Finite Geometries

Focus

Traditional Euclidean Geometry has an infinitude of points, lines, and planes as well as a sizable collection of theorems that continues to grow. A "miniature" geometry that has a small number of axioms, few theorems, and most importantly a finite number of elements is known as a finite geometry.

We shall look at such geometries. They provide a rich opportunity from which to study geometric structure, as well as serve as an excellent springboard into discussion of other geometries, such as transformational, and projective geometry.

Discussion

Axiomatics

As stated in the Introduction, the general structure of an axiomatic system includes:

- undefined terms
- definitions
- axioms or postulates
- theorems
- logic

Undefined terms refer to objects and concepts that are basic to the discussion. They are left undefined to avoid the creation of circular definitions. Undefined terms include: point, line, plane, prime (space), on, and through.

Definitions provide the technical language for dealing with objects, numbers, concepts, and relationships.

An axiom is a self-evident truth. It is not proven.

Theorems are provable statements. The proof of such statements rely on definitions, axioms, and logic. Proofs can also be based on previously proven theorems.

Axiomatic systems are described using the terms consistent, independent, and complete.

> **Definition:** An axiomatic system is said to be consistent if neither the axioms or theorems contradict one another.

Definition: An axiom is described as independent of other axioms if it cannot be derived from them.

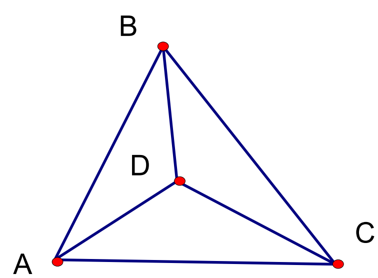
Definition: An axiomatic system is complete, if for all correctly posed statements can be proven or disproven. Another way of describing completeness, if it is impossible to add a new independent axiom to the system.

FOUR-POINT GEOMETRY

Axioms

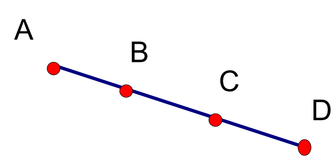
- There exists exactly four points.
- Any two distinct points are on exactly one line.
- Each line is on exactly two points.

Consistency for an axiomatic system can be established by creating a model and verifying each axiom.



$$P = \{A, B, C, D\}$$
$$L = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$$

Independence is much more difficult to establish. A possible approach is to replace an axiom with its negation and creating a model. You must do this for each axiom.



Inserting the negation of the last axiom.

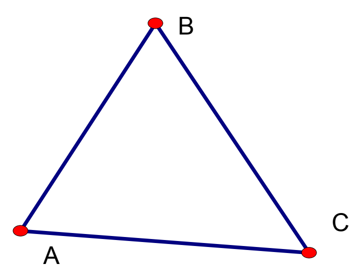
Completeness for a rich and comprehensive system such as arithmetic and geometry is impossible. Kurt Gödel proved this in 1931.

Finite Geometries

All the finite geometries of this discussion have *point* and *line* as undefined terms, and *on* as the main relation. Note that figures in geometry are sets, thus, in a finite geometry each figure is a "finite" set. Your perception of a line must change.
THREE-POINT GEOMETRY

Axioms

- There exists exactly three distinct points in the geometry.
- Any two distinct points are on exactly one line.
- Not all points are on the same line.
- Any two distinct lines are on at least one point.



$$P = \{A, B, C\}$$
$$L = \{\{A, B\}, \{A, C\}, \{B, C\}\}$$

Definition: Two distinct lines with a common point on each are said to *intersect* or are said to be *intersecting lines*.

Conjectures and Theorems

- What kind of drawing can be made to illustrate the geometry?
- How many lines are in the geometry?
- What, if any, theorems can be proved?
- What other objects can be used besides points and lines to represent the geometry?
- Are there any properties or theorems from Euclidean geometry that apply to this geometry?

Theorem 1 Any two distinct lines are on exactly one point.

Proof

Let g and h be two distinct lines. By axiom there exists at least one point P on them. Suppose there exists point Q on both g and h distinct from P. By axiom there is exactly one line on P and Q. This is a contradiction to g and h being distinct. Thus, there can be exactly one point on g and h.

□

Theorem 2 There are exactly three lines in the geometry.

Proof

From the three given points, there are three lines when points are taken two at a time, $\binom{3}{2} = 3$. Suppose there is a fourth line. It must have a distinct point in common with each of the other three lines. Thus, the line must be on two of the given three points, and therefore must be one of the other three lines (otherwise there would be two lines on two distinct points). Therefore, there are exactly three lines.

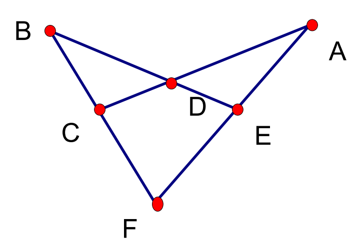
□

Theorem 3 Each line contains only two points.

FOUR-LINE GEOMETRY

Axioms

- There exists exactly four lines.
- Any two distinct lines intersect in one point.
- Each point is on exactly two lines.



$$P = \{A,\ B,\ C,\ D,\ E.\ F\}$$
$$L = \{\{A, D, C\}, \{A, E, F\}, \{B, D, E\}, \{B, C, F\}\}$$

Conjectures and Theorems

Theorem 1 There exists exactly six points.

Proof

By axiom there are four lines. There are $\binom{4}{2} = 6$ pairing of lines, thus, there are at least six points. If the six points were not distinct (that is suppose that two points are the same) then the point would be on at least three lines which would be a contradiction. Suppose there is a seventh point. Then there would exist at least one more line other than the four used to get the original six points. There would be at least five lines, which is a contradiction. Thus there are exactly six points.

□

Theorem 2 Each line contains exactly three points.

Proof

Let g, h, j and k be the four distinct lines. Consider g paired with the other three lines. There would be at least three points on g. Further suppose that there is a fourth point on the line g. This would mean that there is a line distinct from h, j, and k on the point; a fifth line which is a contradiction. Therefore, there are exactly three points on each line.

□

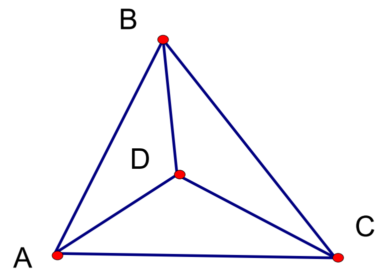
- Do two distinct points determine a line?
- How many triangles exists? (Definition: A triangle consists of three distinct lines that intersect in three distinct points, vertices.)
- Are there parallel lines? (Definition: Two lines are said to be *parallel* if they do not intersect.)

Definition: The *plane dual* of a statement is formed by exchanging the words *point* and *line* in the statement. By exchanging these words you can create the axioms for a four-point geometry.

FOUR-POINT GEOMETRY

Axioms

- There exists exactly four points.
- Any two distinct points are on exactly one line.
- Each line is on exactly two points.



$$P = \{A, B, C, D\}$$
$$L = \{\{A,\ B\}, \{A,\ C\}, \{A,\ D\}, \{B,\ C\},\ \{B,\ D\}, \{C,\ D\}\}$$

Theorems & Conjectures

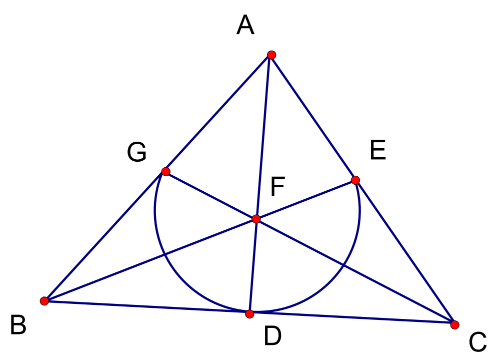
Theorem 1 There exists exactly six lines.

Theorem 2 Each point is on exactly three lines.

FANO'S FINITE GEOMETRY

Axioms

- There exists at least one line.
- Every line contains exactly three points.
- Not all points are on the same line.
- Two distinct points uniquely determine a line.
- Two distinct lines intersect in at least one point.



Theorems

Theorem 1 Two distinct lines intersect in exactly one point.

Proof

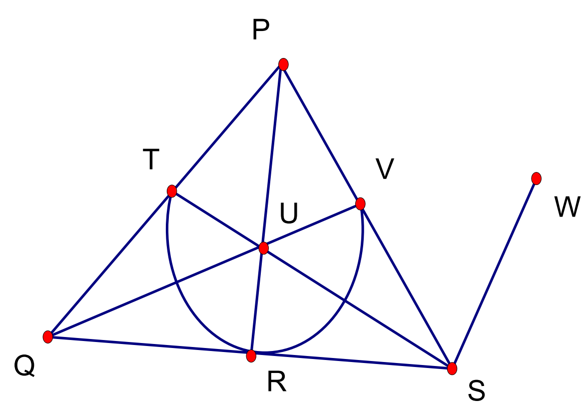
By axiom two distinct lines (say g and h) intersect in at least one point P. Suppose there is a second point, Q, at which the lines intersect. ($P, Q \in g \cap h$). By axiom P and Q uniquely determine a line, therefore, $P, Q \in g \cap h$ is a contradiction. Thus, g and h intersect in exactly one point.

□

Theorem 2 There exists exactly seven points and seven lines.

Proof

Consider line g and point P not on g. There exists points Q, R, and S on g. There are three unique lines on P and Q, P and R, and P and S. There are three distinct points T, U, V on $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$, $\overleftrightarrow{PS}$ respectively. Thus there are at least seven points in the geometry. Suppose there exists an eighth point W.



Consider the model above. Let there exists line $\overleftrightarrow{SW}$. By axiom $\overleftrightarrow{QTP}$ must intersect $\overleftrightarrow{SW}$ at a distinct point, which is a contradiction to three points on $\overleftrightarrow{QTP}$. If $\overleftrightarrow{SW}$ intersects $\overleftrightarrow{QTP}$ at one of its three points, then $\overleftrightarrow{SW}$ is not unique which is also a contradiction. Therefore, there exists exactly seven points.
□

YOUNG'S FINITE GEOMETRY

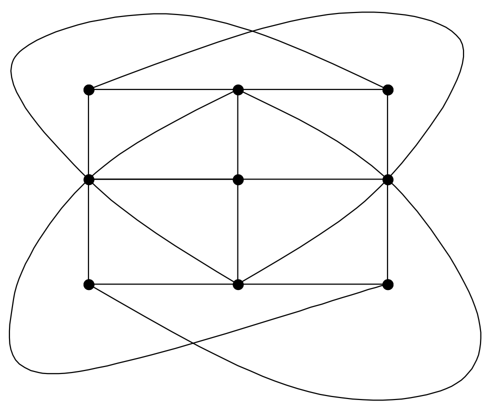
Axioms

- There exists at least one line.
- Every line contains exactly three points.
- Not all points are on the same line.
- Two distinct points uniquely determine a line.
- If a point is not on a given line, then there exists a unique line on the point that does not intersect the given line.

Model for Young's Finite Geometry

1	2	3	4	5	6	7	8	9	10	11	12
A	A	A	A	B	B	B	C	C	C	D	G
B	E	D	H	E	D	F	F	E	H	E	H
C	I	G	F	H	I	G	I	G	D	F	I

Table 1

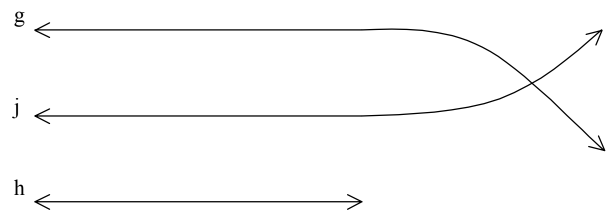


Theorems

Theorem 1 If two lines are each parallel to a third line, then they are parallel to each other.

Proof

Let line g be parallel to line h $(g \parallel h)$ and line $j \parallel h$. Suppose $g \nparallel j$ then g would intersect j $(g \cap j \neq \emptyset)$



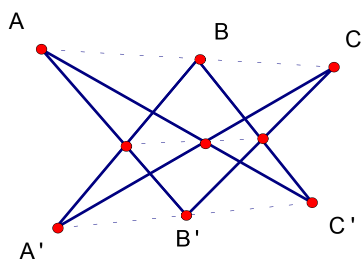
Let $P, Q,$ and R be on g $(P, Q, R \in g)$ and suppose $g \cap j = \{R\}$. $R \notin h$ by definition of parallel, therefore, both g & j are on R and parallel to h which is a contradiction. Thus $g \cap j = \emptyset$ or $g \parallel j$.

□

FINITE GEOMETRIES OF PAPPUS & DESARGUES

Theorem of Pappus (From Euclidean Geometry)

Given points A, B, C as distinct points on line g, and A', B', C' as distinct points on line g', then $\overleftrightarrow{AB'} \cap \overleftrightarrow{A'B}$, $\overleftrightarrow{AC'} \cap \overleftrightarrow{A'C}$, and $\overleftrightarrow{BC'} \cap \overleftrightarrow{B'C}$ are collinear.



This Euclidean theorem leads to the finite geometry of Pappus.

Axioms for the Finite Geometry of Pappus

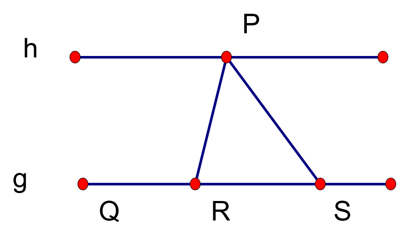
- There exists (∃) at least one line.
- Every line has exactly three points.
- Not all points are on the same line.
- Given point P and line g. If $P \notin g$ then there exists exactly one (∃!) line h such that ($\ni$) $P \in h$ and $h \parallel g$.
- Given P and g $\ni P \notin g$, $\exists!$ $P' \in g$ $\ni$ ***no*** lines contain both P and P'.
- With exception for the previous axiom, two distinct points uniquely determine a line.

Theorems

Theorem 1 Each point is on exactly three lines.

Proof

Let P be a point, then $\exists$ line $g \ni P \notin g$.



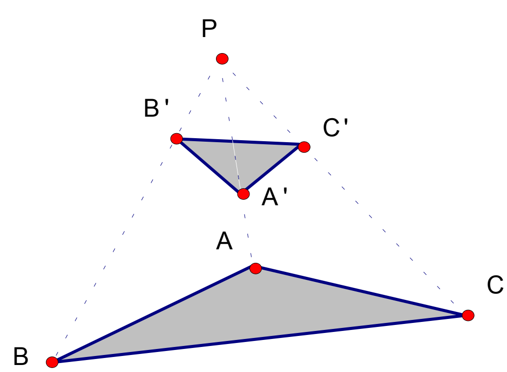
$\exists$ $Q, R, S \in g$. Let Q be such a point that no line contains both P and Q. Thus, $\exists$ $\overleftrightarrow{PR}$ and $\overleftrightarrow{PS}$. By axiom $\exists$ $h \in P \ni h \parallel g$. Therefore, there exists at least three lines on P. Suppose $\exists$ j on P. $j \cap g \neq \emptyset$ implies that $j \cap g = Q$ which is a contradiction or $j \cap g = T$ which is also a contradiction. Therefore, $\exists$ exactly three lines on P.

$\square$

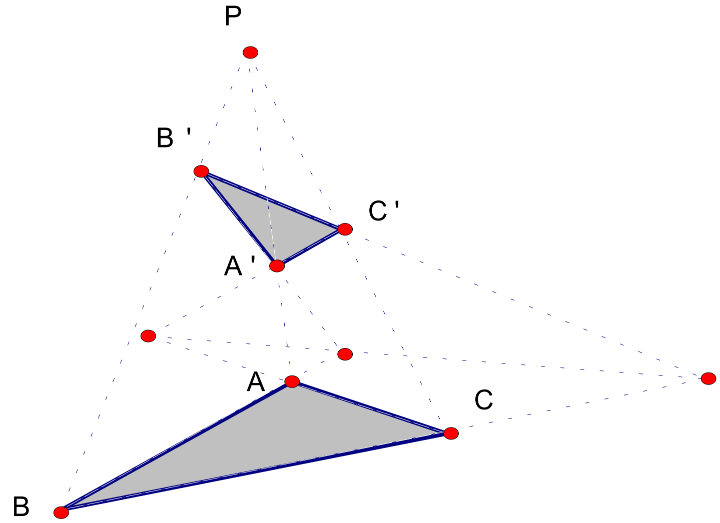
Theorem 2 $\exists$ exactly nine points and nine lines.

Definitions: (From Euclidean Geometry)

Triangle ABC ($\triangle ABC$) and $\triangle A'B'C'$ are perspective from point P if $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are concurrent at P.



Two triangles perspective from a point are also perspective from a line. Two triangles are perspective from a line if the intersection of their corresponding sides are collinear.



If a point is a point of perspectivity for two triangles and a line is the line of perspectivity for the same two triangles, then the point is the *pole* and the line is the *polar*.

Finite Geometry of Desargues

Definitions: The line g is a polar of ponit P if no lines contain P and a point on g.

A point P is a pole of line g if no lines contain P and a point on g.

Axioms for the Finite Geometry of Desargues

- $\exists$ at least one point.
- Each point has at least one polar.
- Each line has at most one pole.
- Two distinct points are on at most one line.
- Every line has exactly three distinct points.
- If a line does not contain a certain point, then there is a point of intersection for the line and any polar of the point. (Let p be the polar of P, if $P \notin g$ then $p \cap g \neq \emptyset$)

Theorems

Theorem 1 Each line has exactly one pole.

Theorem 2 Each point has exactly one polar.