties, a knowledge of which may be

useful. We proceed, therefore, to consider them. It is necessary here to recollect how the asymptotes are determined. See art. 147.

Fig. 70. We now refer each point E (*fig.* 70) of the hyperbola to the two asymptotes CLO, $CL'\,o$, by drawing the line EQ parallel to one of them, and we seek the relation subsisting between the lines EQ and CQ.

In order to find this relation, we draw through any point E, the line $OE\,o$ parallel to the conjugate axis DD', and the line ES parallel to CLO; through the vertex A we draw AG parallel to $CL'\,o$, and we designate CA, CD, &c., as follows, namely,

$$CA = \tfrac{1}{2}a, \quad CD \text{ or } AL \text{ or } AL' = \tfrac{1}{2}b, \quad CP = z,$$
$$PE = y, \ AG = m, \quad GL = n, \ CQ = t, \quad QE = u,$$

The similar triangles CPO, CAL, give

$$CA : AL :: CP : PO,$$

or $\quad \tfrac{1}{2}a : {}^{1}b$, or $a : \ b \ :: \ z \ : PO$;

whence we have

$$PO = Po = \frac{bz}{a},$$

also

$$EO = \frac{bz}{a} - y, \text{ and } E\,o = \frac{bz}{a} + y,$$

and consequently

$$EO \times E\,o = \frac{b^2 z^2}{a^2} - y^2,$$

or, putting for y^2 its value $\frac{b^2}{a^2}(z^2 - \tfrac{1}{4}a^2)$ and reducing,

$$EO \times E\,o = \tfrac{1}{4}b^2,$$

that is,

$$EO \times E\,o = \overline{CD}^2 = \overline{AL}^2,$$

a property which belongs to every point of the hyperbola, since the point E has been taken at pleasure in any part of the curve.

163. The similar triangles QEO, $ES\,o$, and AGL, give

$$AL : AG :: EO : EQ,$$
$$AL : GL :: E\,o : ES;$$

whence, multiplying the proportions in order, we have

$$\overline{AL}^2 : AG \times GL :: EO \times E\,o : EQ \times ES,$$

that is,

$$\tfrac{1}{4}b^2 : mn :: \tfrac{1}{4}b^2 : ut;$$

whence $\quad ut = mn,$

the equation of the hyperbola with reference to its asymptotes. Thus at any point E of the hyperbola, we have always $EQ \times ES$ or rather $EQ \times CQ = AG \times GL$.

If now we suppose the point E to fall upon A, we shall have

$$CQ = CG, \text{ and } EQ = AG;$$

and consequently,

$$CG \times AG = AG \times GL,$$

from which we deduce

$$CG = GL;$$

and, since the point G thus becomes the middle of CL, we shall have

$$CG = AG = GL;$$

for the circle, described upon CL, as a diameter, and which would consequently have CG for its radius, must pass through the point A, since the angle at A is a right angle, we have therefore

$$= n,$$

and hence $$u\,t = m^2 = \overline{CG}^2.$$

This constant square m^2 or $\overline{CG}^2$, to which the product $u\,t$ or $CQ \times QE$ is always equal, is called the *power* of the hyperbola.

164. By means of the property just demonstrated, we can deduce this other; namely, *If from any point* E *of the hyperbola, we draw any straight line* RE r *terminated by the asymptotes, the parts* RE, m r, *intercepted between the curve and the asymptotes, will be equal.* For, if through the point m we draw $h\,m\,H$, parallel to $OE\,o$, the similar triangles REO, $R\,m\,H$, give

$$ER : R\,m :: EO : H\,m;$$

and the similar triangles $r\,h\,m$, $r\,o\,E$, give

$$E\,r : m\,r :: E\,o : m\,h;$$

multiplying these two proportions in order, we have

$$ER \times E\,r : R\,m \times m\,r :: EO \times E\,o : H\,m \times m\,h.$$

Now the two products $EO \times E\,o$, $H\,m \times m\,h$, are equal each to $\overline{CD}^2$ (162); consequently,

$$ER \times E\,r = R\,m \times m\,r,$$

or

$$ER \times (E\,m + m\,r) = (ER + E\,m) \times m\,r,$$

the multiplications indicated being performed, and $ER \times m\,r$ being suppressed in each member, we have

$$ER \times Em = Em \times mr,$$

therefore $$ER = mr.$$

165. From what has been proved, we infer, that every tangent Tt to the hyperbola, terminated by the asymptotes, is bisected at the point of contact M.

166. If through the point M, we draw IMi parallel to DD', and if through any point E we draw REr parallel to the tangent Tt, the similar triangles TMI, REO, give

$$TM : MI :: RE : EO;$$

and the similar triangles Mit, Eor, give

$$Mt \text{ or } TM : MI :: Er : Eo;$$

Multiplying these two proportions in order, we have

$$\overline{TM}^2 : MI \times Mi :: RE \times Er : EO \times Eo.$$

Now the two products $MI \times Mi$, $EO \times Eo$, are each equal to $\overline{CD}^2$; therefore,

$$\overline{TM}^2 = RE \times Er.$$

167. If from the centre C we draw the diameter CMV, it will bisect the line Rr parallel to Tt, since it passes through the middle M of Tt (165, *and Geom.* 211); accordingly, designating CM, TM, &c., as follows, namely,

$CM = \frac{1}{2} a'$, $TM = \frac{1}{2} q$, $CV = z'$, and the ordinate $VE = y'$;
we shall have, from the similar triangles CMT, CVR,

$$CM : TM :: CV : VR,$$

that is,

$$\tfrac{1}{2} a' : \tfrac{1}{2} q, \text{ or } a' : q :: z' : VR;$$

whence

$$VR = Vr = \frac{qz'}{a'};$$

consequently

$$RE = \frac{qz'}{a'} - y', \text{and } Er = \frac{qz'}{a'} + y';$$

and, since $RE \times Er = \overline{TM}^2 = \frac{1}{4} q^2$,
we shall have

$$\frac{q^2 z'^2}{a'^2} - y'^2 = \tfrac{1}{4} q^2;$$

but

$$y'^2 = \frac{b'^2}{a'^2}(z'^2 - {}^{1} a'^2) \; (154);$$

therefore, by substituting this value for y'^2, we have

$$\frac{q^2 z'^2}{a'^2} - \frac{b'^2 z'^2}{a'^2} + \tfrac{1}{4} b'^2 = \tfrac{1}{4} q^2,$$

or

$$(q^2 - b'^2) \frac{z'^2}{a'^2} = \tfrac{1}{4} (q^2 - b'^2);$$

and

$$(q^2 - b'^2) \frac{z'^2}{a'^2} - \tfrac{1}{4} (q^2 - b'^2) = 0,$$

or

$$(q^2 - b'^2 \left(\frac{z'^2}{a'^2} - \tfrac{1}{4}\right) = 0;$$

and, dividing by $\frac{z'^2}{a'^2} - \frac{1}{4}$, we obtain

$$q^2 - b'^2 = 0,$$

which gives

$$q = b', \text{ or } \tfrac{1}{2} q = \tfrac{1}{2} b';$$

that is, MT is equal to CN, CN being the semiconjugate of CM, which we promised some time since to show (158). We have therefore (*fig.* 68) $MI = CN$. Fig. 68.

168. We have moreover for every straight line $RE\,r$, parallel to the conjugate CN (*fig.* 70), $RE \times E\,r = \overline{CN}^2$. Fig. 70.

169. We see, therefore, that knowing two semiconjugate diameters CM, CN (*fig.* 71), and the angle contained by them, it is easy to describe the hyperbola by points. Indeed it is evident, from what has been said, art. 165, 167, that, drawing from the origin M of the semidiameter CM, the line $TM\,t$ parallel to CN, and taking in each direction from the point M the parts MT, $M\,t$, equal each to CN, if, through the centre C we draw the lines CT, $C\,t$, these will be the asymptotes. And, from what has been demonstrated (164), it will be seen that, if through the point M, we draw as many lines PMQ, PMQ, as we please, making in each $PO = MQ$, the points O thus found will belong to the hyperbola sought. We can then, by means of the points O, find other points, as V, V, &c., by drawing the straight lines ROS, ROS, &c., and making $SV = RO$. Fig. 71.

170. We hence see also how, between two lines given for asymptotes, we can describe an hyperbola which shall pass through a given point between these lines.

171. Lastly, by bisecting the angle of the asymptotes and its supplement, we shall have the directions of the two axes, the

magnitude of which may be determined in the manner already explained (161), which furnishes another method of resolving the question considered in the article here referred to.

Of the Parabola.

172. We now propose to find the properties of the curve, in
Fig. 72. which each point is equally distant from a fixed point F (*fig.* 72), and a straight line XZ, the position of which is known; that is, of a curve, from each point of which letting fall the perpendicular MH, we have throughout $MF = MH$.

From the point F we draw FV perpendicular to XZ, and bisect FV in A; A will be a point in the curve, since $AV = AF$; this point is the *vertex*.

In order to investigate the properties of this curve, which is called a *parabola*, we proceed to find an equation, which shall express the relation between the perpendiculars MP, let fall upon FV, and their distances AP from the point A. We designate the lines to be used, as follows, namely,

$$AV \text{ or } AF = c, \quad AP = x, \quad PM = y;$$

and we shall accordingly have

$$VP = AV + AP = c + x = MH;$$

also, since $MF = MH$,

$$MF = c + x;$$

moreover

$$FP = AP - AF = x - c.$$

Now the triangle FPM gives

$$\overline{FP}^2 + \overline{PM}^2 = \overline{FM}^2,$$

that is, the algebraic values being substituted,

$$x^2 - 2\,c\,x + c^2 + y^2 = c^2 + 2\,c\,x + x^2,$$

or, transposing and reducing,

$$y^2 = 4\,c\,x;$$

the equation of the curve, from which we learn its properties.

1. This equation gives $y = \pm \sqrt{4\,c\,x}$; whence, for the same value of x or AP, we have two equal values of y or PM; but, as one is positive and the other negative, they fall upon opposite sides of the indefinite line API, called the *axis*, that is, they are PM and PM'; the curve, therefore, has two branches AM, AM', perfectly equal, and of unlimited extent, since it is evident, that the more we increase x, the more we increase $\sqrt{4\,c\,x}$ or y.

2. If we make x negative, we shall have

$$y = \pm \sqrt{-4\,c\,x},$$

that is, imaginary; the curve, therefore, does not extend above the point A.

3. If we make $x = c$, for the purpose of obtaining the ordinate passing through the point F, which is called the *focus*, we shall have

$$y = \pm \sqrt{4\,c^2} = \pm 2\,c,$$

that is, $F\,m''$ is equal to twice AF, and $m''\,m'''$ equal to four times AF. This line which passes through the focus is called the *parameter* of the axis. Thus, *in the parabola, the parameter of the axis is quadruple the distance of the vertex from the focus.*

4. Accordingly, if we designate this parameter by p, we shall have $4\,c = p$, and the equation of the parabola will become

$$y^2 = p\,x.$$

173. Having the equation of a parabola, we can easily describe the curve by points; we have only to give to x different values, successively, and to calculate the corresponding values of y.

174. We can moreover describe the curve by points, in this manner; having selected the point A, to be taken as the vertex, and the indefinite TVI, as the direction of the axis, we take the parts AV, AF, equal each to $\frac{1}{4}\,p$, the point F will be the focus; we then raise upon each point of the axis the indefinite perpendiculars MM', and describe from the point F, as a centre, and with the distance VP, as a radius, two small arcs cutting each perpendicular in two points M and M'; these points will form the parabola; since FM, thus made equal to VP, will be equal to MH, the straight line VH being supposed perpendicular to the axis. This straight line XVH is called the *directrix.*

175. Lastly, we can describe the parabola by a continued motion with a *square* VHf. We attach to some point f of one of the branches of this square the extremity of a thread equal in length to $f\,H$; and, having fixed the other extremity to the point F, we apply, by means of a style M, a part of the thread against $f\,H$, and keeping the thread always stretched, we slide the other side of the square along ZX; the style M will thus trace the parabola MA.

176. From the equation $y^2 = p\,x$, we learn that, for each point M, *the square of the ordinate* MP *is equal to the rectangle of the corresponding abscissa and the parameter.*

It is evident also from the same equation, that *the squares of the ordinates are to each other as the abscissas*, that is,

$$\overline{PM}^2 : \overline{pm}^2 :: AP : Ap;$$

for $\overline{PM}^2 = p \times AP,$

and $\overline{pm}^2 = p \times Ap,$

therefore $\overline{PM}^2 : \overline{pm}^2 :: p \times AP : p \times Ap,$
$$:: AP : Ap.$$

If, in the equation of the ellipse

$$y^2 = \frac{4ac - 4c^2}{a^2}(ax - x^2) \quad (104),$$

we suppose the transverse axis a to be infinite, x^2 may be suppressed, as too small to affect ax, the same may be said of $4c^2$ with respect to ac; the equation then reduces itself to

$$y^2 = \frac{4ac \times ax}{a^2} = \frac{4a^2cx}{a^2} = 4cx,$$

which is the equation of the parabola; therefore, *the parabola is simply an ellipse of which the transverse axis is infinite.*

Fig. 72. 177. If, having joined the points F, H (*fig.* 72), we draw to the line FH from the point M the perpendicular MOT, this line will be a tangent to the parabola, that is, it will meet it in only one point M. From any other point N of this line, NF, NH, being drawn, and the line NZ perpendicular to XZ, if the point N belonged to the curve, we should have $NF = NZ$; but NZ is less than NH, which, by construction is equal to NF.

178. The angle FMO being, by construction, equal to OMH, which is equal to its opposite fMN, it follows that FMO is equal to fMN; rays of light, therefore, proceeding from the point F, and falling upon the curve $M'AM$, would be reflected in lines parallel to the axis; and reciprocally, rays of light coming in lines parallel to the axis would, upon being reflected, be concentrated in the focus F.

179. The line MH being parallel to VP, the triangles HOM, TOF, are similar; they are moreover equal, since HO is equal to OF; accordingly

$$FT = MH = PV = x + c;$$

consequently

$$PT = FT + FP = x + c + x - c = 2x;$$

therefore, *the subtangent* PT *of the parabola is double of the abscissa* AP.

180. If at the point M we draw to the tangent TM, the perpendicular MI, the similar triangles TPM, PMI, give

$$TP : PM :: PM : PI,$$

that is,

$$2x : y :: y : PI;$$

whence

$$PI = \frac{y^2}{2x},$$

or, since $y^2 = px$,

$$PI = \frac{px}{2x} = \tfrac{1}{2} p.$$

Therefore, *in the parabola, the subnormal is the same for each point, and equal to half the parameter.*

181. Every line MX (*fig.* 73), drawn from a point M, of the parabola, parallel to the axis AQ, is called a *diameter;* and each diameter has a parameter, which is quadruple the distance MF of the origin of this diameter from the focus. Every straight line mO drawn from a point m of the parabola, parallel to the tangent TM that passes through the origin M of this diameter, is called an *ordinate* to this diameter. We shall now show that the ordinates to any diameter whatever have the same properties, as the ordinates to the axis. Fig. 73.

We draw the ordinate MP to the axis, and from the points m, O, the lines mp, OQ, parallel to MP, also mS parallel to the axis. Designating AP by x, PM by y, Qp by g, AQ by k, we have $Ap = k - g$. The similar triangles TPM, mSO, give

$$TP : PM :: mS : SO,$$

that is,

$$2x : y :: g : SO;$$

whence

$$SO = \frac{gy}{2x};$$

consequently

$$pm = QS = QO - SO = PM - SO = y - \frac{gy}{2x}.$$

Now, since the point m belongs to the curve, we have

$$\overline{pm}^2 : \overline{PM}^2 :: Ap : AP,$$

that is,

$$\left(y - \frac{g\,y}{2\,x}\right)^2 : y^2 :: k - g : x,$$

or,

$$y^2 - \frac{2\,g\,y^2}{2\,x} + \frac{g\;y^2}{4\,x^2} : y^2 :: k - g : x;$$

whence, taking the product of the extremes and means, we have

$$x\,y^2 - g\,y^2 + \frac{g^2\,y^2}{4\,x} = k\,y^2 - g\,y^2,$$

which reduces itself to

$$x + \frac{g^2}{4\,x} = k,$$

or

$$\frac{g^2}{4\,x} = k - x.$$

If now we designate the abscissa, MO by x', and the ordinate $m\,O$ by y', we shall have

$MO = PQ = AQ - AP = k - x$, that is, $x' = k - x$;

and consequently

$$\frac{g^2}{4\,x} = x', \text{ or } g^2 = 4\,x\,x';$$

but the right-angled triangle $m\,SO$ gives

$$\overline{m\,S}^2 + \overline{SO}^2 = \overline{m\,O}^2,$$

that is,

$$g^2 + \frac{g^2\,y^2}{4\,x} = y'^2.$$

Substituting, therefore, for g^2 its value $4\,x\,x'$, above found, and for y^2 its value $p\,x$ (172), we shall have

$$4\,x\,x' + \frac{4\,x^2\,x'\,p}{4\,x^2} \text{ or } 4\,x\,x' + p\,x' \text{ or } (4\,x + p)\,x' = y'^2.$$

If we designate the parameter to the diameter MX by p', we shall have, by hypothesis,

$$p' = 4\,FM,$$

that is, since FM is equal to $x + c$ (172),

$$p' = 4\,x + 4\,c = 4\,x + p;$$

substituting p' for $4\,x + p$ in the above result, we obtain

$$p'\,x' = y'^2.$$

The equation, with regard to any diameter, therefore, is the same as that respecting the axis. Accordingly, *in the parabola, the square of the ordinate* m O *to any diameter* MX, *is equal to the rectangle of the abscissa and parameter to this diameter; and the squares of the ordinates to any diameter are to each other as the corresponding abscissas.*

182. If we would describe a parabola, which has an indefinite line MX for a diameter, a given line p' for the parameter to this diameter, and the angle contained by the diameter and its ordinates also given; according to what precedes, we draw through the origin M a line NMT, making with MX the angle NMX, equal to the given angle. Through the same point M we draw MF, making with MT the angle FMT equal to NMX; and, having made MF equal to $\frac{1}{4}p'$, the point F will be the focus of the parabola (178, 181); consequently, if we draw through the point F the indefinite line TFQ parallel to MX, and meeting TM in T, this will be the direction of the axis; in which the vertex A is determined by letting fall the perpendicular MP and bisecting PT in A (179). The focus and vertex being known, the parabola is easily described.

183. The three curves which have been the subject of consideration are called *conic sections*, because they are obtained from a cone by means of a plane cutting it. The section is an ellipse (*plate* iv. *fig.* 75), as $AMmB$, for example, when the cutting plane meets the two sides CH, CI, on this side the vertex C, the single case excepted, in which this plane makes with the side CI the same angle, as the other side CH makes with the base, when the section is a circle. Fig. 75.

If, on the contrary, the cutting plane meets only one of the sides of the cone, or meets the second CH (*fig.* 76), only according as it is produced, the section is a hyperbola, as AMm. Fig. 76.

If the cutting plane be parallel to one of the sides of the cone, the section is a parabola, as AMm (*fig.* 77). Fig. 77.

Let the cone CHI (*fig.* 75, 76), be supposed to be cut by a plane passing through the vertex and the centre of the base, or through the axis; the section will be a triangle. Let the cone now be cut by three planes AMm, FMG, HmI, perpendicular to this triangle, the last two being also parallel to the base of the cone. The two sections FMG, HmI, will be circles, meeting the section AMm, in M and m. The intersections FG, HI, of the planes of these circles with the triangle through the axis, will be the diameters of these same circles. The intersections PM, pm, of these circles with the plane AMm, will be perpendicular to the plane of the triangle, and they will at the same time be ordinates of the circle and of the section AMm. Fig. 75, 76.

This being supposed, the similar triangles APG, Apl, give

$$AP : Ap :: PG : pI;$$

and the similar triangles BFP, BHp, give

$$PB : pB :: FP : Hp;$$

multiplying these two proportions in order, we have

$$AP \times PB : Ap \times pB :: FP \times PG : Hp \times pI.$$

But, by the nature of the circle,

$$FP \times PG = \overline{PM}^2, \quad \text{and} \quad Hp \times pI = \overline{pm}^2,$$

consequently

$$AP \times PB : Ap \times pB :: \overline{PM}^2 : \overline{pm}^2;$$

therefore the squares of the ordinates of the section AMm are to each other, as the rectangles of the abscissas. Now these abscissas fall on different sides of the ordinate in figure 75, and on the same side in figure 76; wherefore AMm in the former is an ellipse, and AMm in the latter is a hyperbola.

As to figure 77, the same things being supposed as above, we have, by the nature of the circle, $\overline{PM}^2 = FP \times PG$;

and $$\overline{pm}^2 = Hp \times pI,$$

or, on account of the parallels Pp, FH, and FP, Hp, which give $FP = Hp$,

$$\overline{pm}^2 = FP \times pI;$$

consequently

$$\overline{PM}^2 : \overline{pm}^2 :: FP \times PG : FP \times pI,$$
$$:: PG : pI;$$

or, on account of the similar triangles APG, ApI,

$$\overline{PM}^2 : \overline{pm}^2 :: AP : Ap,$$

that is, the squares of the ordinates are to each other as the abscissas; therefore the curve is a parabola.

KEY

TO THE MORE DIFFICULT FORMULAS OF PAGES 24, 25,

ANALYTICAL GEOMETRY.

THE formula, $\sin a^2 - \sin b^2 =$ &c., 5th line from the bottom of page 24th, is found thus ;

Multiply the two first formulas of art. 11, member by member, and we shall have

$$\sin(a+b)\sin(a-b) = \frac{\sin a\cos b + \sin b\cos a}{R} \times \frac{\sin a\cos b - \sin b\cos a}{R}$$

$$= \frac{\sin a^2\cos b^2 - \sin b^2\cos a^2}{R^2}$$

$$= \frac{\sin a^2(R^2 - \sin b^2) - \sin b^2(R^2 - \sin a^2)}{R^2}$$

$$= \sin a^2 - \sin b^2.$$

Also by substituting, in the second of the above equations, $R^2 - \cos a^2$ for $\sin a^2$, and $R^2 - \cos b^2$ for $\sin b^2$, we shall have, in like manner,

$$\sin(a+b)\sin(a-b) = \cos b^2 - \cos a^2.$$

The formula of the 4th line from the bottom is found in the same way.

In order to obtain the formula of the top line of page 25th, we have

$$\operatorname{tang} a = \frac{R\sin a}{\cos a}\ (8),\ \operatorname{tang} b = \frac{R\sin b}{\cos b};$$

whence, by adding,

$$\operatorname{tang} a + \operatorname{tang} b = \frac{R\sin a}{\cos a} + \frac{R\sin b}{\cos b}$$

$$= \frac{R\sin a\cos b + R\sin b\cos a}{\cos a\cos b}$$

$$= \frac{R(\sin a\cos b + \sin b\cos a)}{\cos a\cos b}$$

$$= \frac{R^2\sin(a+b)}{\cos a\cos b}\quad (11).$$

The three following formulas are found in a similar manner. For the 5th we take the product of the two first, thus;

$$(\text{tang } a + \text{tang } b)\ (\text{tang } a - \text{tang } b)$$

or

$$\text{tang } a^2 - \text{tang } b^2 = \frac{R^4 \sin(a+b) \sin(a-b)}{\cos a^2 \cos b^2}$$

For the 7th we have

$$\frac{\sin a + \sin b}{\sin a - \sin b} = \frac{\frac{2}{R} \sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}{\frac{2}{R} \cos \frac{1}{2}(a+b) \sin \frac{1}{2}(a-b)} \quad (27)$$

$$= \text{tang } \frac{1}{2}(a+b) \cot \frac{1}{2}(a-b)$$

$$= \text{tang } \frac{1}{2}(a+b) \frac{1}{\text{tang } \frac{1}{2}(a-b)}.$$

$$= \frac{\text{tang } \frac{1}{2}(a+b)}{\text{tang } \frac{1}{2}(a-b)}.$$

The first formula of the 8th line is obtained in a similar manner, and the 2d formula of the 8th line is derived from the 1st, by supposing $b = 0$.

The second formula of the 12th line is found by means of the formula near the bottom of 24th page, namely,

$$\sec a = \frac{R^2}{\cos a}, \text{ or } \cos a = \frac{R^2}{\sec a};$$

thus

$$\frac{\cos a + \cos b}{\cos a - \cos b} = \frac{\frac{R^2}{\sec a} + \frac{R^2}{\sec b}}{\frac{R^2}{\sec a} - \frac{R^2}{\sec b}} = \frac{\frac{\sec a + \sec b}{\sec a \sec b}}{\frac{\sec a - \sec b}{\sec a \sec b}}$$

$$= \frac{\sec a + \sec b}{\sec b - \sec a} = -\frac{\sec a + \sec b}{\sec a - \sec b}.$$

The first formula of the 13th line is deduced from the formula

$$\text{tang } a = \frac{R \sin a}{\cos a}, \text{ which gives}$$

$$\sin a = \frac{\text{tang } a \cos a}{R},$$

or, since by art. 8, $\cos a = \frac{R^2}{\sec a}$,

$$\sin a = \frac{\text{tang } a\ R^2}{R \sec a}$$

$$= \frac{R \text{ tang } a}{\sec a}$$

$$= \frac{R \text{ tang } a}{\sqrt{R^2 + \text{tang } a^2}} \text{ [see fig. 3.]}$$

Moreover, the above formula for the tang. gives

$$\cos a = \frac{R \sin a}{\text{tang } a},$$

and, by substituting for sin a the value just found, we have

$$\cos a = \frac{R \times R \text{ tang } a}{\text{tang } a \sqrt{R^2 + \text{tang } a^2}}$$

$$= \frac{R^2}{\sqrt{R^2 + \text{tang } a^2}}.$$

In the 15th line ;—since the formula $\sec a = \frac{R^2}{\cos a}$, when $a = \frac{2}{3} q$, gives

$$\sec \tfrac{2}{3} q = \frac{R^2}{\cos \frac{2}{3} q} = \frac{R^2}{\frac{1}{2} R} = 2 R, \text{ we have } \tfrac{1}{2} \sec \tfrac{2}{3} q = R.$$