A TEXT-BOOK

OF

ANALYTIC GEOMETRY;

ON THE BASIS OF

PROFESSOR PEIRCE'S TREATISE.

BY

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PREFACE.

The immediate design of this book is to fill a want which has grown out of changes in the programme of instruction at Harvard College. Professor Peirce's treatise on Analytic Geometry, which forms the first book of the *Curves and Functions*, was written at a time when this part of the mathematical course was confined to students who pursued it of their own choice. Since that time, Analytic Geometry has been included among the required studies; and thus has arisen the need of a text-book better suited than any we have been able to find to the purposes of general instruction.

In respect of methods of treatment, Professor Peirce's work has generally been closely followed. While that book, however, is designed as an introduction to the higher mathematics, for students of special mathematical taste, and is meant to be used in connection with lectures, the purpose of this book is to close the mathematical course of general students, and it aims at a fulness of treatment which will leave the necessity for only such explanation and comment as may suitably be given in the recitation-room.

It will be seen that I have departed somewhat from the common course of mathematical text-books in the occasional introduction of illustrations drawn from Physics, and in sometimes following out to a considerable length merely speculative views, such as the relation between the forms of the conic section. I cannot but think it a mistake to confine the general student to the methods of Mathematics, — to drill him in processes which are to him dry technicalities, without calling his attention either to its purely intellectual value, or to its importance as an instrument in Physical Science. Whether I have, in any degree, succeeded in avoiding the consequences of this course, which seems to me unjust to the powers of the pupil and to the true character of the science, experiment alone can show.

I would acknowledge, in closing, my obligations for the aid and encouragement which I have received from others. Professor Peirce has given me the benefit of his advice in repeated instances. Whatever merit the book may have is owing, in a great degree, to the assistance of Mr. C. W. Eliot, who, besides many less definite, but important services, has read and criticised a considerable part of the manuscript before it was sent to press. To other friends I am indebted for suggestions which have added value to these pages.

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REFERENCES.

The references are made to Professor Peirce's "Course of Pure Mathematics."

Geom.	signifies	Geometry.
Tr.	"	Plane Trigonometry.
Alg.	"	Algebra.

ERRATA.

In § 106, substitute the following values for those which are there given: —

Ex. 6, $\tan \frac{x_1}{x} = - \frac{3}{2} - \frac{1}{2} \sqrt{13}$.

Ex. 9, $y^\circ = \frac{1}{8}$;

" *Ans.* $y = \pm \frac{1}{4\sqrt{5}}$.

ANALYTIC GEOMETRY.

CHAPTER I.

GENERAL REMARKS.

1. The modern mode of treating Geometry took its beginning and has its foundation in the application of the algebraic method of reasoning, first effected in a systematic form by Descartes, to the solution of geometrical questions. Some of the merits of the algebraic method, such as clearness and conciseness in statement, in operation, and in result, and the exclusion from the process of all irrelevant and superfluous conditions, must have occurred to you in solving the examples in Algebra; and many other advantages are brought more clearly to light, when it is applied to Geometry. Especially may be mentioned the property of generality, which enables us, by the substitution of letters for numbers, to pass from the separate consideration of particular problems to the investigation of questions which include the particular problems as nothing but special cases, and which has given the dignity and value of a science to a system originally proposed as a convenient method of overcoming some difficulties in the theory of curves. Under the algebraic

treatment of Geometry, the study of details and of isolated propositions becomes subservient to the comprehension of the profound and central principles of geometric truth, and to the full and harmonious development of our conception of the nature and the laws of space, its parts, and its contents.

A good illustration of the manner in which Algebra may be made to lead to general results, and in which these results are often advantageously discussed for certain marked and (so called) *singular* cases, is afforded by Alg. arts. 126 (ex. 25, 31, 38, 39, 41, &c.) – 134.

2. The name ANALYSIS is given to any method which is applied to develop a mathematical science in a general form, and on the basis of universal and fundamental principles. This word, however, when not otherwise limited, is commonly to be understood in a technical and confined sense, for what is properly called *Algebraic* Analysis, or the species in which the instrument employed is the algebraic process. Thus, ANALYTIC GEOMETRY is the Science of Space (that is, of Position and Extension) as unfolded under the forms and by the operations of Algebra.

It will be seen that the above definition of Analytic Geometry covers Trigonometry; for there also the reasonings are principally conducted, not, as in Elementary Geometry, by direct and constant reference to the figures, but through symbols and according to algebraic rules. The scientific value of Trigonometry consists in its method of denoting two simple but essentially distinct classes of quantities, namely, lines and angles, by symbols which may be brought into the same formulæ, and by means of which the fundamental properties of those quantities, considered relatively to each other, may be investigated. Hence, Trigonometry, when looked on as *pure science* (that is, not from the practical point of view), is an introduction to Analytic Geometry, though it wants the characteristic features of the Cartesian system.

3. The process of solving any problem by Algebra consists of these three parts: —

I. The putting of the question into equations;

II. The solution of these equations;

III. The interpretation of the results.

The first step is taken by representing the known and unknown quantities by letters (or numbers), and expressing the conditions which affect them by means of equations in terms of these letters (Alg. arts. 101, 102). The second is exclusively algebraical, depending wholly on the form of the equations, and not at all on the nature of the problem.

The equations

$$a x = b y, \qquad x + y = c,$$

will always lead to the result

$$x = \frac{b c}{a + b}, \qquad y = \frac{a c}{a + b},$$

whether they arise from a problem in Geometry, or in Physics, or in trade, whether the letters represent all one kind of quantity, or different kinds, and whether lines, surfaces, dollars, or ounces, whatever be their absolute or relative magnitude, and whether the conditions which the equations express are accidental or arise from the nature of the quantities.

The interpretation of the result of an algebraic process requires only a knowledge of the notation adopted at the outset of that process.

If in the above example the letters denote lines, those which are unknown may be found by measuring a, b, and c, and substituting their numerical values in the expressions for x and y. Thus, if a denotes a line 2 inches long, b a line 4 inches long, and c a line 3 inches long, x and y will denote respectively lines of 2 inches and of 1 inch.

Thus the algebraic method, by separating the reasoning on a problem from its subject-matter, becomes equally applicable to all classes of questions in which the conditions can be reduced to the form of equations. Hence we may say, in general, that whenever the first part of an algebraic solution is possible, the whole is possible; for though a problem in (for instance) Geometry may lead to equations which cannot be solved by the ordinary methods of Algebra, such difficulties do not arise from the *geometric* nature of the problem, but are as likely to be met with in one application of Algebra as another.

4. The mode of representing the quantities in any problem by letters is, of course, arbitrary; but, for uniformity's sake, a certain notation has been agreed on and is used throughout Analytic Geometry. A peculiar method has also been invented, by which the most common algebraic quantities may be interpreted without regard to their numerical values; by which, for instance, the lengths of x and y, in the case just mentioned, may be found without measuring a, b, and c. This method necessarily rests on the conventional system of notation, which is conveniently explained in connexion with it. The interpretation of algebraic expressions is, in Analytic Geometry, called the GEOMETRIC CONSTRUCTION of those expressions.

5. It has already been pointed out (Alg. art. 139) that some conditions which may limit the values of

quantities cannot be expressed by equations, so that analysis is unavailable in problems which depend on such conditions. Some properties of geometric quantities are apparently of this kind. We can easily express by an equation that a line is three inches long; but that it is straight, or is the circumference of a circle, cannot directly be put into algebraic language. The quantities considered in Geometry have two fundamental and essentially distinct properties, namely, Magnitude and Figure. Now, in Algebra, as the letters denote only magnitudes, equations can express nothing but relations between magnitudes; so that a difficulty arises in the application of Algebra to those problems (the most important in Geometry) in which Figure is concerned, and it is felt in the attempt to put the problems into equations. The method by which this difficulty has been overcome is the remarkable invention of Descartes, on which Analytic Geometry rests, and which has suggested a new view of all those branches of Mathematics which depend on Geometry. It will be explained, after those problems which admit of direct algebraic treatment have been considered.

CHAPTER II.

THE GEOMETRIC CONSTRUCTION OF ALGEBRAIC QUANTITIES.

6. *Rule.* In Analytic Geometry, every letter properly denotes a line.

7. *Scholia.* *a.* Each *Roman* or *Italic* letter expresses the *length* of the line, without (primarily) any regard to its *direction.* Thus, if AB (Fig. 1) be denoted by a, we may also take a (at least when considered independently of its sign) to represent any line of the same length as AB, whatever be its position, and whether it be straight, broken, or curved. In a few instances, however, which will readily be recognized when they occur, the direction of a line is implied in the Roman or Italic letter the value of which expresses its length.

b. The numerical measure of the line may, when known, be substituted, at pleasure, for the letter which stands for the line; but it must always be remembered that what the letter denotes is not the number which measures the length, but the length itself. Thus, if AB (Fig. 1) is 2 inches long, and an inch is the unit of length, we shall have $a = 2$; but if half an inch is the unit, $a = 4$. Here a has two different numerical values, while that which a really represents, the actual length of the line, is in both cases the same.

c. Greek letters, on the other hand, denote lines, with reference only to their *directions.* Thus, if the line OA (Fig. 8) be called α, any line parallel to OA, whatever be its length, may also be denoted by α.

8. *Problem.* To construct $a\,b$.

Solution. Draw (Fig. 2)

$$AB = a,$$

and perpendicular to it (Geom. § 133)

$$AC = b.$$

Through A and B draw any two lines parallel to each other, and through C a line parallel to AB, and meeting the former parallels at D and E respectively. $ABED$ is, then, a parallelogram; and, because the area of a parallelogram is the product of its base by its altitude (Geom. § 247), it is one value of the expression $a\,b$. Since, however, we may conceive a surface of any figure whatever, such as a triangle or a circle, which shall be equivalent to this parallelogram (i. e. also equal to $a\,b$), and since nothing but a surface can be equivalent to it, we may conclude, in general, that the geometric construction of $a\,b$, and therefore of any expression which may be resolved into two factors, each of which denotes a line, is a surface.

9. *Problem.* To construct $a\,b\,c$.

Solution. This expression may be constructed by a parallelopiped of which the base is equal to $a\,b$ and the altitude to c (Geom. § 362). Hence, by reasoning similar to that used in § 8, $a\,b\,c$ represents, in general, a solid.

10. *Rule.* An angle is denoted by writing the letters which represent its sides, one above the other. Thus, AOB (Fig. 8) is $\overset{\beta}{\alpha}$, which may be read, *the angle which β makes with α*, or, more simply, *alpha-beta.* Such symbols must be carefully distinguished from fractions.

11. *Scholium.* Instead of measuring an angle by degrees, minutes, &c., it is commonly measured, in Analytic Geometry, by the ratio of the arc which subtends it in any circle described about its vertex as a centre to the radius of that circle.

To show that this method is legitimate, it must be proved that any two angles are to each other as the ratios by which we propose to measure them, and that the proposed measure of any angle has the same value in all circles described about its vertex as a common centre.

Now, if, in any circle, we take two angles, $\genfrac{}{}{0pt}{}{\beta}{\alpha}$ and $\genfrac{}{}{0pt}{}{\beta'}{\alpha}$, which have their vertices at its centre, and represent the arcs which subtend them respectively by s and s', and the radius of the circle by r, we shall have (Geom. § 97; Alg. art. 55)

$$\genfrac{}{}{0pt}{}{\beta}{\alpha} : \genfrac{}{}{0pt}{}{\beta'}{\alpha} = s : s',$$

$$\genfrac{}{}{0pt}{}{\beta}{\alpha} : \genfrac{}{}{0pt}{}{\beta'}{\alpha} = \frac{s}{r} : \frac{s'}{r};$$

the first part of the required proof.

Again, if we take a single angle $\genfrac{}{}{0pt}{}{\beta}{\alpha}$, but two different circles, having their centres at its vertex, then, denoting the radii by r and r_1, and the arcs which subtend $\genfrac{}{}{0pt}{}{\beta}{\alpha}$ respectively by s and s_1, we have, since similar arcs are to each other as their radii (Geom. §§ 170, 234),

$$s : s_1 = r : r_1,$$

$$s : r = s_1 : r_1;$$

so that the ratios corresponding to an angle in two circles of any radii are equal, and the justification of the proposed method is completed.

12. *Corollary.* Using the notation of § 11, we may write

$$\genfrac{}{}{0pt}{}{\beta}{\alpha} = s : r;$$

since the principles of algebraic notation allow us to form an equation between the symbol which represents a quantity and that which represents its measure. Hence we have

$$s = r \genfrac{}{}{0pt}{}{\beta}{\alpha};$$

that is, an arc may be denoted by the product of the angle which it subtends at its centre by its radius.

If r is the unit of length, we have

$$r = 1,$$

$$\frac{\beta}{\alpha} = s;$$

so that the measure of an angle may be said, in this system, to be the length of the arc which subtends it in the circle whose centre is at its vertex and whose radius is unity.

13. *Corollary.* In Analytic Geometry, π is commonly used, as in Geom. § 237, to denote the ratio of the circumference of a circle to its diameter, or of the semicircumference to its radius,* so that, by the rule of § 11, π represents the angle subtended by a semicircumference, that is, one of 180°, $\frac{1}{2}\pi$ an angle of 90°, or a right angle, 2π an angle of 360°, &c.

14. *Problem.* To construct $\frac{a}{b}$.

Solution 1st. From any point C (Fig. 3), with a radius equal to b, describe an arc AB, making it equal to a. Connect A and B with C. The measure of the angle ACB is, by § 11,

$$\frac{AB}{CA} = \frac{a}{b};$$

so that the angle ACB is the required construction.

The arc AB may be made equal to a in the following manner: — Draw the straight line

$$A'B' = a.$$

Divide it into very small parts, $A'a'$, $a'b'$, $b'c'$, &c. Describe the indefinite arc AD. From A as a centre, with a radius equal to

* This is an exception to the general rule of § 7, c. To avoid possibility of mistake, π will only be used as directed in § 13.

$A'a'$, describe an arc cutting AD at a. Then from a as a centre, with a radius equal to $a'b'$, describe an arc cutting AD at b. Go on in the same manner, finding the points c, d, &c.; and the last point thus obtained is the extremity of the required arc. For if the chords Aa, ab, &c., be drawn, the sum of these chords is, by construction, equal to $A'B'$. But there is no sensible difference between each of these small chords and its arc; hence the sum of the small arcs, or the arc $AB = A'B'$. This method is only an approximative one; but the smaller we take the portions $A'a'$, &c., the higher will be the degree of accuracy. If it were practicable to take the portions infinitely small, there would be no error whatever, since (Geom. § 203) an infinitely small arc is equal to its chord. In practice, it will be found best to make $A'a'$, $a'b'$, &c. of the same length, laying off this length on $A'B'$ as many times as it will go, and leaving the remainder $f'B'$, because then it will only be necessary to adjust the compasses once for measuring off all the equal arcs, Aa, ab, &c.

Solution 2d. Draw (Fig. 4)

$$AB = a,$$
$$BC = b,$$

perpendicular to each other. Connect AC. The tangent of the angle C or the cotangent of A will then be equal to $\frac{a}{b}$. (Tr. §§ 6, 9.) This construction is evidently possible for any values of a and b.

Solution 3d. A different construction, which we can illustrate by the same figure, may be employed, if it is preferred. Suppose we have

$$AB = a,$$
$$AC = b.$$

Then, $\frac{a}{b}$ denotes the sine of C or the cosine of A. (Tr. §§ 5, 9.) It is evident that this construction is possible only when $a < b$.

Solution 4th. We have also another solution for the case in which $a > b$. Suppose (Fig. 4)

$$AC = a,$$
$$BC = b.$$

Then, $\frac{a}{b}$ denotes the secant of C or the cosecant of A. (Tr. §§ 7, 9.)

Hence, an expression of the form $\frac{a}{b}$ may be constructed geometrically by an angle or by a trigonometric function of an angle.

Though, for convenience, the same figure has been employed in explaining the last three constructions, it must carefully be observed, that either the third solution or the fourth may always be employed for the same values of a and b as those assumed in solution second; and the student should draw for solution third a right triangle having one leg equal to a and the hypothenuse equal to b, and, for solution fourth, a right triangle having one leg equal to b and the hypothenuse equal to a. (Geom. § 145.)

15. *Scholium.* An examination of the preceding constructions (see §§ 6, 8, 9, 14) brings to notice a relation between the degrees of the algebraic expressions and the nature of their geometric equivalents. Thus, a, $a\,b$, $a\,b\,c$, are examples of the simplest monomials of, respectively, the first, the second, and the third degrees, or of one (algebraic) dimension, two dimensions, and three dimensions (Alg. art. 15); and they are constructed, respectively, by a line, which has one (geometric) dimension, a surface, which has two, and a solid, which has three (Geom. §§ 4, 5, 6). Also, $\frac{a}{b}$, being equal to $a\,b^{-1}$ (Alg. art. 38), is of the simplest form of monomials of the zero

degree, and it is constructed by an angle, a geometric quantity which cannot be said to have any of the dimensions of extension (being, in fact, only a *geometric relation* between two lines, as $\frac{a}{b}$ is an *algebraic* one), or else by a trigonometric function, which cannot properly be called a geometric quantity at all, but is an algebraic ratio of certain lengths. What has here been shown to be true of those expressions which have been constructed, will be found to hold good for all monomial expressions whatever, and therefore for polynomials as well, since a polynomial is only the sum of positive or negative monomials, and the sum or the difference of lines is a line, of surfaces a surface, &c.; but as geometric quantities of different kinds cannot be added together (e. g. a line and a surface, or a solid and an angle), no polynomial can be constructed, unless it consists of terms of the same degree, that is, unless (Alg. art. 16) it is homogeneous. Hence the following

Rule. Every homogeneous algebraic quantity of the zero degree denotes an angle or a trigonometric function; of the first degree, a line; of the second, a surface; of the third, a solid.

16. *Problem.* To construct $a + b$.

Solution. Measure off (Fig. 5)

$$AB = a,$$
$$BC = b;$$

and we have, by inspection,

$$AC = AB + BC = a + b;$$

so that AC is the required value of $a + b$.

17. *Problem.* To construct $a - b$.

Solution. Lay off (Fig. 6)

$$AB = a,$$

and from B, in the opposite direction,

$$CB = b;$$

and we have

$$AC = AB - CB = a - b;$$

so that AC is the required value of $a - b$.

18. *Corollary.* If §§ 16 and 17 be compared, it will be seen that the only difference between the algebraic expressions is in the *sign* of b, and that the only difference in the geometric constructions is in the *direction* of b.

Suppose that, in the expression $a + b$, we take b of less and less value, $a + b$ will differ less and less from a, and, in Fig. 5, the point C will move towards A, passing through the positions C', C'', &c., but keeping always on the right of B. At the moment, however, at which $b = 0$, the point C will fall on B, and

$$AC = AB = a + b = a.$$

Now, if b be taken still smaller than zero, i. e. negative, $a + b$ becomes of the form of § 17, and the point C approaches still nearer to A, and therefore necessarily passes to the left of B.

Again, if, in the expression $a - b$, we take $a = 0$, the two extremities of the line a, i. e. the points A and B, must coincide, so that we shall have (Fig. 7)

$$AC = BC = a - b = -b.$$

Hence, the only difference between the line $+b$ and the line $-b$, or between the line denoted by any letter and that denoted by the negative of the same letter, is that they point in opposite directions to each other.

We deduce from the reasoning of this article, and universally adopt in Analytic Geometry, the following

Rule. The geometric construction of the negative sign is opposite direction; that is, direction opposite to that which, in any case, we arbitrarily choose to call positive.

19. *Corollary.* If, in the fraction of § 14, we change the numerator from a to $-a$, we must, by the rule of § 18, alter the first construction by laying off the arc AB'' (Fig. 3), instead of AB, equal in length to AB, but in the opposite direction; and therefore the value of the fraction will be the angle ACB'', which only differs from the angle ACB in being estimated from the line CA in the opposite direction. But, as the proposed change in the fraction has no other effect than to reverse its sign, ACB'' is the negative of ACB; so that the above method of interpreting the negative sign may be applied to angles as well as lines.

20. *Scholium.* It is convenient to introduce and explain, in connexion with the construction of the negative sign, a mode of conceiving lines and angles which will also be found useful in other parts of Analytic Geometry.

I. *a.* The length of a line may properly be conceived as *the amount by which a point changes its place*, in other words, the distance which it traverses, in passing along the line from one extremity of it to the other; and this amount will be positive or negative, according to the direction in which the point is supposed to move. For example, the positive line b is drawn (Fig. 5) by moving the pencil to the *right* from its starting-point B (to which it had been brought from A by describing the line a), by an amount equal to the length of b; and therefore, in order to draw $-b$, we must move the pencil to the *left* from B (Figs. 6, 7), and by the same amount. Here the tip of the pencil represents the moving point.

b. Hence it appears that *the same line may be considered either as positive or as negative, according to the extremity from which we conceive it to begin.* Thus, if, when the line AB (Fig. 1) is conceived to be generated by the motion of a point from A to B, we call it positive, we must regard it as negative when it is conceived to be generated by the motion of a point from B to A. For the sake of distinguishing these two cases, a line from A to B is, in this book, indicated by AB, and a line from B to A by BA; so that, if $AB = a$, $BA = -a$. The starting-point from which a line is supposed to be directed may be called its *origin*.

c. According to this notation, it is evident, even without a figure, that, in all cases, it will be true, on any given line, that

$$AB + BC = AC;$$

since, if a point moves in the given path from A to B (i. e. generates the line AB), and then from B to C (i. e. generates the line BC), the distance traversed is the distance, measured on the same line, from A to C. In Fig. 6, where

$$BC = -b,$$

the above equation becomes

$$AB + BC = a - b = AC.$$

Here, indeed, the *actual* change of place which the generating point undergoes in moving from A to B, and then back to C, is greater than if it had moved only from A to C. But all difficulty is removed by considering, as we may, that the backward motion from B to C undoes, or negatives, that part of the forward motion which is from C to B, so that the whole *resulting* motion, the (reduced) sum of the component motions, is that represented by AC.

II. *d*. In like manner, an angle may be conceived as *the amount by which a line changes in direction* in turning from one position into another; and this amount will be positive or negative, according to the direction in which the line is supposed to rotate. I shall go on to develop a few of the consequences of this view, by reference to Fig. 8, in which

$$AOB = COA,$$

$$AOD = A'OD' = EOA;$$

and in which the straight arrows, with the accompanying Greek letters, indicate the assumed directions of the several lines. Let rotation in the direction indicated by the curved arrow θ be taken as positive. Then the arrow $-\theta$ points to negative rotation. Now, if we suppose a line to turn in the direction of positive rotation from its original position OA by an amount measured by the angle $\frac{\beta}{\alpha}$, it will come into the position OB, so that the angle which OB makes with OA (or AOB) is $\frac{\beta}{\alpha}$. If the line, however, be supposed to turn through the same amount, but in the negative direction, it will assume the position OC, so that the angle which OC makes with OA (or AOC) is $-\frac{\beta}{\alpha}$.

e. If the line be supposed to turn from the position OB, which we now take as its original position, to the position OA, and not, as before, from OA to OB, the absolute amount of the rotation will be the same, but its sign will be reversed, so that the change of direction which the line undergoes, that is, the angle which OA makes with OB, is $-\frac{\beta}{\alpha}$, instead of $\frac{\beta}{\alpha}$. Thus, *the same angle will be regarded either as positive or as negative, according as we conceive the change of direction to begin from the one side or from the other.* These two views are, in this book, distinguished by indicating the angle between OA and OB by AOB, or $\frac{\beta}{\alpha}$, when the change of direction is conceived to begin from OA, or α, and by BOA, or $\frac{\alpha}{\beta}$, when it begins from OB, or β; so that

$$AOB = \frac{\beta}{\alpha}, \qquad BOA = \frac{\alpha}{\beta} = -\frac{\beta}{\alpha},$$

$$AOC = \frac{\gamma}{\alpha} = -\frac{\alpha}{\gamma}, \qquad COA = \frac{\alpha}{\gamma} = -\frac{\gamma}{\alpha}.$$

Hence, if the symbol which denotes an angle be inverted and its sign changed, the value of the symbol is not thereby affected.

The side of an angle from which its change of direction is conceived to begin may be called its *axis*.

f. To find the *sum* of any number of angles, we must consider what single rotation will be equivalent to the sum of the rotations corresponding to the several angles, performed successively. Thus we may write

$$\genfrac{}{}{0pt}{}{\zeta}{\delta} + \genfrac{}{}{0pt}{}{\alpha}{\zeta} + \genfrac{}{}{0pt}{}{\gamma}{\alpha} + \genfrac{}{}{0pt}{}{\beta}{\gamma} = \genfrac{}{}{0pt}{}{\beta}{\delta}.$$

For if a line rotate from δ to ζ, then from ζ to α, then from α to γ, and lastly from γ to β, the composition of these rotations comes to the same thing as if the line had turned immediately from δ to β. In general, *if a polynomial, which consists of angular symbols, can be so arranged that, when all its terms are made positive* (by inverting them, if necessary), *the upper letter of the first term is the lower letter of the second, the upper of the second the lower of the third, &c., the polynomial may,* by the principles of the notation, *be constructed by the angle made by the upper line of the last term with the lower line of the first.* For example,

$$\genfrac{}{}{0pt}{}{\alpha}{\zeta} - \genfrac{}{}{0pt}{}{\alpha}{\gamma} - \genfrac{}{}{0pt}{}{\delta}{\zeta} + \genfrac{}{}{0pt}{}{\beta}{\gamma} = \genfrac{}{}{0pt}{}{\zeta}{\delta} + \genfrac{}{}{0pt}{}{\alpha}{\zeta} + \genfrac{}{}{0pt}{}{\gamma}{\alpha} + \genfrac{}{}{0pt}{}{\beta}{\gamma} = \genfrac{}{}{0pt}{}{\beta}{\delta}.$$

g. By reversing the above rule, we may, at pleasure, *decompose* any angle. Thus,

$$\genfrac{}{}{0pt}{}{\varepsilon}{\alpha} = \genfrac{}{}{0pt}{}{\beta}{\alpha} + \genfrac{}{}{0pt}{}{\gamma}{\beta} + \genfrac{}{}{0pt}{}{\delta}{\gamma} + \genfrac{}{}{0pt}{}{\varepsilon}{\delta} = \genfrac{}{}{0pt}{}{\beta}{\alpha} + \genfrac{}{}{0pt}{}{\varepsilon}{\delta} - \genfrac{}{}{0pt}{}{\beta}{\gamma} - \genfrac{}{}{0pt}{}{\gamma}{\delta}.$$

h. If the line turn from OA by the amount denoted by π (§ 13), it will assume the direction OA', or $-\alpha$ (the reverse of OA); so that

$$\pi = \genfrac{}{}{0pt}{}{-\alpha}{\alpha}.$$

If this rotation be repeated, the line returns to its original position; and we have (see *f.*)

$$2\,\pi = \genfrac{}{}{0pt}{}{-\alpha}{\alpha} + \genfrac{}{}{0pt}{}{\alpha}{-\alpha} = \genfrac{}{}{0pt}{}{\alpha}{\alpha}.$$

Indeed, since, at the end of a complete revolution, the position of a line is the same as at the beginning, *we may, in estimating the angle which a line makes with itself, suppose it to have returned to its original position after making any arbitrary number of com-*

plete revolutions, either in the positive or in the negative direction. Then, as a complete revolution is measured by 2π,

$$\begin{aligned} \frac{\alpha}{\alpha} &= 0 \ (= 0\,\pi, \text{ or } = -0\,\pi), \\ &\text{or } = 2\,\pi, \text{ or } = -2\,\pi, \\ &\text{or } = 4\,\pi, \text{ or } = -4\,\pi, \\ &\text{or } = 2\,n\,\pi; \end{aligned}$$

in which n denotes any integer positive or negative (including zero).

i. Any angle, as $\frac{\beta}{\alpha}$, may be decomposed in the following manner:

$$\frac{\beta}{\alpha} = \frac{\alpha}{\alpha} + \frac{\beta}{\alpha} = \frac{\alpha}{\alpha} + AOB = 2\,n\,\pi + AOB;$$

where AOB is used to denote the positive concave angle of β with α. Thus, if $AOB = 30°$, the above becomes

$$\frac{\beta}{\alpha} = 2\,n\,\pi + 30°.$$

If we take $n = -1$, we have

$$\frac{\beta}{\alpha} = -2\,\pi + 30° = -360° + 30° = -330°;$$

that is, if a line rotate negatively from the position α through 330°, it will reach the position β, as is also evident from the figure.

Observe, however, that, since all the trigonometric functions of $2\,\pi + AOB$ are the same as those of AOB (Tr. § 69), it is unnecessary to distinguish between these different values of $\frac{\beta}{\alpha}$ in any case in which the result depends on the values of the trigonometric functions, and not directly on that of the angle itself.

j. If the revolving line turn from the direction OA, or α, to OD, or $-\delta$, and then to $D'O$, or δ, the whole rotation is

$$\frac{-\delta}{\alpha} + \frac{\delta}{-\delta} = \frac{\delta}{\alpha} = AOD';$$

so that *the sides of an angle always may and should be conceived to radiate from the vertex.* The negative of AOD' is

$$-\frac{\delta}{\alpha}=\frac{\varepsilon}{\alpha}.$$

k. In determining the trigonometric functions of an angle in a right triangle, the sides which include the angle must be conceived to be directed from the vertex (as in *j.*), and the opposite side to conform to the direction of the *rotation* belonging to the angle. If, in Fig. 4, A is taken as BAC, the opposite side is BC; if it is taken as CAB ($=-BAC$) the opposite side is CB ($=-BC$); but in both cases the including sides are the same, namely, AB and AC.

Hence, if $AB=\alpha$, $AC=\beta$, we have

$$\sin\frac{\beta}{\alpha}=\frac{BC}{AC}, \qquad \operatorname{cosec}\frac{\beta}{\alpha}=\frac{AC}{BC},$$

$$\tan\frac{\beta}{\alpha}=\frac{BC}{AB}, \qquad \operatorname{cotan}\frac{\beta}{\alpha}=\frac{AB}{BC},$$

$$\sec\frac{\beta}{\alpha}=\frac{AC}{AB}, \qquad \cos\frac{\beta}{\alpha}=\frac{AB}{AC};$$

but

$$\sin\left(-\frac{\beta}{\alpha}\right)=\sin\frac{\alpha}{\beta}=\frac{CB}{AC}=-\frac{BC}{AC}=-\sin\frac{\beta}{\alpha},$$

$$\operatorname{cosec}\left(-\frac{\beta}{\alpha}\right)=\operatorname{cosec}\frac{\alpha}{\beta}=\frac{AC}{CB}=-\frac{AC}{BC}=-\operatorname{cosec}\frac{\beta}{\alpha},$$

$$\tan\left(-\frac{\beta}{\alpha}\right)=\tan\frac{\alpha}{\beta}=\frac{CB}{AB}=-\frac{BC}{AB}=-\tan\frac{\beta}{\alpha},$$

$$\cot\left(-\frac{\beta}{\alpha}\right)=\cot\frac{\alpha}{\beta}=\frac{AB}{CB}=-\frac{AB}{BC}=-\cot\frac{\beta}{\alpha},$$

$$\sec\left(-\frac{\beta}{\alpha}\right)=\sec\frac{\alpha}{\beta}=\frac{AC}{AB}=\sec\frac{\beta}{\alpha},$$

$$\cos\left(-\frac{\beta}{\alpha}\right)=\cos\frac{\alpha}{\beta}=\frac{AB}{AC}=\cos\frac{\beta}{\alpha};$$

which agrees with Tr. § 64.

l. Since the angles of the lines of Fig. 8 are equal to those of any lines parallel to them, *the reasoning of arts. d. — k. is equally applicable, if the lines do not proceed from a common point.*

21. *Problem.* To construct $\frac{a\,b}{c}$.

Solution. If we put

$$x = \frac{a\,b}{c},$$

we have

$$c\,x = a\,b,$$

$$\frac{c}{a} = \frac{b}{x},$$

$$\text{or } c : a = b : x.$$

Therefore, $\frac{a\,b}{c}$ is constructed by finding a fourth proportional to the lines c, a, and b. (Geom. §§ 165, 166.)

22. *Problem.* To construct any monomial of the first degree, which does not involve radicals.

Solution. Since the monomial does not involve radicals, it must be a fraction of which the numerator contains one dimension more than the denominator. It may then be represented by the expression

$$\frac{a\,b\,c\,d \ldots\ldots \&c.}{a'\,b'\,c' \ldots\ldots \&c.} = \frac{a\,b}{a'} \times \frac{c}{b'} \times \frac{d}{c'} \ldots \&c.$$

Construct first $\frac{a\,b}{a'}$ by § 21, and let m be the line which it represents. The given expression becomes $\frac{m\,c}{b'} \times \frac{d}{c'} \ldots\ldots \&c.$

Again, construct $\frac{m\,c}{b'} = m'$;

then, $\frac{m'\,d}{c'} = m''$;

and so on, and the last line thus obtained is plainly the result.

Polynomial, free from Radicals.

23. *Problem.* To construct a homogeneous polynomial of the first degree, which does not involve radicals.

Solution. If the polynomial consists of, or can be reduced to, a series of monomial terms, each of these terms may be constructed by § 22, and then the value of the polynomial may be found.

If the expression consists of or contains a fraction which cannot be reduced to the form of § 22, it may be constructed by an artifice which can best be explained through an example. Let it be required to construct the fraction $\frac{abc}{de-fg}$; in which the letters denote the same lines as in Fig. 9. Represent by o a length which may be taken at pleasure. Divide the numerator of the given fraction by that power of o which will make the product of the first degree. Since abc is of the third degree, we must divide by o^2; and the result, $\frac{abc}{o^2}$, being of the first degree, denotes a line, and may be constructed by § 22. Divide the denominator by the next lower power of o, namely, in this case, o, and the result is $\frac{de}{o}-\frac{fg}{o}$, which, being also of the first degree, may be constructed by § 22. The fraction has now been reduced to the form $\frac{\frac{abc}{o^2}}{\frac{de}{o}-\frac{fg}{o}}, = \frac{\frac{abc}{o}}{de-fg}$, and may obviously be restored to its original form by multiplying by o; so that, if we adopt the notation

$$m = \frac{abc}{o^2}, \quad n = \frac{de}{o} - \frac{fg}{o},$$

we have $\frac{abc}{de-fg} = \frac{mo}{n}$. Hence, the value of the given fraction is obtained by finding a fourth proportional to n, m, and o. The whole construction is exhibited in a compact form in Fig. 10, in which we have

$$AO = o, AB = a, AC = b, AD = \frac{a\,b}{o},$$

$$AE = c, AF = m;$$

$$AG = d, AH = e, AI = \frac{d\,e}{o},$$

$$AK = f, AL = g, AM = \frac{f\,g}{o},$$

$$AN = MI = \frac{d\,e}{o} - \frac{f\,g}{o} = n;$$

$$AP = \frac{m\,o}{n} = \frac{a\,b\,c}{d\,e - f\,g}.$$

It is to be observed that the value of the line found by the above process is independent of the length assigned to o, which is introduced only to facilitate the construction, and disappears from the result.

24. *Problem.* To construct $\sqrt{a\,b}$.

Solution. If we put

$$x = \sqrt{a\,b},$$

we have

$$x^2 = a\,b,$$

$$a : x = x : b;$$

so that $\sqrt{a\,b}$ is constructed by finding a mean proportional between the lines a and b. (Geom. §§ 183 – 188.)

25. *Corollary.* Since

$$\sqrt[4]{a\,b\,c\,d} = a^{\frac{1}{4}}\,b^{\frac{1}{4}}\,c^{\frac{1}{4}}\,d^{\frac{1}{4}} = \sqrt{(a\,b)^{\frac{1}{2}}\,(c\,d)^{\frac{1}{2}}},$$

this expression may be constructed by finding the values of $(a\,b)^{\frac{1}{2}}$ $(= \sqrt{a\,b})$ and of $(c\,d)^{\frac{1}{2}}$ $(= \sqrt{c\,d})$, by § 24, and taking the mean proportional betweeen the lines thus obtained. A similar construction may be applied to any radical monomial of the first degree in which the index of the root is an integral power of 2.

26. *Problem.* To construct $\sqrt{a^2 + b^2}$.

Solution. Find the side of that square which is equivalent to the sum of the two squares described on the given lines a and b. (Geom. §§ 243, 256, 261.)

27. *Problem.* To construct $\sqrt{a^2 - b^2}$.

Solution 1*st.* Find the side of that square which is equivalent to the difference of the two squares described on the given lines a and b. (Geom. § 262.)

Solution 2*d.* Since

$$\sqrt{a^2 - b^2} = \sqrt{(a + b)(a - b)},$$

it may be constructed as a case under § 24, by constructing $a + b$ and $a - b$, and finding a mean proportional between them.

The following form of construction applies to either of the above solutions: — From any point A (Fig. 11), with a radius equal to a, describe a circumference. Draw any diameter BB'; lay off on it from A a length AC equal to b; and at C erect a perpendicular meeting the circumference at D and d. Then CD or Cd is the construction of $\pm \sqrt{a^2 - b^2}$. Equal values, $C'D'$ and $C'd'$, would have been obtained, if b had been laid off negatively.

28. *Corollary.* If

$$b = 0,$$

$$\sqrt{a^2 - b^2} = \sqrt{a^2} = \pm a;$$

and, in the construction, the point C (Fig. 11) falls at A, giving $AD'' = a$, and $Ad'' = -a$, for the perpendiculars corresponding to CD and Cd.

The greater the value of b, if positive, and the less its value, if negative, the greater will be the value of b^2 (Alg. art. 194), and the less the absolute value of $\sqrt{a^2 - b^2}$; and, in the figure, the greater will be the length of AC or AC', and the less that of CD, Cd, $C'D'$, and $C'd'$.

If

$$b^2 = a^2,$$

$$\text{i. e. if } + b = a, \text{ or } - b = a,$$

$$\sqrt{a^2 - b^2} = 0;$$

and, in this case, C will fall at B or at B', and $D d$ and $D' d'$ become tangents to the circle, so that CD, Cd, $C'D'$, and $C' d'$ are reduced to nothing.

If we increase b still more, making

$$b^2 > a^2,$$

which will be the case whenever b is *absolutely* greater than a, then

$$a^2 - b^2 < 0, \text{ i. e. is negative,}$$

so that $\sqrt{a^2 - b^2}$ is imaginary (Alg. art. 197). Now, in this case, C falls without the circumference, on the diameter produced, as at E or E', and the perpendicular erected at either of these points will evidently have no point of intersection with the circumference; so that there can be no length whatever corresponding to CD. Thus, when the expression we are now discussing becomes imaginary, we find a geometric impossibility in its construction. Since, however, all imaginary quantities may be resolved into the form $\sqrt{a^2 - b^2}$, it is true, in general, that

Rule. Imaginary quantities do not admit of geometric construction.

29. *Scholium.* Throughout the Analytic Geometry, it will be found of interest and advantage to *discuss* algebraic expressions, i. e. to trace (as in § 28) the changes of value in them and their geometric equivalents, with special reference to any values which exhibit algebraic or

geometric peculiarities. Remarkable cases in geometric constructions are sometimes indicated by remarkable forms of their algebraic expressions, as by the occurrence of imaginary quantities; and sometimes no such indications exist. It is recommended to the student to mark the difference between these classes of cases as they arise.

30. *Problem.* To construct a homogeneous polynomial of the first degree which involves no radicals except those whose indices are integral powers of 2.

Solution. If the expression can be reduced to a series of monomial terms, each term may be constructed by one of the §§ 22, 24, 25; and thus the value of the polynomial may be found.

Otherwise, the form of the polynomial may be changed by the introduction of a new line, as in § 23.

It is sometimes convenient to substitute the methods of §§ 26 and 27 for that of § 24.

31. *Scholium.* In some of the following examples, the length represented by each letter is denoted by its numerical value, which expresses the ratio of the line to the standard length, or *unit*, which may be chosen arbitrarily. Where the numerical value of a letter is not given, it is to be constructed by the line which it indicates in Fig. 9.

32. Examples.

1. Construct $\sqrt{(a\,b + \frac{b^3}{c} - d^2)}$, in which $a = 2, b = 3, c = 1, d = 4$.

Solution. If we take

$$m = \sqrt{(a\,b)}, \quad m' = \sqrt{\left(\frac{b^3}{c}\right)},$$

we have
$$m^2 = a\,b, \quad m'^2 = \frac{b^3}{c},$$
$$\sqrt{\left(a\,b + \frac{b^3}{c} - d^2\right)} = \sqrt{(m^2 + m'^2 - d^2)};$$
and if we take
$$n = \sqrt{(m'^2 - d^2)},$$
we have
$$n^2 = m'^2 - d^2,$$
$$\sqrt{\left(a\,b + \frac{b^3}{c} - d^2\right)} = \sqrt{(m^2 + n^2)};$$
so that the required length is the hypothenuse of a right triangle, of which m and n are the legs.

In Fig. 12, where a quarter of an inch is the unit,
$$CA = a, \quad AB = b, \quad AD = \sqrt{(a\,b)} = m;$$
$$AE = c,\ AF = b,\ AH = AG = \frac{b^2}{c},\ AI = \sqrt{\left(\frac{b^2}{c} \times b\right)} = m';$$
$$IK = d, \quad AK = \sqrt{(m'^2 - d^2)} = n;$$
$$AL = AK, \quad LD = \sqrt{(m^2 + n^2)} = \text{answer}.$$

The given expression might also be constructed by assuming any line o, constructing
$$\frac{a\,b}{o} + \frac{b^3}{c\,o} - \frac{d^2}{o},$$
which, being of the first degree, denotes a line, and finding a mean proportional between the result and o, which will be
$$\sqrt{\left[\left(\frac{a\,b}{o} + \frac{b^3}{c\,o} - \frac{d^2}{o}\right) o\right]} = \sqrt{\left(a\,b + \frac{b^3}{c} - d^2\right)}.$$

2. Construct the roots of the equation, $x^2 - 2\,d\,x - c^2 = 0$.

Solution. The equation gives, by reduction,
$$x = d \pm \sqrt{(d^2 + c^2)}.$$

In Fig. 13,
$$BC = c, \quad BA = d, \quad AC = \sqrt{(d^2 + c^2)},$$
$$D'A = AD = d, \quad x = D'C \text{ or } = CD.$$

3. Construct $\frac{ab}{c}$, in which $a = 4, b = 3, c = 6$.

4. Construct $\sqrt{(a^2 - ad)}$.

5. Construct $\sqrt{(ag + fe - g^2)}$.

6. Construct $-\frac{a^5}{b^4} + \frac{ab}{\sqrt{(eg - g^2)}}$, in which $a = 1, b = 2, e = 7, g = 4$.

7. Construct the roots of the equations,

$$ax = by, \qquad x + y = c.$$

8. Construct the roots of the equation, $x^2 - 2ax + c^2 = 0$.

9. Construct the roots of the equation, $x^2 - 2cx + a^2 = 0$.

10. Construct the roots of the equation, $x^2 - 2ax + a^2 = 0$.

11. Construct the roots of the equation, $x^2 - 2ax = 0$.

33. *Scholium.* Since the numerical value of the expression of Ex. 3 is 2, the required construction might be performed, more simply than by the method of § 21, by drawing a line twice as long as the unit of length. In the construction of the results of actual algebraic processes, the student should in each case use the method which is most convenient in that case; but the examples of this chapter are to be solved by the geometrical, and not by the arithmetical, process.

34. *Scholium.* Expressions for which no geometric method of construction has been devised may be interpreted by first obtaining their numerical values. Thus, to construct $\sqrt[3]{abc}$; find, by actual measurement, the numerical values of a, b, and c, multiply them together, and draw a line the ratio of which to the unit is equal to the cube root of their continued product.

35. *Scholium.* To construct a homogeneous expression of the second degree, we may, if no simpler method occurs, divide the

expression by some letter, as o (thus reducing it to the first degree), construct the quotient, and find the surface which is equivalent to the product of the resulting line and that represented by o. Homogeneous expressions of the third and of the zero degrees may be constructed in like manner.

36. Examples.

1. Construct the angle of which the cosine is $\frac{c\,g}{c^2 + g^2}$.

Solution. We might construct the line $\frac{c\,g\,o}{c^2 + g^2}$, then draw a right triangle having this line for one leg and o for the hypothenuse, and the included angle would be the angle required, since its cosine is $\frac{c\,g\,o}{(c^2 + g^2)\,o} = \frac{c\,g}{c^2 + g^2}$.

The following, however, is a somewhat more elegant construction. Let

$$\frac{c\,g}{\sqrt{(c^2 + g^2)}} = m,$$

then

$$\frac{c\,g}{c^2 + g^2} = \frac{c\,g}{\sqrt{(c^2 + g^2)}\sqrt{(c^2 + g^2)}} = \frac{m}{\sqrt{(c^2 + g^2)}}.$$

Then, in Fig. 14,

$$AB = c,\ \ BC = g,\ \ AC = \sqrt{(c^2 + g^2)},$$

$$AD = g,\ AE : AD = AB : AC,\ AE = \frac{c\,g}{\sqrt{(c^2 + g^2)}} = m;$$

$$AF = AC = \sqrt{(c^2 + g^2)}, \cos EAF = \frac{AE}{AF} = \frac{m}{\sqrt{(c^2 + g^2)}};$$

so that EAF is the required angle.

2. Construct the regular hexagon $\frac{a\,b^2}{c}$, in which $a = 2$, $b = 5$, $c = 4$.

Solution. Find the parallelogram $\frac{a\,b^2}{c}$; then, by Geom. § 214, draw any regular hexagon, *ABC*, &c. (Fig. 15); inscribe a circle within it by Geom. § 229; and the polygon drawn by Geom. §§ 290, 292, 295, equivalent to the parallelogram $\frac{a\,b^2}{c}$, and similar to *ABC*, &c., will be the required hexagon. This method may be abbreviated as follows: — Let m denote one side of the given hexagon; then $6\,m$ will be its perimeter; and, if r denotes the radius of its inscribed circle, its area will be (Geom. § 278) $3\,m\,r$. By Geom. § 268, the given hexagon and that required are to each other as the squares of their homologous sides; i. e. if x is one side of the required hexagon,

$$3\,m\,r : \frac{a\,b^2}{c} = m^2 : x^2,$$

or

$$x^2 = \frac{a\,b^2\,m^2}{3\,c\,m\,r} = \frac{a\,b^2\,m}{3\,c\,r},$$

$$x = \sqrt{\frac{a\,b^2\,m}{3\,c\,r}}.$$

Then, taking a quarter of an inch for the unit, we have (Fig. 15),

$$HI = c,\ HK = a,\ HL = b,\ HM = \frac{a\,b}{c},$$

$$HN = 3 \times OG = 3\,r,\ HP = \frac{HM \times HL}{HN} = \frac{a\,b^2}{3\,c\,r};$$

$$QH = AB = m,\ HR = \sqrt{(QH \times HP)} = \sqrt{\frac{a\,b^2\,m}{3\,c\,r}}.$$

HR is then the value of x; and *AB'C'D'E'F'*, whose sides are each equal to *HR*, and parallel respectively to those of *ABCDEF*, is the required regular hexagon.

3. Construct the right triangle $\sqrt{(a^4 - b^4)}$, in which $a = 5$, $b = 3$.

4. Construct the square $a\,b$; also the regular hexagon $a\,b$.

5. Construct the angle of which the tangent is $\frac{a^2 + b^2}{a^2 - b^2}$.

6. Construct the angle of which the cosecant is a^0.

7. Construct the angle $\frac{(a^2\,b\,c)^{\frac{1}{2}}}{a\,b - c^2}$, in which $a = 1$, $b = 5$, $c = 2$.

37. *Corollary.* If, in an expression of the form of § 21, c is equal to the unit of length, we have

$$\frac{a\,b}{c} = \frac{a\,b}{1} = a\,b;$$

so that the fourth proportional to c, a, and b, though necessarily a *line*, is represented by $a\,b$, an expression of the *second degree*. Thus, when the unit of length (represented algebraically by 1) is so involved in the conditions of any question as to enter into its algebraic result as a factor or divisor, an apparent exception to the general rule of § 15 arises; and, on the other hand, when such an exception occurs, it may be explained by supposing the disappearance of some (positive or negative) power of the factor 1. Hence, an expression of any degree, even if not homogeneous, may be constructed as either an angle (or trigonometric function), a line, a surface, or a solid, by introducing 1 as a factor or divisor into each term as many times as may be necessary to raise it to the zero, the first, the second, or the third degree.

38. EXAMPLES.

1. Construct the line $a\,g$.

Solution. If o is taken for the unit of length, we have

$$a\,g = \frac{a\,g}{1} = \frac{a\,g}{o},$$

which, being of the first degree, may be constructed as a line.

If twice o is taken for the unit, we have

$$a\,g = \frac{a\,g}{1} = \frac{a\,g}{2\,o} = \frac{1}{2}\cdot\frac{a\,g}{o};$$

so that the line now obtained is one half as large as the former value of $a\,g$. The same is also true if $a\,g$ is constructed arithmetically. Thus, if a is twice as long as o, and g one third as long, then, taking o as the unit, we have

$a = 2, g = \frac{1}{3}, a\,g = 2 \times \frac{1}{3} = \frac{2}{3}$, i. e. $\frac{2}{3}$ of the unit, or $\frac{2}{3}\,o$.

But if $2\,o$ is the unit,

$a = 1,\ g = \frac{1}{6},\ a\,g = 1 \times \frac{1}{6} = \frac{1}{6}$, i. e. $\frac{1}{6}$ of the unit, or $\frac{1}{6} \times 2\,o = \frac{1}{3}\,o = \frac{1}{2}\,(\frac{2}{3}\,o)$.

2. Construct the square $\frac{a^4 - b}{c + d^2}$.

This expression, made homogeneous of the second degree, becomes $\frac{a^4 - b\,(1)^3}{c\,(1) + d^2}$.

3. Construct the line $\sqrt[4]{a}$, in which $a = 10$.

4. Construct the equilateral triangle $\frac{1}{a}$.

5. Construct the line $\sqrt{(c - c^2)}$, first taking the unit greater than c, then less than c.

6. Construct the angle of which the cotangent is

$$\frac{a}{b\,c} + \sqrt{d - e^2 - 6}.$$

39. *Scholium.* The principles of Analytic Geometry are sometimes advantageously applied to the solution of problems in Algebra and Arithmetic. Thus, the methods of §§ 24, 25, 26, 27, 37 may be used for extracting the square roots of numbers, as in the examples of the following section. Observe that, while in theory this method is perfect, it is practically only an approximative one, since it is impossible to avoid some degree of error in the drawing of the requisite figure.

40. Examples.

1. Find the fourth root of 10 by geometric construction.

Solution.

$$\begin{aligned}\sqrt[4]{10} &= \sqrt[4]{(5\times 2)} = \sqrt{[\sqrt{5}\times\sqrt{2}]}\\ &= \sqrt{[\sqrt{(9-4)}\times\sqrt{(1+1)}]}\\ &= \sqrt{[\sqrt{(3^2-2^2)}\times\sqrt{(1^2+1^2)}]}.\end{aligned}$$

In Fig. 16, we have (taking half an inch for the unit)

$$\begin{aligned}&AB = AC = 1,\ BC = \sqrt{(1^2+1^2)} = \sqrt{2};\\ &DB = 2,\ DE = 3,\ EB = \sqrt{(3^2-2^2)} = \sqrt{5};\\ &BF = \sqrt{(EB\,.\,BC)} = \sqrt{[\sqrt{5}\times\sqrt{2}]} = \sqrt[4]{10}.\end{aligned}$$

The length of BF, applied to the scale of equal parts, is found to be 1.775; and the fourth root of 10, extracted by logarithms, is 1.778.

2. Find the square root of 2; of 5; of 6; of 19; of 27.

Ans. 1.414; 2.236; 2.449; 4.359; 5.196.

3. Find the fourth root of 6; of 27; of 15; of 21; of 19.

Ans. 1.565; 2.279; 1.968; 2.141; 2.088.

N.B. It will be seen that the square root of any number may be constructed as the third side of a right triangle, of which the hypothenuse and one leg are respectively the halves of the numbers next above and next below the given number. Thus

$$\sqrt{5} = \sqrt{\left[\left(\tfrac{6}{2}\right)^2 - \left(\tfrac{4}{2}\right)^2\right]};\quad \sqrt{6} = \sqrt{\left[\left(\tfrac{7}{2}\right)^2 - \left(\tfrac{5}{2}\right)^2\right]};$$

and, in general,

$$\begin{aligned}&\sqrt{\left[\left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2\right]}\\ &= \sqrt{\left[\frac{(a^2+2a+1)-(a^2-2a+1)}{4}\right]} = \sqrt{a}.\end{aligned}$$

CHAPTER III.

DETERMINATE PROBLEMS.

41. We have been led in Algebra to divide problems into two classes: *determinate problems*, or those which give as many equations as unknown quantities, and *indeterminate problems*, or those which give fewer equations than unknown quantities. (Alg. arts. 138, 143, 144.)

42. Determinate geometric problems are solved by the ordinary rules of Algebra. Having adopted for each problem a notation which is conformable to the principles of the preceding chapter and to the usage of Algebra, put into the form of equations the expressed conditions of the question, and the implied geometric relations which exist between its known and unknown quantities; solve the equations; and construct the values of the unknown quantities, choosing that form of construction which is the most direct, the neatest, or the best adapted to the particular case. The difficulties which occur are chiefly practical and special ones, for which no general rule can be given.

43. Examples.

1. To inscribe in a given triangle a rectangle, with its base and altitude in a given ratio.

Solution. Let the triangle be ABC, and HA its altitude; let $DEFG$ represent the required rectangle; and let the following notation be adopted:

$$BC = b, \qquad HA = h, \qquad DE = x,$$

$m : n =$ the given ratio of the base of the required rectangle to its altitude.

Then we have

$$IA = HA - HI = h - x,$$

$$(GD = FE) : (DE = x) = m : n,$$

$$FE = \frac{m\,x}{n}.$$

ABC and AFE, being (Geom. § 172) similar triangles, give (Geom. § 199)

$$BC : FE = HA : IA,$$

or

$$b : \frac{m\,x}{n} = h : h - x;$$

which gives for the value of x

$$x = \frac{b\,n\,h}{b\,n + m\,h} = \frac{b\,h}{b + \frac{m\,h}{n}}.$$

Construction. Let ABC in Fig. 18 represent the same triangle as in Fig. 17. Produce BC to K, making CK equal to $\frac{m\,h}{n}$, constructed by § 21. Draw AL and KL, parallel respectively to BK and CA. Drop on BK the perpendicular LM, draw BL, and from E, its point of intersection with CA, drop ED perpendicular to BK. Then

$$BK = b + \frac{m\,h}{n}, \; ML = h,$$

and the similar triangles LBK and EBC give

$$BK : BC = ML : DE,$$

or

$$b + \frac{m\,h}{n} : b = h : DE,$$

$$DE = \frac{b\,h}{b + \frac{m\,h}{n}} = x;$$

so that $DEFG$ is the rectangle required.

Scholium. In order to guard against any possible confusion between the drawing of the rectangle *required* (that is, $DEFG$ in Fig. 18) and the drawing of a rectangle (that is, $DEFG$ in Fig. 17) *representing* the required one, to aid in *forming equations* between the known and unknown quantities, I have given separate figures for the Solution and the Construction. In future problems only one figure will be used; but the student must observe, that *to suppose* the required parts drawn, for the purpose of effecting a solution, does not involve a knowledge of the method by which those parts may *actually* be drawn, which is the result of the solution. Indeed, it will be seen, on comparing Figs. 17 and 18, that the rectangle in the former figure, though it answers all the purposes of that figure, is not the rectangle required.

Corollary. If $m = n$, the required rectangle is a square; and, in that case, we have (Fig. 18)

$$CK = \frac{m\,h}{n} = h = HA.$$

Corollary. Since b and h are the only parts of the given triangle which enter into the value of x, it follows that the rectangles inscribed in triangles of equal bases and altitudes are equal. If, however, either of the angles at the base of the triangle is obtuse (Fig. 19), the rectangle falls partly without the triangle; so that, unless we give to the word *inscribe* a larger meaning than it commonly has, the problem has a limitation which is not indicated in the algebraic value of x. (Compare § 29.) This is because the value of x depends on the similarity of ABC and AFE, and these triangles are still similar in Fig. 19.

2. To divide a given straight line into two such parts that the difference of the squares described on the two parts may be equal to a given surface.

Solution. Let AB (Fig. 20) be the given line, E being its middle point; let the given surface be twice the square described on AC; and let D represent the required point of division. Let

$$AE = EB = a, \qquad AC = b, \qquad ED = x.$$

We have then

$$AD = a + x, \qquad DB = a - x;$$
$$(a + x)^2 - (a - x)^2 = 4\,a\,x = 2\,b^2;$$
$$x = \frac{b^2}{2\,a}.$$

Construction. Draw EB' in any direction whatever, and take

$$EB' = AB = 2\,a,$$
$$EC' = EC'' = AC = b.$$

Join $B'C'$, and draw $C''D$ parallel to it. Then we have

$$ED : EC' = EC'' : EB';$$
$$ED = \frac{b^2}{2\,a} = x;$$

so that D is the required point of division.

Corollary. When the given surface is less than the square on the given line, we have

$$2\,b^2 < (2\,a)^2,$$
i. e. $$2\,b^2 < 4\,a^2,$$
i. e. $$b^2 < 2\,a^2,$$
i. e. $$\frac{b_2}{2\,a} < a,$$
i. e. $$x < a,$$

i. e. the point D falls somewhere between E and B.

When the given surface is equal to the square on the given line,

$$2\,b^2 = (2\,a)^2,$$
i. e. $$x = a,$$

i. e. the point D falls at B, and the greater of the required parts is the whole line, and the less is nothing.

When the given surface is greater than the square on the given line,

$$2\,b^2 > (2\,a)^2,$$
$$\text{i. e.} \quad x > a,$$

i. e. the point of division falls beyond the extremity of the line, at D', for instance. In this case, the required parts may be taken to be AD' and $D'B$, the sum of which is, by § 20. c,

$$AD' + D'B = AD' - BD' = AB;$$

and it is evident, from inspection, that the difference of the squares on AD' and $D'B$ is greater than the square on the given line.

3. To divide a given straight line into two such parts that the sum of the squares described on the two parts may be equal to a given surface.

Solution. Adopting for Fig. 21 the notation of the last example, we have

$$(a + x)^2 + (a - x)^2 = 2\,a^2 + 2\,x^2 = 2\,b^2,$$
$$x = \pm \sqrt{(b^2 - a^2)}.$$

Construction. Erect at E a perpendicular to AB; from A as a centre, with a radius b, describe an arc cutting the perpendicular at F; EF is the value of x, and may be laid off positively or negatively from E, giving two possible positions for D.

Corollary. When the given surface is greater than the square on the given line, we have

$$2\,b^2 > (2\,a)^2,$$
$$\text{i. e.} \quad 2\,b^2 > 4\,a^2,$$
$$\text{i. e.} \quad b^2 > 2\,a^2,$$
$$\text{i. e.} \quad b^2 - a^2 > a^2,$$
$$\text{i. e.} \quad x^2 > a^2,$$
$$\text{i. e. (absolutely)} \quad x > a,$$

i. e. the required point falls beyond the limits of the line. This case admits of an interpretation like that applied to the corresponding case in the last example.

When the given surface is equal to the square on the given line,

$$2\,b^2 = (2\,a)^2,$$

i. e. $$x^2 = a^2,$$

i. e. $$x = \pm\, a,$$

i. e. the point of division falls at B or A.

When the given surface is less the square on the given line, but greater than twice the square on half of the given line,

$$2\,b^2 < (2\,a)^2, \qquad 2\,b^2 > 2\,a^2,$$

i. e. $$b^2 < 2\,a^2, \qquad b^2 > a^2,$$

i. e. $$b^2 - a^2 < a^2, \qquad b^2 - a^2 > 0,$$

i. e. $$x^2 < a^2, \qquad x^2 > 0,$$

i. e. (absolutely) $x < a$, and is real,

i. e. the point of division falls between E and B or between E and A.

When the given surface is equal to twice the square on half of the given line,

$$2\,b^2 = 2\,a^2,$$

i. e. $$x^2 = 0,$$

i. e. $$x = \pm\, 0,$$

i. e. D falls at E, and the parts of the line are its two halves.

When the given surface is less than twice the square described on half of the given line,

$$2\,b^2 < 2\,a^2,$$

i. e. $x^2 < 0$, i. e. is negative,

i. e. x is imaginary, i. e. the problem is, by § 28, impossible.

4. To describe the circumference of a circle through two given points and tangent to a given line.

Solution. Let A and B (Fig. 22) be the given points, and CD the given line. If we can find the point of contact of the required circumference with CD, the problem can be solved by Geom. §§ 148, 149. Draw BA, meeting CD at E; let F represent the point of contact, and let

$$EA = a,\ EB = b,\ EF = x.$$

We have, by Geom. § 191,

$$EA : EF = EF : EB,$$
$$a : x = x : b,$$
$$x = \pm \sqrt{(ab)}.$$

Construction. A construction better adapted to this case than that of § 24 is the following. Describe any circumference through A and B; through E draw a tangent EG to it, by Geom. § 150. EG is a mean proportional between EA and EB, and F is found by laying off EG on either side of E, so that there are two circles which answer the required conditions.

Corollary. For all positions of the given points on the same side of the given line, a and b have the same sign, either positive or negative, and the problem admits of solution.

If A and B coincide (Fig. 23), the problem is reduced to drawing a circumference through one given point and tangent to a given line, and we have

$$a = b,$$
$$x = \pm \sqrt{(ab)} = \pm \sqrt{a^2} = \pm a;$$

so that EF must be taken equal to EA, and the required centre O will be at the intersection of perpendiculars to BE and CD, erected respectively at B and F. But since the line BA may now have any direction whatever, the point E may be anywhere in the line CD; so that, in this case, the problem becomes indeterminate, admitting of an infinite variety of solutions.

If either point, as A, is in CD (Fig. 24), the problem is reduced to drawing a circumference through a given point and tangent to a given line at a given point, and we have

$$a = 0,$$
$$x = \pm \sqrt{(ab)} = \pm 0;$$

so that both the points F fall at E, i. e. at A, and, in this case, there is only one solution. The centre O is at the intersection of perpendiculars erected to CD at A, and to AB at its middle point.

If the given points are on different sides of the given line, a and b have opposite signs, so that their product is negative, and the value of x imaginary. In this case, therefore, the problem is, by § 28, impossible.

5. To draw a common tangent to two given circles.

Solution 1*st.* Let the given circles be those described about O and O' (Fig. 25), and let MT represent the required tangent. If we find the point T, the tangent can be drawn by Geom. § 150. Let

$$OM = r, \quad O'M' = r', \quad OO' = a, \quad OT = x.$$

Since (Geom. §§ 120, 174) the triangles OMT and $O'M'T$ are similar, we have

$$OT : O'T' = OM : O'M',$$
$$x : x - a = r : r',$$
$$x = \frac{a\,r}{r - r'}.$$

Construction. Draw any two parallel radii, ON and $O'N'$; draw $NN'T$, and T is the required point. For if we draw $N'D$, parallel to $O'O$, and cutting ON at D, we have

$$DN' = OO' = a, \quad OD = O'N' = r', \quad DN = r - r',$$
$$DN : DN' = ON : OT,$$
$$r - r' : a = r : OT,$$
$$OT = \frac{a\,r}{r - r'} = x.$$

Solution 2*d.* The common tangent may also be drawn, as in Fig. 26, so that the points of contact shall be on opposite sides of OO'. Adopting the same notation as before, except that for r' we will take $M'O'$, instead of $O'M'$, (because $O'M'$ is now oppositely directed to OM,) we have

$$OT : T'O' = OM : M'O',$$
$$x : a - x = r : r',$$
$$x = \frac{a\,r}{r + r'}.$$

Construction. Draw any two oppositely directed radii, ON and $O'N'$. NN' will cut OO' at the required point T. For if we draw $N'D$ parallel to $O'O$, and cutting NO produced at D, we have

$$DN' = OO' = a,\quad DO = N'O' = r',\quad DN = r + r',$$
$$DN : DN' = ON : OT,$$
$$r + r' : a = r : OT,$$
$$OT = \frac{a\,r}{r + r'} = x.$$

Corollary. Since

$$r + r' > r,$$

we have, in the second value of x,

$$\left(x = a\frac{r}{r + r'}\right) < \left(a\frac{r}{r} = a\right);$$

so that, since x is, in this expression, necessarily positive, the point must lie between O and O'. When $r' < r$, we have

$$r + r' < 2\,r,$$
$$\frac{a\,r}{r + r'} > \frac{a\,r}{2\,r},$$
$$x > \tfrac{1}{2}\,a;$$

so that T will lie nearer to O' than to O. When $r' = r$, $x = \frac{1}{2}\,a$; so that T will lie half-way between O and O'. When $r' > r$, $x < \frac{1}{2}\,a$; so that T will lie nearer to O than to O'.

For the first value of x,

$$r - r' < r.$$

Then, when $r' < r$, so that $r - r'$ is positive,

$$\frac{a\,r}{r - r'} > \frac{a\,r}{r},$$
$$x > a;$$

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so that T must lie beyond O'; and, the greater the value of r' (r and a remaining the same), the greater the value of x, till, when $r' = r$, so that $r - r' = \pm\ 0$,

$$x = \frac{a\,r}{r - r'} = \frac{a\,r}{\pm\ 0} = \pm\ \infty;$$

that is, the distance OT becomes infinitely great. If, now, r' is made still larger, so that $r - r'$ becomes negative, $\frac{a\,r}{r - r'}$, or x, becomes negative; i. e. T reappears on the left of O, and, the greater the value of r' (r and a remaining the same), the greater the absolute value of $r - r'$, and the less the absolute value of x; i. e. the nearer does T approach O. Thus, as $r - r'$ changes its sign by undergoing gradual diminution, and passing through the value *zero*, x simultaneously changes its sign, but by undergoing gradual increase, and passing through the value *infinity*. It seems as if, during the increase of the circle round O', the point T is moving off from O towards the right, till, when $r' = r$, it vanishes at an infinite distance on the right, and, at the next moment, reappears, still moving towards the right, but from an infinite distance on the left (mark the force of the double sign in $x = \pm\ \infty$) and towards the centre O. This mode of changing the sign is frequent in Analytic Geometry. Its *algebraic* explanation is that (Alg. art. 124)

$$+\,0 = -\,0,$$

therefore

$$\infty = \frac{1}{+\,0} = \frac{1}{-\,0} = -\,\infty.$$

In *Geometry*, this leads to the inconceivable result that a point which moves positively to an infinite distance reaches the same position as one which moves negatively to an infinite distance; but when this paradox arises, it admits of *indirect* explanation. Thus, in the present case (compare Fig. 25), it is evident that, as r' increases, the mutual inclination of the lines MM' and OO'

becomes less and less, and their point of meeting at a greater and greater distance; when $r' = r$, these lines become parallel, and may be conceived as meeting infinitely far off, either on the right or on the left; and when $r' > r$, the tangent slants the other way, and intersects OO' on the left of O.

Corollary. Either value of x admits of construction for all values of a, r, and r'. But the drawing of the tangent is impossible, if T falls within either of the circles. This is the case, for the first method, when

$$r' < r, \text{ and } x < a + r',$$

or

$$r' > r, \text{ and } x > -r,$$

and, for the second, when

$$r' < r, \text{ and } x > a - r',$$

or

$$r' > r, \text{ and } x < r.$$

The discussion of these cases is left to the student.

CHAPTER IV.

INTRODUCTION TO INDETERMINATE GEOMETRY.

44. SUCH problems as those of the last chapter readily admit of solution by analysis, but, as it has been remarked in § 5, the most important problems of Geometry are of a kind to which Algebra cannot directly be applied. Of this class of problems the following are examples: — *To determine a line which is throughout at a given distance from a given straight line;* and, *To determine a line which contains the centres of all circles which can be drawn tangent to two given circles.* What is required, in each of these cases, is *the form of a line*, drawn under certain conditions; and, since this is something which cannot (directly at least) be represented by *the value of an unknown quantity*, the first step in the algebraic solution of either problem cannot be taken.

45. To solve the above problems by Algebra, we change their statement without altering their meaning, so that they shall read as follows: — *To find a point which is at a given distance from a given straight line;* and, *To find the centre of a circle tangent to two given circles.* What is now directly required is not *the form of a line*, but *the position of a point* which must lie in the line that was originally to be found, and by which, therefore, we can determine it. Then, supposing that we

have the means of expressing in algebraic language the position of a point, it will be found in the case of the first problem (by methods presently to be explained) that the required point must be so situated that its direction from any point at the given distance from the given straight line is the same as that of this line itself; that is, (Geom. §§ 17, 27,) that the line which we ultimately seek to determine must be a straight line passing through a point at the given distance from the given straight line, and parallel to it. Likewise, the solution of the second problem will show that the required centre must lie, according to the relative position and magnitude of the given circles, in some one or other of a class of curves hereafter to be investigated, which, when the given circles are concentric, will be the circumference of another circle also concentric with them and of a radius equal to the arithmetical mean (Alg. art. 252) of their radii.

46. Thus, all such problems as relate to *the forms of lines* are, in Analytic Geometry, resolved into problems relating to *the positions of points*. It is to be remarked, however, that, while this change has no effect on the meaning of a problem, it makes a difference in its character. Either of the above examples, as stated in § 44, is, in a geometrical sense, determinate, since the line sought is fixed by the conditions both in form and in position; but when the same problem is stated as in § 45, we are required to find a point which is, indeed, restricted to a certain line, but of which the position is not absolutely fixed, since it may lie anywhere in that line, and so may occupy any one of an infinite series of places. Thus, in its final statement, the problem becomes, in a geometri-

cal sense, partially *indeterminate*, and it will also be found to be indeterminate in the technical algebraic sense of giving fewer equations than unknown quantities. We see, then, that Analytic Geometry renders indeterminate those questions which concern the forms of lines; so that, in this application of Algebra, the investigation of indeterminate problems acquires a high degree of importance.

47. It appears from §§ 44 – 46 that it is a necessary introduction to the study of Indeterminate Geometry (i. e. of the Geometry of *forms*, as distinguished from *magnitudes*), —

I. To explain the methods of denoting the position of a point by the magnitudes of lines and angles, so that it can be expressed algebraically;

II. To show how these methods enable us to deduce from an indeterminate equation the line which must contain every position of the undetermined point.

And I shall

III. Take up some general problems which it is convenient to bring together and investigate at the outset.

I.

POSITION.

48. We conceive of a point as having an absolute position, and also a position relative to that of other points; but it is only when understood in the latter sense that the position of a point is susceptible of algebraic expression. Moreover, all that is required, in the beginning,

is to denote algebraically the positions of points as compared with some one assumed point, for this will enable us to infer their positions as compared among themselves or with other points.

Thus, one ordinarily estimates and describes the position of objects about him by their distances and directions from himself; and the astronomer refers the positions of the heavenly bodies to that of the sun.

49. The words *Origin* and *Axis* are used, in Indeterminate Geometry, somewhat as in § 20; the former word denoting any assumed point to which positions are referred, the latter a line passing through the origin in any fixed direction. The positions of points are referred to the origin by means of certain lines and angles, the magnitude of which varies for different points, and which are called *Coördinates.* Thus the position of a point, as expressed in Analytic Geometry, depends partly on fixed, or constant elements, namely, the positions of the origin and axis or axes, and partly on variable elements, namely, the coördinates; and all these elements, in combination, constitute a *Coördinate System.*

We shall, at present, confine ourselves to points lying in the same plane; and this plane we may regard as given.

50. *Polar Coördinates.* If a straight line, indefinitely produced from the fixed point which is taken as the origin, be conceived to turn in a plane about that point, it will evidently, in the course of its revolution, sweep the whole plane, and strike every point in it. Now, if we suppose that the line may, as it turns, be extended or shortened at pleasure, we can make its extremity coincide

with any point in the plane, by turning the line from its primary position through the proper amount, and in the proper direction, and giving to it the proper length. Hence, the position of any point in the plane is *determined*, if we know the *angle* which measures this rotation (compare § 20. $d-l$), and the *distance* of the point from the origin. Thus, if A (Fig. 27) is the origin, and AR the primary direction of the revolving line, the positions of B', B'', and B''' are fully determined and denoted by the values of AB' and RAB', AB'' and RAB'', AB''' and RAB'''; and these values are called the *polar coördinates* of B', B'', and B'''.

The direction AR, from which the rotation is conceived to begin, is called the *axis*, and denoted by ϱ.

The revolving line is called the *radius vector*, and is denoted in general by r. The lengths which it has for particular points are called the radii vectores of those points, and are denoted by r^0, r', r'', &c.

The angle $\overset{r}{\underset{\varrho}{}}$ which the radius vector makes with the axis is called the *polar angle*.

Positive rotation is assumed, as in § 20, to be that represented by the hands of a watch, supposed to be seen from behind.

51. *Corollary.* For a point situated at the origin, we have $r = 0$; for one in the axis, $\overset{r}{\underset{\varrho}{}} = 0$.

52. *Corollary.* The coördinates of any point may be conceived to have either of two sets of values. If, for instance, B''' (Fig. 27) is situated on the same straight line as B' and A, we may take for its coördinates, either,

$$r''' = AB''', \qquad \genfrac{}{}{0pt}{}{r'''}{\rho} = RAB''' = \pi + \genfrac{}{}{0pt}{}{r'}{\rho},$$

or, making $B'''A$ the *positive* direction of r''',

$$r''' = AB''' = -B'''A, \qquad \genfrac{}{}{0pt}{}{r'''}{\rho} = RAB' = \genfrac{}{}{0pt}{}{r'}{\rho}.$$

But it agrees best with the conception of polar coördinates, and it is, besides, most convenient, to denote the direction of a point from the origin wholly by the value of the polar angle, taking the radius vector to express only the absolute value of the distance. *Hence, in polar coördinates, negative radii vectores, and negative lines in general, admit of no geometric construction.*

This rule is generally construed so as even to exclude the construction of negative values of r which result from an algebraic process. In such cases, however, it will be found of advantage to note what the corresponding points would be if the rule were disregarded.

53. *Corollary.* The angle $\genfrac{}{}{0pt}{}{r}{\rho}$ may, for any point, have any one of the series of values attributed to a single angle in § 20. *i.*

54. *Corollary.* The quantities r and $\genfrac{}{}{0pt}{}{r}{\rho}$ evidently come under the definition of the *variable;* while the values r', $\genfrac{}{}{0pt}{}{r'}{\rho}$, &c., which they have for particular points, are *constants.* (Alg. art. 164.)

55. *Problem.* To find the distance between two points in terms of polar coördinates.

Solution. Let B' and B'' (Fig. 28) be the two points, whose coördinates are respectively r', $\frac{r'}{\varrho}$, and r'', $\frac{r''}{\varrho}$. The triangle $B'AB''$ gives (Tr. § 87)

$$\overline{B'B''}^2 = \overline{AB'}^2 + \overline{AB''}^2 - 2\,AB'\,.\,AB''\,.\cos B'AB''$$

$$= r'^2 + r''^2 - 2\,r'r''\cos\left(\frac{r''}{\varrho} - \frac{r'}{\varrho}\right),$$

$$B'B'' = \sqrt{\left[r'^2 + r''^2 - 2\,r'r''\cos\left(\frac{r''}{\varrho} - \frac{r'}{\varrho}\right)\right]}. \quad (1.)$$

56. *Corollary.* If B' is in the axis, we have

$$\frac{r'}{\varrho} = 0,$$

and (1) becomes

$$B'B'' = \sqrt{\left[r'^2 + r''^2 - 2\,r'r''\cos\frac{r''}{\varrho}\right]}. \quad (2.)$$

57. *Corollary.* If B' is at the origin, we have

$$r' = 0,$$

and (1) becomes

$$B'B'' = \sqrt{r''^2} = r''. \quad (3.)$$

58. *Corollary.* If the two points are on the same radius vector, we have

$$\frac{r'}{\varrho} = \frac{r''}{\varrho},$$

and (1) becomes (Tr. § 55)

$$B'B'' = \sqrt{[r'^2 - 2\,r'r'' + r''^2]} = r' - r''. \quad (4.)$$

59. *Corollary.* If the two points are on opposite radii vectores, we have

$$\frac{r''}{\varrho} = \pi + \frac{r'}{\varrho},$$

and (1) becomes (Tr. § 56)

$$B'B'' = \sqrt{[r'^2 + 2\,r'r'' + r''^2]} = r' + r''. \quad (5.)$$

60. *Scholium.* Formulæ (2) - (5) are readily seen to be true, independently of (1); but they are here exhibited as special

cases under the general problem of § 55. This illustrates the remark of § 1, that the analytic treatment of Geometry, by taking up questions in their most comprehensive forms, establishes general relations between problems which would otherwise seem unconnected, and thus leads deeper into the true principles of the science.

61. *Scholium.* If, in § 55, we attend to the ambiguous sign of the square root (Alg. art. 197), we have (compare § 20. *b*)

$$\sqrt{\overline{B'B''}^2} = \pm B'B'' = B'B'' \text{ or } B''B';$$

and (1) becomes

$$B'B'' = \pm \sqrt{\left[r'^2 + r''^2 - 2\, r'r'' \cos\left(\frac{r''}{\varrho} - \frac{r'}{\varrho}\right)\right]},$$

or

$$B''B' = \pm \sqrt{\left[r'^2 + r''^2 - 2\, r'r'' \cos\left(\frac{r''}{\varrho} - \frac{r'}{\varrho}\right)\right]}.$$

This ambiguity, however, introduces no uncertainty into the solution of the problem; for it only expresses that the distance between B' and B'' may be measured by a line running from B' to B'', or by one running from B'' to B', and that either of these lines may be conceived at pleasure to run positively or negatively; but this sign may be neglected, since, in any case, the distance *between* the points (i. e. the absolute value of $B'B''$) is the same.

62. *Rectilinear Coördinates.* The method of Polar Coordinates is the most natural one, because it denotes the position of a point in terms of what are, in the simplest view, the elements of its position (Geom. § 8), its distance and direction from the origin. For some purposes, this method is to be preferred to others; but, usually, a method of denoting position, not by a line and angle, but by two lines, is found, in practice, to be most elegant and convenient.

Let A (Fig. 29) be the origin of a system of coördinates through which are drawn two axes, $X'X$ and $Y'Y$, at a certain arbitrary angle; and let the arrows denote the positive directions of those lines and of all lines parallel to them. It is evident that the position of any point is determined, if we know the distance from the origin at which each axis is cut by a line passing through the required point parallel to the other axis. Thus, B', B'', B''', B^{IV} are determined respectively by AP' and AR', AP'' and AR'', AP''' and AR''', AP^{IV} and AR^{IV}. $X'X$ is called *the axis of abscissas, or of* x, and $Y'Y$ *the axis of ordinates, or of* y.

The distance on the axis of x, as AP', or its equal $R'B'$, is called the *abscissa*, and is denoted, in general, by x. The distance on the axis of y, as AR', or its equal $P'B'$, is called the *ordinate*, and is denoted, in general, by y. The abscissas and ordinates of particular points are denoted respectively by x', x'', &c., and y', y'', &c.; and they are called the *rectilinear coördinates* of those points.

The positive directions of the axes must be taken so that XAY shall be positive, and less than π; and, in speaking generally of a rectilinear system, we commonly imagine ourselves so situated that the axis of x points to the right, and then the axis of y points upwards.

63. *Corollary.* Since the coördinates are measured on the axes *from* the origin, points on the left of the axis of ordinates, as B'', B''', have their abscissas, AP'', AP''', *negative;* and points below the axis of abscissas, as B''', B^{IV}, have their ordinates, AR''', AR^{IV}, *negative.* Thus, all points in the first division of the plane, XAY, have positive abscissas and positive ordinates; points in the second division, YAX', have negative abscissas and positive

ordinates; points in the third division, $X'AY'$, have negative abscissas and negative ordinates; and points in the fourth division, $Y'AX$, have positive abscissas and negative ordinates.

64. *Corollary*. For a point in the axis of y, as R', the abscissa is

$$x = AA = 0;$$

for a point in the axis of x, as P', the ordinate is

$$y = AA = 0;$$

for a point in both axes, i. e. at their intersection, the origin, both the above equations are true, and we have

$$x = 0, \quad y = 0.$$

Thus, as a point moves towards the left, its abscissa grows less and less; when it reaches the axis of y, x becomes 0; when it passes beyond the axis of y, x becomes less than 0, i. e. negative. A similar change takes place in the value of the ordinate of a point which moves downwards.

65. *Corollary*. We may form a conception of rectilinear coördinates similar to that of polar coördinates presented in § 50. Thus, suppose a line to move in such a manner as always to begin in the axis of x, and to be parallel to the axis of y, and suppose that it may be lengthened and shortened at pleasure, (may even become negative, i. e. may be directed downwards,) so that it can be made to end in any point of the coördinate plane. The moving line represents the *ordinate*, and is denoted by y, while the distance of its initial point from the origin represents the *abscissa*, and is denoted by x; and it is evident, from the nature of the case, that x and y are *variables*, but that x' and y', particular values which they must have to suit a particular point, are *constants*.

66. *Scholium.* It is generally most convenient to take rectilinear axes perpendicular to each other, and, in this case, the axes and the coördinates are called *Rectangular.* Rectilinear axes and coördinates, not perpendicular, are called *Oblique.* Some particular system of oblique coördinates is occasionally best suited to a particular problem; but where nothing is said to the contrary, x and y are to be understood to denote rectangular coördinates.

67. *Projection.* Connected with the subject of position is the theory of Projections, by which, when we conceive a point to be in motion, we refer its change of place to any axis, with the view of ascertaining how much the point advances or retrogrades in the direction of that axis.

By whatever course a point be conceived to pass from C (Fig. 30) to D, it moves in the direction AB by the same amount, which, if CC' and DD' are perpendiculars to AB, is represented by $C'D'$, and this amount is the same for a point moving from any position on the first perpendicular, to any position on the second.

C' and D' are called the *projections* of C and of D; and $C'D'$ the projection of CD (i. e. of any line which represents the motion of a point from C to D).

68. *Corollary.* The projection of a broken line is obviously the same as the sum of the projections of the straight lines which compose it. Thus (Fig. 30), the sum of the projections of CG, GH, and HD is (compare § 20. c)

$$C'G' + G'H' + H'D' = C'G' + G'H' - D'H' = C'D',$$

which is, by definition, the projection of $CGHD$.

69. *Theorem.* The projection of a straight line on an

axis is equal to the product of the line and the cosine of the mutual inclination of the line and the axis.

Proof. Let α denote the direction of the axis AX (Fig. 31), and let the line be $B'B''$, denoted by n. Draw $B'P'$ and $B''P''$ perpendicular to AX and $B'P_1$ parallel to it. Then

$$P_1 B'B'' = \genfrac{}{}{0pt}{}{n}{\alpha},$$

and the projection of n on α is (§ 20. k; Tr. § 32)

$$P'P'' = B'P_1 = n \cos \genfrac{}{}{0pt}{}{n}{\alpha} = n \cos \genfrac{}{}{0pt}{}{\alpha}{n}.$$

70. *Scholium.* Observe that, since $\cos \genfrac{}{}{0pt}{}{n}{\alpha} = \cos \genfrac{}{}{0pt}{}{\alpha}{n}$, it is indifferent in which direction the angle is estimated; and it is therefore called the *mutual* inclination of the line and the axis.

71. *Corollary.* If the projected line is *parallel* to the axis, the whole motion of the point which generates it is in the direction of the axis; and, in this case (Tr. § 55),

$$\genfrac{}{}{0pt}{}{n}{\alpha} = 0, \quad n \cos \genfrac{}{}{0pt}{}{n}{\alpha} = n.$$

If the direction of n is *with* that of α, that is, if $\genfrac{}{}{0pt}{}{n}{\alpha}$ is between 0 and $\frac{1}{2}\pi$, or between $\frac{3}{2}\pi$ and 2π, the projection $n \cos \genfrac{}{}{0pt}{}{n}{\alpha}$ is (Tr. § 67) positive.

If the direction of n is *transverse* to α (Tr. §§ 55, 57),

$$\genfrac{}{}{0pt}{}{n}{\alpha} = \tfrac{1}{2}\pi \text{ or } = \tfrac{3}{2}\pi, \qquad n \cos \genfrac{}{}{0pt}{}{n}{\alpha} = 0.$$

If the direction of n is *against* that of α, $\genfrac{}{}{0pt}{}{n}{\alpha}$ is between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, and $n \cos \genfrac{}{}{0pt}{}{n}{\alpha}$ is (Tr. §§ 62, 66) negative.

If the direction of n is *opposite* to α (Tr. § 56),

$$\genfrac{}{}{0pt}{}{n}{\alpha} = \pi, \qquad n \cos \genfrac{}{}{0pt}{}{n}{\alpha} = -n.$$

72. *Corollary.* Let AX and AY (Fig. 31) be the axes in a rectangular system. Then, by § 20. g,

$$\frac{y}{n} = \frac{x}{n} + \frac{y}{x} = \frac{y}{x} - \frac{n}{x} = \tfrac{1}{2}\pi - \frac{n}{x},$$

and $R'R''$, the projection of $B'B''$ on AY, is, by §§ 69, 70, and Tr. § 8,

$$R'R'' = n \cos \frac{y}{n} = n \cos\left(\tfrac{1}{2}\pi - \frac{n}{x}\right) = n \sin \frac{n}{x}.$$

73. *Corollary.* The projection of a line on either of two rectangular axes is evidently equal to the difference of the abscissas or of the ordinates of its extremities. Thus, if the coördinates of B' and B'' (Fig. 31) are, respectively, x', y', and x'', y'', we have

$$x'' - x' = n \cos \frac{n}{x}, \qquad y'' - y' = n \sin \frac{n}{x}. \qquad (6.)$$

74. *Corollary.* Since (Tr. § 13)

$$\left(n \cos \frac{n}{x}\right)^2 + \left(n \sin \frac{n}{x}\right)^2 = n^2\left(\cos^2 \frac{n}{x} + \sin^2 \frac{n}{x}\right) = n^2,$$

the square of a line is equal to the sum of the squares of its projections on two rectangular axes.

75. *Problem.* To find the distance between two points in terms of rectangular coördinates.

Solution. Let the points be B' and B'' (Fig. 31). §§ 73, 74 give

$$\overline{B'B''}^2 = \left(n \cos \frac{n}{x}\right)^2 + \left(n \sin \frac{n}{x}\right)^2 = (x'' - x')^2 + (y'' - y')^2,$$

$$B'B'' = \sqrt{[(x'' - x')^2 + (y'' - y')^2]}. \qquad (7.)$$

76. *Corollary.* If one of the points B' is the origin, we have

$$x' = 0, \qquad y' = 0,$$

$$B'B'' = AB'' = \sqrt{(x''^2 + y''^2)}.$$

77. *Scholium.* Formula (7) applies, even if B' is so situated that $x' > x''$, or $y' > y''$; for

$$(x'' - x')^2 = (x' - x'')^2, \qquad (y'' - y')^2 = (y' - y'')^2.$$

It applies, also, to two points, such as B' and B''', situated in different quarters of the plane. For $P'''P'$ is still the difference of the abscissas of B' and B''', since $x''' = AP'''$, so that

$$P'''P' = P'''A + AP' = -x''' + x' = x' - x'''.$$

(7), therefore, though obtained for two particular points, is perfectly general in its application.

78. Examples.

1. Find the points of which the polar coördinates are

$$r = 3, \quad \genfrac{}{}{0pt}{}{r}{\rho} = \tfrac{1}{3}\pi, \quad \text{and} \quad r = 4, \quad \genfrac{}{}{0pt}{}{r}{\rho} = \tfrac{5}{6}\pi;$$

and calculate their distance apart.

Answer. Distance = 5.

2. Find the points of which the polar coördinates are

$$r = 8, \quad \genfrac{}{}{0pt}{}{r}{\rho} = \tfrac{3}{4}\pi, \quad \text{and} \quad r = 3, \quad \genfrac{}{}{0pt}{}{r}{\rho} = \tfrac{13}{12}\pi;$$

and calculate their distance apart.

Answer. Distance = 7.

3. Find the points of which the polar coördinates are

$$r = 3, \quad \genfrac{}{}{0pt}{}{r}{\rho} = \tfrac{5}{8}\pi, \quad \text{and} \quad r = 5, \quad \genfrac{}{}{0pt}{}{r}{\rho} = -\tfrac{3}{8}\pi;$$

and calculate their distance apart.

Answer. Distance = 8.

4. Find the points of which the rectangular coördinates are

$$x = 5, \quad y = 0, \quad \text{and} \quad x = 0, \quad y = -12;$$

and calculate their distance apart.

Answer. Distance = 13.

5. Find the points of which the rectangular coördinates are

$$x = 5, \quad y = -10, \quad \text{and} \quad x = -1, \quad y = -2;$$

and calculate their distance apart.

Answer. Distance = 10.

6. Find the points of which the rectangular coördinates are

$$x = b, \quad y = -3\sqrt{ab}, \quad \text{and} \quad x = a, \quad y = -\sqrt{ab};$$

and calculate their distance apart.

Answer. Distance $= a + b$.

79. *Scholium.* A point is said to be *given*, when, in any system, its coördinates are given, because, in that case, its position can be found. The point whose coördinates are r and $\frac{r}{\varrho}$, or x and y, is often designated as the point r, $\frac{r}{\varrho}$, or x, y.

II.

CONSTRUCTION OF EQUATIONS.

80. In either of the systems which have been explained, two coördinates are necessary to determine the position of a point in a given plane. This results from the fact that a plane has two dimensions, and is true of any plane coördinate system whatever. If, however, one coördinate alone is given, since more is known of the point than when neither is given, we shall find that its position is, as we might expect, not indeed fully determined, but still somewhat restricted. In a system of polar coördinates, if a value is assigned to $\frac{r}{\varrho}$, we know the direction of the corresponding point from the origin, and therefore the straight line which contains it. Thus, in Fig. 32, where A is the origin and AR the axis, if $\frac{r}{\varrho} = \frac{1}{3}\pi$, the point

must be some point, as A, B, C, D, E, &c., in the line AE drawn at an angle of 60° with AR, and may be any point in it. But if r is given, we know the distance of the point from the origin, and therefore that it lies in the circumference of the circle which has the origin for its centre and a radius equal to the radius vector. Thus (Fig. 32), if $r = \frac{3}{2} = AF$, the required point must be some point, as F, G, C, H, I, K, &c., and may be any point, in the circumference FGC, &c. Again, in a rectilinear system, a value assigned to x restricts the corresponding point to a straight line parallel to the axis of y, and at a distance from it (measured on the axis of x) equal to the given abscissa; and a value assigned to y restricts it to a straight line parallel to the axis of x, and at a distance from it (measured on the axis of y) equal to the given ordinate. Thus, taking A (Fig. 33) for the origin, and $X'X$ and $Y'Y$ for the axes, if $x = 1 = AP$, the point must be some point, as B, C, P, D, &c., and may be any point, in BD; or, if $y = -\frac{1}{4} = AR$, it must be some point, as E, R, C, F, &c., and may be any point, in EF. Hence, if *one* of the coördinates of a point is given, the point is thereby restricted to a certain *line;* and this is true of any system in a plane. We may, then, conceive the position of a point to be determined by the *intersection* of the two lines which correspond respectively to the values of its two coördinates. For example, if we have $r = \frac{3}{2}$, $\overset{r}{\varrho} = \frac{1}{3}\pi$, the point will be C (Fig. 32), or the intersection of the circumference FGC, &c. and AE. Or, if we have $x = 1$, $y = -\frac{1}{4}$, the point will be C (Fig. 33), or the intersection of BD and EF.

81. Since questions can be solved by Algebra only when they give as many equations as unknown quantities, a problem regarding position which gives two equations between the coördinates of the required point, without other unknown quantities, is *determinate*, and is satisfied by only one point, or, in case of equations of a higher degree than the first, a definite number of points; but if it gives only one equation between the coördinates, it is *indeterminate*, and, by assuming different values of one coördinate, we can calculate different values of the other; so that the problem is satisfied by any one of *an indefinite series of points*. This series of points is, in general, arranged in a line, the form of which depends on the algebraic form of the equation.

Suppose that, in a system of rectangular coördinates (Fig. 35), a line moves in the manner described in § 65, but that, instead of being lengthened and shortened at pleasure, it so varies, as it moves, that between its length and its distance from the origin a certain relation always exists, expressed by the equation $x^2 = 4\,p\,y$, in which p denotes a known length, as $\frac{1}{4}$ inch. It is obvious that, for every new position of the line (i. e. for every new value of x), it will have a new length (i. e. a new value of y), determined by the equation, and, moreover, that, as it slips along the axis of x, it will vary only gradually, as an abrupt change would disturb the prescribed equation. Thus, let $P'M'$ (Fig. 35) and $P''M''$ represent the variable ordinate for two values of x which differ infinitely little from each other; and let the coördinates of M' and M'' be x', y', and x'', y''. The equation gives

$$y'' - y' = \frac{x''^2}{4p} - \frac{x'^2}{4p} = \frac{x''^2 - x'^2}{4\,p}.$$

But since the difference between x'' and x' is infinitely small, the difference of their squares, that is, the numerator of the above fraction, is infinitely small, so that $y'' - y'$ is infinitely small. $P''M''$ differs, therefore, infinitely little from $P'M'$, and M'' is infinitely near to M'. In like manner, if $P''P'''$ is infinitely small, M''' must be infinitely near M''; so that, if the value of x be continuously increased by infinitely small amounts, i. e. if the ordinate be shifted along the axis of x, y also will be changed by infinitely small amounts, the end M of the moving ordinate will pass successively through the positions M', M'', M''', &c., and these points, being contiguous, will form a line which we may regard as representing and represented by the equation $x^2 = 4\,p\,y$.

82. The conclusions of §§ 80, 81 are summed up in the following

Rule. One equation between coördinates in a plane represents a line (or infinite series of points); two equations represent a point (or definite number of points), so situated that its (or their) coördinates shall satisfy both equations, that is, at the intersection (or intersections) of the two lines which they respectively denote.

83. *Scholium.* Apparent exceptions to the above rule occur, because the line represented by an equation is sometimes reduced to a point, in consequence of certain values attributed to the constants, or known quantities, in the equation. Thus, every point for which $r = r'$, where r' denotes any constant, i. e. fixed length, is in the circumference of a circle having the origin for its centre and r' for its radius. But if the value zero be given to r', the equation becomes $r = 0$, and the circle is reduced to its centre; so that $r = 0$ represents, as in § 51, a single point, viz. the origin.

84. The curve of Fig. 35 is called the *locus* of the equation $x^2 = 4\,p\,y$, which is called the *equation of that line.* The circumference FGC, &c. (Fig. 32) is the locus of $r = \frac{a}{2}$; and

$r = \frac{3}{2}$ is the equation of that locus. Also, the locus of the first problem of § 45 is a straight line parallel to the given straight line, and at the given distance from it.

Definitions. A *locus* is a place to which a point is confined, and in any part of which it is allowed to lie, by prescribed conditions.

Any locus in a plane is called a *curve.*

The *equation of a locus* is the equation which limits the values of the coördinates of every point of that locus.

85. *Definitions.* A curve is said to be *continuous,* when its course is uninterrupted both in extent, (so that we can conceive a point to move in the path of the curve from any one position in it to any other,) and in the character of its curvature.

Thus, a straight line, the circumference of a circle, and the curve of Fig. 35, are *continuous* curves; but a broken line (whether made up of straight or curved lines) is discontinuous *in curvature.* Again, in the equation $y^2 = x^2 - a^2$, for values of x between $-a$ and a, y is imaginary and cannot be constructed; while it is real, and therefore admits of construction, for all values of x which are less than $-a$ or greater than a. Hence, the locus of the equation is wholly without these limits, and partly on either side of them. It is, therefore, discontinuous *in extent,* consisting of two separate parts. There are other ways in which discontinuity in extent may be indicated by the algebraic form of an equation.

A continuous part of a locus is called a *branch.*

Thus, the locus of $y^2 = x^2 - a^2$ has two branches. A curve which is continuous throughout may be said to consist of a single branch.

When a branch comes round into itself, so that a point which moves in it from any given position whatever will at last be brought back to that position, it is called an *oval.*

Thus, the circumference of a circle is an oval; but not all curves which enclose space (as that of Fig. 36) are ovals.

When a branch consists of a single point, it is called a *conjugate point*, and it may be regarded as an infinitely small oval.

When the substitution of any value for either coördinate in an equation gives more than one value for the other coördinate, the different points which correspond to these values are said to belong to different *portions* of the locus.

Thus, with reference to the axis of y, the curve of Fig. 35 consists of two portions.

Two portions of a locus are *symmetrically situated* with respect to either axis, in a rectilinear system, if their distances from it, measured on lines parallel to the other axis, are equal throughout; and a curve which consists wholly of such portions is *symmetrical* with respect to the former axis.

For example, the curve of Fig. 35 is symmetrical with respect to the axis of y, because $x = \pm 2\sqrt{py}$, so that every value of y gives two équal values of x, with contrary signs.

86. *Problem.* To find the locus of a given equation.

Solution. For an equation in terms of polar coördinates, if any values of $\frac{r}{\rho}$ make r negative, and therefore incapable of construction (§ 52), find the limits of those values. If the coördinates are rectilinear, find the limits of such values of x or y as make y or x imaginary.

Next, assume, between the limits for which the curve is possible, successive values of one of the coördinates, differing but little from each other, and construct the corresponding values of the other coördinate, as obtained from the given equation. Thus may be determined as many points of the required locus as are desired, and the line drawn through them, by the eye or with the aid of the curve-ruler, cannot differ much from the required line. If the points could be obtained infinitely near each other, the locus would be fully determined.

The arithmetical method of construction (see § 33) is generally most convenient; but where the geometrical is employed, the method of § 37 must be applied to heterogeneous expressions.

87. EXAMPLES.

NOTE. — The drawings of the first eleven curves should be preserved for future reference.

Construct the following equations: —

1. $x^2 - 6xy + y^2 - 6x + 2y + 5 = 0.$

Solution. First, to find the value of y: —

$$y^2 - 6xy + 2y = y^2 - 2(3x - 1)y = -x^2 + 6x - 5.$$

Completing the square (Alg. art. 230),

$$\begin{aligned} y^2 - 2(3x-1)y + (3x-1)^2 &= y^2 - 2(3x-1)y + 9x^2 - 6x + 1 = [y - (3x-1)]^2 \\ &= -x^2 + 6x - 5 + 9x^2 - 6x + 1 = 8x^2 - 4; \end{aligned}$$

$$y - (3x - 1) = \pm 2\sqrt{(2x^2 - 1)};$$

$$y = 3x - 1 \pm 2\sqrt{(2x^2 - 1)}.$$

Thus y has two real values for all values of x which make the radical term real;

i. e. if $\qquad 2x^2 > 1;$

i. e. if $\qquad x^2 > \frac{1}{2};$

i. e. if x is between $-\infty$ and $-\sqrt{\frac{1}{2}}$, or between $+\sqrt{\frac{1}{2}}$ and $+\infty$. But if x is between $-\sqrt{\frac{1}{2}}$ and $+\sqrt{\frac{1}{2}}$, y is imaginary. Hence the curve consists of two infinite branches, and also of two portions, each of which is partly in one branch and partly in the other.

Beginning with the limits of the imaginary values of y, we have,

if
$$x = \sqrt{\tfrac{1}{2}} = .707,$$
$$x^2 = \tfrac{1}{2},$$
$$2x^2 = 1,$$
$$2x^2 - 1 = 0,$$
$$y = 3\sqrt{\tfrac{1}{2}} - 1 = 2.121 - 1 = 1.121.$$

If
$$x = -\sqrt{\tfrac{1}{2}},$$
$$y = -3\sqrt{\tfrac{1}{2}} - 1 = -2.121 - 1 = -3.121.$$

Taking (Fig. 34) $AP = .71$, $PM = 1.12$, $AP_0 = -.71$, $P_0M_0 = -3.12$, we have two points, M and M_0, of the locus.

Taking values of x differing from each other by two tenths, we find successive points of the locus, as follows*: —

x	y			x	y		
.8	2.46	or	.34	— .8	— 2.34	or	— 4.46
1.	4.	or	0.	— 1.	— 2.	or	— 6.
1.2	5.34	or	— .14	— 1.2	— 1.86	or	— 7.34
1.4	6.62	or	— .22	— 1.4	— 1.78	or	— 8.62
1.6	7.86	or	— .26	— 1.6	— 1.74	or	— 9.86
1.8	9.08	or	— .28	— 1.8	— 1.72	or	— 11.08
2.	10.29	or	— .29	— 2.	— 1.71	or	— 12.29

&c., &c.

* The following form of arranging the computation is recommended as a convenient model in constructing subsequent examples. The squares, square roots, &c. may be taken from tables, or found by means of logarithms.

x	.8	— .8	1.	— 1.	&c.
x^2	.64	.64	1.	1.	
$2x^2$	1.28		2.		
$2x^2 - 1$	.28		1.		
$\pm\sqrt{(2x^2-1)}$	± .5292		± 1.		
$\pm 2\sqrt{(2x^2-1)}$	± 1.0584	± 1.0584	± 2.	± 2.	
$3x - 1$	1.4	— 3.4	2.	— 4.	
y	2.46	— 2.34	4.	— 2.	
	or .34	or — 4.46	or 0.	or — 6.	

Examples.

Thus, we are enabled to lay down, in Fig. 34, that part of the required locus which lies between $x = -2$ and $x = 2$.

This curve is called an *hyperbola*.

2. $$2x^2 + 2y^2 - 3x + 4y - 1 = 0.$$

This locus is the *circumference of a circle.*

3. $$3x^2 - 2xy + y^2 - 4x + 2y - 3 = 0.$$

This locus is an oval, and is called an *ellipse.*

4. $$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

This locus is an *ellipse.*

5. $$x^2 - 4xy + 4y^2 - 4x - 2y + 10 = 0.$$

This locus consists of one infinite branch, and is called a *parabola.*

6. $$x^2 + 2xy - 2y^2 - 4x - y + 10 = 0.$$

This locus is an *hyperbola.*

7. $$xy = 16.$$

This locus is an *hyperbola.*

8. $$(x - a)^2 + 2(y - b)^2 = 0.$$

This locus consists of a single *point.*

9. $$2x^2 - 2xy + y^2 - 2x + 4 = 0.$$

This equation has *no locus.*

10. $$2x^2 - 8xy + 8y^2 + x - 2y = 0.$$

This locus consists of *two parallel straight lines.*

11. $$2x^2 - xy - y^2 - 3x + 1 = 0.$$

This locus consists of *two straight lines which cross each other.*

12. $$y = x^3.$$

This locus consists of one infinite branch, and is called a *cubic parabola.*

13. $$y^2 = x^3.$$

This locus has two infinite branches, and is called a *semi-cubic parabola.*

Examples.

14. $$y^2 = x^4 - x^3.$$

This locus has three infinite branches.

15. $$y^2 = x^3 - x^4.$$

This locus consists of one branch. Is it an oval?

16. $$y^2 = x^4 - x^2.$$

This locus has two infinite branches and a conjugate point.

17. $$y^2 = x^2 - x^4.$$

This locus is an oval. It is called a *lemniscate.*

18. $$y^2 = x^4 - x^6.$$

This locus is an oval.

19. $$y^2 = x^2 (1 - x^2)^3.$$

This locus has two branches. Are its portions the same as its branches?

20. $$y^2 = x^4 (1 - x^2)^3.$$

Are the portions of this locus the same as its branches?

21. $$y^2 = x - x^5.$$

This locus has two branches.

22. $$y^2 = x^5 - x^3.$$

This locus has two branches.

23. $$y^2 = x^3 - (b - c)\, x^2 - b\, c\, x.$$

This curve is referred to the parabolic class. It consists, in general, of two branches, one of which is an oval. The equation should be constructed for four different sets of values of b and c, and the corresponding curves compared.

(1.) $b = 3, \quad c = 1.$

In this case, the curve should be drawn as far as $x = 5$.

(2.) $b = 3, \quad c = 0;$

when the oval is reduced to a conjugate point.

(3.) $b = 0, \quad c = 1;$

when the two branches run together into one.

(4.) $b = 0, \quad c = 0;$

when the branches unite, and the oval becomes a point, and the locus becomes a semi-cubic parabola.

24. $y^2 = (2 - x^2)(1 - x^2)(\frac{1}{2} - x^2).$

This locus consists of three ovals.

25. $(y^2 + x^2)^2 - 4x^2 - y^2 = 0.$

This locus consists of an oval and a conjugate point.

26. $y = (x^2 - a^2)(x^2 - b^2).$

This locus has one infinite branch.

27. $a^2 y - x^2 y - a^3 = 0.$

This locus has three infinite branches. It is called a *redundant hyperbola.*

28. $a^2 y^2 - y^2 x^2 - 16 x^4 = 0.$

This locus consists of two infinite branches.

29. $x^2 y^2 + (x^2 - a^2)(x + b)^2 = 0.$

This locus consists of two infinite branches. It is called a *conchoid*, and its property is, that any line drawn through the origin cuts the curve in two points, which are at the distance a from the points at which it cuts the line of which the equation is $x = b$. Let the equation be constructed when $a < b$, when $a = b$, and when $a > b$.

30. $$\left.\begin{aligned} x &= R\frac{\beta}{\alpha} - R\sin\frac{\beta}{\alpha}, \\ y &= R - R\cos\frac{\beta}{\alpha}; \end{aligned}\right\}$$

in which $\frac{\beta}{\alpha}$ denotes a variable angle. A single equation may be obtained by eliminating $\frac{\beta}{\alpha}$, but it is not so convenient for use as the combination of the given equations. The locus consists of an infinite succession of finite branches. It is called the *cycloid*, and is the locus of a point in the circumference of a wheel of the radius R, which rolls in the coördinate-plane on the axis of x. It is, with its various modifications, a very celebrated curve.

Examples.

31. $$y = \log x.$$

Since negative numbers have no logarithms, the curve will lie wholly on the right of the origin. It will be below the axis of x, for values of x less than 1; when $x = 1$, it will cross the axis, and afterwards continue above it. The curve is called a *logarithmic.*

32. $$y = [a + \sqrt{(a^2 - x^2)}] \log [a + \sqrt{(a^2 - x^2)}].$$

This locus is an oval.

33. $$r = a + \sin m \frac{r}{\varrho}.$$

(1.) Since the sine of an angle cannot be less than -1, r is positive, if $a > 1$; and r will, by Tr. § 69, continually repeat the series of values which it goes through as $m\frac{r}{\varrho}$ passes from 0 to 2π, that is, as $\frac{r}{\varrho}$ passes from 0 to $\frac{2\pi}{m}$. Hence, when $a > 1$, the curve returns into itself after as many revolutions of the radius vector as there are units in the denominator of m. When m is an integer, this denominator is 1. First, let

$$a = 2, \quad m = \tfrac{3}{2}.$$

The curve might be constructed geometrically. But it is better to lay off values of $\frac{r}{\varrho}$ in degrees with the protractor, and find the corresponding values of $\sin m\frac{r}{\varrho}$ from the table of natural sines. The curve is exhibited in Fig. 37.

(2.) $$a = 2, \quad m = 3.$$
(3.) $$a = 2, \quad m = \tfrac{1}{2}.$$
(4.) $$a = 2, \quad m = \tfrac{1}{3}.$$
(5.) $$a = 2, \quad m = \tfrac{2}{3}.$$
(6.) $$a = 2, \quad m = \tfrac{1}{4}.$$
(7.) $$a = 2, \quad m = \tfrac{3}{4}.$$

Examples.

(8.) If $a = 1$, $r = 0$, whenever $\sin m \frac{r}{\varrho} = -1$; so that the curve returns to the origin as many times as there are units in the numerator of m. Let

$$a = 1, \quad m = 1.$$

(9.) $$a = 1, \quad m = 3.$$

(10.) $$a = 1, \quad m = \tfrac{3}{2}.$$

(11.) $$a = 1, \quad m = \tfrac{1}{3}.$$

(12.) $$a = 1, \quad m = 2.$$

(13.) $$a = 1, \quad m = \tfrac{2}{3}.$$

(14.) If $a < 1$, r is sometimes negative. It will be well, however, to retain the negative values of r, indicating the corresponding parts of the curve by dotted lines. Let

$$a = \tfrac{1}{2}, \quad m = 1.$$

(15.) $$a = \tfrac{1}{2}, \quad m = 2.$$

(16.) $$a = \tfrac{1}{2}, \quad m = \tfrac{2}{3}.$$

34. $$r = 2\,R\left(1 + \cos \frac{r}{\varrho}\right).$$

This curve is called an *epicycloid*. It is the locus of a point in the circumference of a wheel, of the radius R, which rolls around another wheel. In this case, where the wheels are equal, the curve is called, from its form, the *cardioid*.

35. $$r^2 = a^2 \cos 2\,\frac{r}{\varrho}.$$

This locus is a *lemniscate*.

36. $$r = \frac{4}{1 - \cos \frac{r}{\varrho}}.$$

This locus is a *parabola*.

37. $$r = \frac{3}{2 - \cos \frac{r}{\varrho}}.$$

This locus is an *ellipse*.

88. *Scholium.* A curve is, in Analytic Geometry, said to be *given*, when its *equation* is given.

89. *Scholium.* The principles of this chapter may be used to find the real roots of equations which involve no more than two unknown quantities. If two equations are given, containing two unknown quantities, construct each equation by § 86, as if x and y denoted the coördinates in a rectangular system; find, by measurement, the values of x and y for the points of intersection of the two lines thus obtained; and these values are the required roots (Alg. art. 108). For, since the coördinates of every point in a line satisfy the equations of that line, the coördinates of the points of intersection of two lines, i. e. of the points common to them, must satisfy the equations of both lines, i. e. are the roots of those equations.

If one equation, with one unknown quantity, x, is given, reduce it as in Alg. art. 118; form a new equation by putting the first member equal to y; construct the new equation; and the abscissas of those points at which the locus cuts the axis of abscissas are the required values of x; since, for every point on the axis of x, $y = 0$, and, for every point in the constructed line, the first member of the original equation is equal to y.

This method, of course, is, practically, only an approximative one.

90. Examples.

1. Find the real roots of the equation

$$x^4 - 4x^3 + x^2 + 6x - 3 = 0,$$

knowing that they are contained between -1.5 and 3.5.

Solution. Fig. 39 shows the locus of the equation

$$y = x^4 - 4x^3 + x^2 + 6x - 3;$$

and the required roots are

$$x' = AP' = -1.19, \qquad x'' = AP'' = .56,$$
$$x''' = AP''' = 1.44, \qquad x^{\text{IV}} = AP^{\text{IV}} = 3.19.$$

If the equation had been

$$x^4 - 4x^3 + x^2 + 6x - 3 = a,$$

it might have been solved by finding the intersection of the curve of Fig. 39 with the straight line determined by the equation $y = a$; and since the equation is of the fourth degree, there will ordinarily be four roots. But, if $a = 1 = AA_1$, the points P''_1, P'''_1 coincide, so that two of the roots become *equal;* and, if $a > 1$, there are no points of intersection corresponding to P'', P''', and two of the roots become *imaginary.*

2. Find the real root of the equation

$$x^3 - 3x + 1 = 0,$$

knowing that they are contained between -2 and 2.

Ans. -1.879, 0.347, 1.532.

3. Find the real roots of the equation

$$5x^3 - 6x + 2 = 0,$$

knowing that they are contained between -2 and 1.

Ans. -1.2345, 0.3785, 0.856.

4. Find the real roots of the equation

$$x^5 + 4x^4 + 4x^3 - 6x^2 - 24x - 24 = 0,$$

knowing that they are contained between -3 and 3.

Ans. -2, 1.817.

5. Find the real roots of the equations

$$x + y = xy,$$
$$x + y + x^2 + y^2 = 12.$$

Ans. Fig. 38 gives

$$\left.\begin{array}{l} x = 2, \\ y = 2; \end{array}\right\} \text{ or } \left\{\begin{array}{l} x = 0.791, \\ y = -3.791; \end{array}\right\} \text{ or } \left\{\begin{array}{l} x = -3.791, \\ y = 0.791. \end{array}\right.$$

6. Find the real roots of the equations

$$x^2 + y^2 = 3,$$
$$x^2 - y^2 = 2.$$

Ans. $\begin{cases} x = \pm 1.581, \\ y = \pm 0.707. \end{cases}$

7. Find the real roots of the equations

$$x^2 y + x y^2 = 4,$$
$$x^3 y^2 + x^2 y^3 = 5.$$

Ans. $\left\{\begin{matrix} x = 13.046, \\ y = -9.846; \end{matrix}\right\}$ or $\begin{cases} x = -9.846, \\ y = 13.046. \end{cases}$

III.

TRANSFORMATION OF COÖRDINATES.

91. Each kind of coördinates is best suited to particular classes of problems; and, among systems of the same kind, one position of the origin and of the axis (or axes) may be convenient for one purpose, and for another purpose a different position. Hence, it is often desirable to change the form of a given equation, so as to obtain from it the equation of the same locus referred to a new system of coördinates. This change in the reference of position from one system of coördinates to another, is called the *Transformation of Coördinates.*

92. *Scholium.* If it is desired to refer the curve of Fig. 35 to a certain polar system, this transformation may be accomplished by finding expressions for the values of the variables x and y, in the rectangular system of Fig. 35,

in terms of the variables r and $\genfrac{}{}{0pt}{}{r}{\varrho}$ of the proposed polar system, and then substituting these values in $x^2 = 4py$; and, after this, the equation may be reduced to its simplest form. It is evident that, in all cases, as in this, the transformation of coördinates may be effected by means of formulæ which express the *old* coördinates in terms of the *new*.

93. *Scholium.* Coördinates of the same kind, but belonging to different systems, are sometimes distinguished from each other by numbers placed below the letters. Thus, if x and y are the coördinates in one rectilinear system, those in another may be denoted by x_1 and y_1. These numbers, being used only to prevent confusion, may evidently be dropped, when we cease to consider the coördinates of the former system.

94. *Problem.* To transform from a system of polar coördinates to a system of rectangular coördinates.

Solution. Let A (Fig. 40) be the polar origin, and AR the polar axis. Let A_1 be the rectangular origin, A_1X the axis of abscissas, and A_1Y the axis of ordinates. Let B represent any point whatever, so that its coördinates will be, in the first system,

$$AB = r, \qquad RAB = \genfrac{}{}{0pt}{}{r}{\varrho},$$

and, in the second system,

$$A_1P = x, \qquad PB = y.$$

We are to find r and $\genfrac{}{}{0pt}{}{r}{\varrho}$ in terms of x and y, and of such constant lines and angles as will determine the position of the new system with reference to the old. These constants are the coördinates of the new origin in the old system, viz.

$$AA_1 = r^\circ, \qquad RAA_1 = \genfrac{}{}{0pt}{}{r^\circ}{\varrho},$$

and the inclination $\genfrac{}{}{0pt}{}{x}{\varrho}$ of the axis of x to the polar axis.

By § 20. g, l, the inclinations of r and of r° to the axis may be decomposed as follows: —

$$\genfrac{}{}{0pt}{}{r}{x} = \genfrac{}{}{0pt}{}{\varrho}{x} + \genfrac{}{}{0pt}{}{r}{\varrho} = \genfrac{}{}{0pt}{}{r}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho},$$

$$\genfrac{}{}{0pt}{}{r^\circ}{x} = \genfrac{}{}{0pt}{}{\varrho}{x} + \genfrac{}{}{0pt}{}{r^\circ}{\varrho} = \genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}.$$

The projections of AA_1, A_1P, and PB on the axis of x are, by §§ 69, 71,

$$r^\circ \cos \genfrac{}{}{0pt}{}{r^\circ}{x} = r^\circ \cos \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right), \quad x, \quad \text{and} \quad 0;$$

and their projections on the axis of y are, by § 72,

$$r^\circ \sin \genfrac{}{}{0pt}{}{r^\circ}{x} = r^\circ \sin \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right), \quad 0, \quad \text{and} \quad y.$$

The projection of AB, or r, on any line, is, by §§ 67, 68, the same as the sum of the projections on the same line of AA_1, A_1P, and PB; so that its projections on the axes of x and of y are

$$r \cos \genfrac{}{}{0pt}{}{r}{x} = r \cos \left(\genfrac{}{}{0pt}{}{r}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right) = r^\circ \cos \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right) + x, \qquad (8.)$$

$$r \sin \genfrac{}{}{0pt}{}{r}{x} = r \sin \left(\genfrac{}{}{0pt}{}{r}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right) = r^\circ \sin \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right) + y.$$

The sum of the squares of these projections is (Tr. § 13)

$$\begin{aligned} r^2 &= x^2 + 2\, r^\circ x \cos \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right) + r^{\circ 2} \cos^2 \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right) \\ &\quad + y^2 + 2\, r^\circ y \sin \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right) + r^{\circ 2} \sin^2 \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right) \\ &= x^2 + y^2 + r^{\circ 2} + 2\, r^\circ \left[x \cos \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right) + y \sin \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right)\right], \end{aligned}$$

$$\begin{aligned} r = \sqrt{\Big(} x^2 + y^2 + r^{\circ 2} + 2\, r^\circ \Big[x \cos \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right) \\ + y \sin \left(\genfrac{}{}{0pt}{}{r^\circ}{\varrho} - \genfrac{}{}{0pt}{}{x}{\varrho}\right)\Big]\Big). \qquad (9.) \end{aligned}$$

Also we have (Tr. § 11),

$$\frac{r \sin \left({r \atop \varrho} - {x \atop \varrho}\right)}{r \cos \left({r \atop \varrho} - {x \atop \varrho}\right)} = \tan \left({r \atop \varrho} - {x \atop \varrho}\right) = \frac{y + r^\circ \sin \left({r^\circ \atop \varrho} - {x \atop \varrho}\right)}{x + r^\circ \cos \left({r^\circ \atop \varrho} - {x \atop \varrho}\right)}. \qquad (10.)$$

(9) and (10) are the required equations, but (8) is, in most of the problems which are considered in this book, more convenient for use than (10).

95. *Corollary.* If the new origin is in the polar axis, and the axis of x has the same direction as the polar axis (Tr. § 55),

$$ {r^\circ \atop \varrho} = 0, \qquad {x \atop \varrho} = 0,$$

$$\sin \left({r^\circ \atop \varrho} - {x \atop \varrho}\right) = 0, \qquad \cos \left({r^\circ \atop \varrho} - {x \atop \varrho}\right) = 1;$$

and (8), (9), and (10) become

$$\left.\begin{array}{l} r \cos {r \atop x} = r \cos {r \atop \varrho} = x + r^\circ, \\ r = \sqrt{(x^2 + y^2 + r^{\circ 2} + 2\, r^\circ x)}, \quad \tan {r \atop \varrho} = \dfrac{y}{x + r^\circ}. \end{array}\right\} \qquad (11.)$$

96. If the origins are the same,

$$r^\circ = 0;$$

and (8) and (9) become

$$r \cos {r \atop x} = r \cos \left({r \atop \varrho} - {x \atop \varrho}\right) = x, \qquad r = \sqrt{(x^2 + y^2)}. \qquad (12.)$$

97. *Problem.* To transform from rectangular coördinates to polar.

Solution. Let AX and AY (Fig. 41) be the rectangular axes. Let A_1 be the polar origin of which the coördinates in the old system are

$$AA' = x^\circ, \quad \text{and} \quad A'A_1 = y^\circ;$$

and let the polar axis be A_1R.

The values of the rectangular coördinates

$$AP = x, \quad \text{and} \quad PB = y,$$

are to be found in terms of the polar coördinates

$$A_1B = r, \quad \text{and} \quad RA_1B = \genfrac{}{}{0pt}{}{r}{\varrho},$$

and of x°, y°, and $\genfrac{}{}{0pt}{}{\varrho}{x}$.

The inclination of A_1B to AX is, by § 20. *g*,

$$\genfrac{}{}{0pt}{}{r}{x} = \genfrac{}{}{0pt}{}{\varrho}{x} + \genfrac{}{}{0pt}{}{r}{\varrho};$$

so that its projections on the axes of x and y are, by (6),

$$x - x^\circ = r\cos\left(\genfrac{}{}{0pt}{}{\varrho}{x} + \genfrac{}{}{0pt}{}{r}{\varrho}\right), \quad \text{and} \quad y - y^\circ = r\sin\left(\genfrac{}{}{0pt}{}{\varrho}{x} + \genfrac{}{}{0pt}{}{r}{\varrho}\right);$$

whence, by transposition,

$$x = x^\circ + r\cos\left(\genfrac{}{}{0pt}{}{\varrho}{x} + \genfrac{}{}{0pt}{}{r}{\varrho}\right), \tag{13.}$$

$$y = y^\circ + r\sin\left(\genfrac{}{}{0pt}{}{\varrho}{x} + \genfrac{}{}{0pt}{}{r}{\varrho}\right). \tag{14.}$$

98. *Problem.* To transform from one system of rectangular coördinates to another.

Solution. Let AX and AY (Fig. 42) be the old axes, and A_1X_1 and A_1Y_1 the new. Let the coördinates of the new origin be

$$AA' = x^\circ, \quad \text{and} \quad A'A_1 = y^\circ.$$

The values of the coördinates

$$AP = x, \quad \text{and} \quad PB = y,$$

are to be found in terms of the coördinates

$$A_1P_1 = x_1, \quad \text{and} \quad P_1B = y_1,$$

and of x°, y°, and $\genfrac{}{}{0pt}{}{x_1}{x}$.

The inclination of P_1B to AX is, by § 20. *g*,

$$\genfrac{}{}{0pt}{}{y_1}{x} = \genfrac{}{}{0pt}{}{x_1}{x} + \genfrac{}{}{0pt}{}{y_1}{x_1} = \tfrac{1}{2}\pi + \genfrac{}{}{0pt}{}{x_1}{x};$$

so that the projections of x_1 and y_1 on the old axis of x are respectively (§ 69; Tr. § 63)

$$x_1 \cos \frac{x_1}{x}, \quad y_1 \cos \left(\tfrac{1}{2}\pi + \frac{x_1}{x}\right) = -y_1 \sin \frac{x_1}{x};$$

and their projections on the old axis of y are respectively (§ 72)

$$x_1 \sin \frac{x_1}{x}, \quad y_1 \sin \left(\tfrac{1}{2}\pi + \frac{x_1}{x}\right) = y_1 \cos \frac{x_1}{x}.$$

Hence, the projections of A_1P_1B on AX and AY are respectively (§§ 68, 73)

$$x - x^\circ = x_1 \cos \frac{x_1}{x} - y_1 \sin \frac{x_1}{x},$$

$$y - y^\circ = x_1 \sin \frac{x_1}{x} + y_1 \cos \frac{x_1}{x};$$

whence

$$x = x^\circ + x_1 \cos \frac{x_1}{x} - y_1 \sin \frac{x_1}{x}, \qquad (15.)$$

$$y = y^\circ + x_1 \sin \frac{x_1}{x} + y_1 \cos \frac{x_1}{x}. \qquad (16.)$$

99. *Corollary.* If the origins are the same,

$$x^\circ = 0, \quad \text{and} \quad y^\circ = 0;$$

and (15) and (16) become

$$x = x_1 \cos \frac{x_1}{x} - y_1 \sin \frac{x_1}{x}, \qquad (17.)$$

$$y = x_1 \sin \frac{x_1}{x} + y_1 \cos \frac{x_1}{x}. \qquad (18.)$$

100. *Corollary.* If the directions of the axes are the same, we have (Tr. § 55)

$$\frac{x_1}{x} = 0, \quad \sin \frac{x_1}{x} = 0, \quad \cos \frac{x_1}{x} = 1;$$

and (15) and (16) become

$$x = x^\circ + x_1, \quad y = y^\circ + y_1. \qquad (19.)$$

If, at the same time, the new origin is in the axis AX,

$$y^\circ = 0;$$

and (19) becomes

$$x = x^\circ + x_1, \quad y = y_1. \qquad (20.)$$

But if the new origin is in the axis AY,

$$x^\circ = 0;$$

and (19) becomes

$$x = x_1, \quad y = y^\circ + y_1. \qquad (21.)$$

101. *Problem.* To transform from any system of rectilinear coördinates to any other.

Solution. Let the original axes be AX and AY (Fig. 43), and the new axes A_1X_1 and A_1Y_1. Let the coördinates of the new origin be

$$AA' = x^\circ, \text{ and } AA'' = y^\circ.$$

The values of the coördinates

$$AP' = RB = x, \text{ and } AR = PB = y,$$

are to be found in terms of the coördinates

$$A_1P_1 = x_1, \text{ and } P_1B = y_1,$$

and of the constants, $x^\circ, y^\circ, \frac{x_1}{x}, \frac{y_1}{x}$, and $\frac{y}{x}$.

$$x = x^\circ + A'P, \quad y = y^\circ + A''R.$$

To obtain the value of $A'P$, we will find $A'P'$, the projection of A_1P_1B on a line perpendicular to the axis AY, and then, in the right triangle $P'A'P$, find $A'P$; and, to obtain $A''R$, we will find $A''R'$, the projection of A_1P_1B on a line perpendicular to AX, and then solve the right triangle $A''R'R$.

Let ι denote the direction of $A'P'$, and $\varkappa$ that of $A''R'$, so that

$$\frac{y}{\iota} = \tfrac{1}{2}\pi, \quad \frac{\varkappa}{x} = \tfrac{1}{2}\pi.$$

The inclinations of A_1P_1 and P_1B to $A'P'$ are respectively, by § 20. g,

$$\frac{x_1}{\iota} = \frac{y}{\iota} + \frac{x}{y} + \frac{x_1}{x} = \frac{y}{\iota} - \frac{y}{x} + \frac{x_1}{x} = \tfrac{1}{2}\pi - \left(\frac{y}{x} - \frac{x_1}{x}\right),$$

$$\frac{y_1}{\iota} = \frac{y}{\iota} + \frac{x}{y} + \frac{y_1}{x} = \frac{y}{\iota} - \frac{y}{x} + \frac{y_1}{x} = \tfrac{1}{2}\pi - \left(\frac{y}{x} - \frac{y_1}{x}\right);$$

and the inclinations of $A''R'$ to A_1P_1 and P_1B are respectively, by § 20. g;

$$\frac{\varkappa}{x_1} = \frac{x}{x_1} + \frac{\varkappa}{x} = \frac{\varkappa}{x} - \frac{x_1}{x} = \tfrac{1}{2}\pi - \frac{x_1}{x},$$

$$\frac{\varkappa}{y_1} = \frac{x}{y_1} + \frac{\varkappa}{x} = \frac{\varkappa}{x} - \frac{y_1}{x} = \tfrac{1}{2}\pi - \frac{y_1}{x}.$$

Hence the projections of x_1 and y_1 on ι are respectively (Tr. § 8)

$$x_1 \cos \frac{x_1}{\iota} = x_1 \cos \left[\tfrac{1}{2}\pi - \left(\frac{y}{x} - \frac{x_1}{x}\right)\right] = x_1 \sin \left(\frac{y}{x} - \frac{x_1}{x}\right),$$

$$y_1 \cos \frac{y_1}{\iota} = y_1 \cos \left[\tfrac{1}{2}\pi - \left(\frac{y}{x} - \frac{y_1}{x}\right)\right] = y_1 \sin \left(\frac{y}{x} - \frac{y_1}{x}\right);$$

and their projections on $\varkappa$ are respectively (§ 70)

$$x_1 \cos \frac{\varkappa}{x_1} = x_1 \cos \left(\tfrac{1}{2}\pi - \frac{x_1}{x}\right) = x_1 \sin \frac{x_1}{x},$$

$$y_1 \cos \frac{\varkappa}{y_1} = y_1 \cos \left(\tfrac{1}{2}\pi - \frac{y_1}{x}\right) = y_1 \sin \frac{y_1}{x};$$

so that the projections of $A_1P_1 + P_1B$ are (§ 68)

$$A'P' = x_1 \sin \left(\frac{y}{x} - \frac{x_1}{x}\right) + y_1 \sin \left(\frac{y}{x} - \frac{y_1}{x}\right),$$

$$A''R' = x_1 \sin \frac{x_1}{x} + y_1 \sin \frac{y_1}{x}.$$

In the right triangles $A'P'P$, $A''R'R$ (§ 20. g, k; Tr. §§ 7, 8, 10)

$$\frac{x}{\iota} = \frac{y}{\iota} + \frac{x}{y} = \frac{y}{\iota} - \frac{y}{x} = \tfrac{1}{2}\pi - \frac{y}{x},$$

Rectangular to Oblique.	Oblique to Rectangular.

$$\frac{\varkappa}{y} = \frac{x}{y} + \frac{\varkappa}{x} = \frac{\varkappa}{x} - \frac{y}{x} = \tfrac{1}{2}\pi - \frac{y}{x};$$

$$\sec\frac{x}{\iota} = \operatorname{cosec}\frac{y}{x} = \frac{A'P}{A'P'},$$

$$\sec\frac{\varkappa}{y} = \operatorname{cosec}\frac{y}{x} = \frac{A''R}{A''R'};$$

$$A'P = A'P' \operatorname{cosec}\frac{y}{x} = \frac{x_1 \sin\left(\frac{y}{x} - \frac{x_1}{x}\right) + y_1 \sin\left(\frac{y}{x} - \frac{y_1}{x}\right)}{\sin\frac{y}{x}},$$

$$A''R = A''R' \operatorname{cosec}\frac{y}{x} = \frac{x_1 \sin\frac{x_1}{x} + y_1 \sin\frac{y_1}{x}}{\sin\frac{y}{x}}.$$

Whence

$$x = x^\circ + \frac{x_1 \sin\left(\frac{y}{x} - \frac{x_1}{x}\right) + y_1 \sin\left(\frac{y}{x} - \frac{y_1}{x}\right)}{\sin\frac{y}{x}}, \qquad (22.)$$

$$y = y^\circ + \frac{x_1 \sin\frac{x_1}{x} + y_1 \sin\frac{y_1}{x}}{\sin\frac{y}{x}}. \qquad (23.)$$

102. *Corollary.* If the original axes are rectangular, then (Tr. § 55)

$$\frac{y}{x} = \tfrac{1}{2}\pi, \qquad \sin\frac{y}{x} = 1;$$

and (22) and (23) become

$$x = x^\circ + x_1 \cos\frac{x_1}{x} + y_1 \cos\frac{y_1}{x}, \qquad (24.)$$

$$y = y^\circ + x_1 \sin\frac{x_1}{x} + y_1 \sin\frac{y_1}{x}. \qquad (25.)$$

103. *Corollary.* If the new axes are rectangular, then (§ 20. g; Tr. §§ 8, 63, 64)

The Formulæ are of General Application.

$$\frac{y_1}{x} = \frac{x_1}{x} + \frac{y_1}{x_1} = \tfrac{1}{2}\pi + \frac{x_1}{x}, \quad \sin\frac{y_1}{x} = \cos\frac{x_1}{x},$$

$$\sin\left(\frac{y}{x} - \frac{y_1}{x}\right) = \sin\left(\frac{y}{x} - \tfrac{1}{2}\pi - \frac{x_1}{x}\right) = \sin\left[-\left(\tfrac{1}{2}\pi - \frac{y}{x} + \frac{x_1}{x}\right)\right]$$

$$= -\sin\left[\tfrac{1}{2}\pi - \left(\frac{y}{x} - \frac{x_1}{x}\right)\right] = -\cos\left(\frac{y}{x} - \frac{x_1}{x}\right);$$

and (22) and (23) become

$$x = x^\circ + \frac{x_1 \sin\left(\frac{y}{x} - \frac{x_1}{x}\right) - y_1 \cos\left(\frac{y}{x} - \frac{x_1}{x}\right)}{\sin\frac{y}{x}}, \qquad (26.)$$

$$y = y^\circ + \frac{x_1 \sin\frac{x_1}{x} + y_1 \cos\frac{x_1}{x}}{\sin\frac{y}{x}}. \qquad (27.)$$

104. *Scholium.* It is left as an exercise for the student to find what (24) and (25) become when the new axes are rectangular, and what (26) and (27) become when the original axes are rectangular. The resulting formulæ in each case should, of course, be identical with (15) and (16).

105. *Scholium.* For simplicity's sake, the new origin and axes and the point B are, in Figs. 40–43, supposed to be so situated that each italic letter shall denote a positive line, and that all the trigonometric functions which enter into the result shall be positive. But the solutions, and therefore the formulæ, depend on general reasoning, and are equally applicable to any other case that can arise under their respective problems. Thus, for the position of the axes and of B represented in Fig. 44, where x° and y_1 are negative and $\frac{x_1}{x}$ is between $\frac{3}{2}\pi$ and 2π, so that (Tr. § 67) $\sin\frac{x_1}{x}$ is negative, x comes out negative in equation (15).

106. Examples.

NOTE.—In the following examples, it is recommended to lay down the new origin and axes on the previous drawings of the curves of § 87.

1. Transform the equation of Ex. 1, § 87 to a new system of rectangular coördinates such that

$$x^\circ = 0, \quad y^\circ = -1, \quad \tan \frac{x_1}{x} = -1.$$

Solution. A_1 (Fig. 34), determined by the values of x° and y° is the new origin, and A_1X_1 and A_1Y_1, determined by the value of $\tan \frac{x_1}{x}$, are the new axes.

Tr. §§ 10, 11, 14, give

$$\sec \frac{x_1}{x} = \sqrt{\left(1 + \tan^2 \frac{x_1}{x}\right)} = \pm \sqrt{2},$$

$$\cos \frac{x_1}{x} = \frac{1}{\sec \frac{x_1}{x}} = \pm \sqrt{\tfrac{1}{2}},$$

$$\sin \frac{x_1}{x} = \tan \frac{x_1}{x} \cos \frac{x_1}{x} = \mp \sqrt{\tfrac{1}{2}};$$

in which, if A_1X_1, and not $A_1X'_1$, is taken for the direction of the axis of x, we must make (Tr. §§ 64, 67) $\cos \frac{x_1}{x}$ positive, and $\sin \frac{x_1}{x}$ negative. Then (15) and (16) become

$$x = 0 + x_1 \sqrt{\tfrac{1}{2}} + y_1 \sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{1}{2}}\,(x_1 + y_1),$$
$$y = -1 - x_1 \sqrt{\tfrac{1}{2}} + y_1 \sqrt{\tfrac{1}{2}} = -1 - \sqrt{\tfrac{1}{2}}\,(x_1 - y_1).$$

Whence

$$x^2 = \tfrac{1}{2}(x_1 + y_1)^2 = \tfrac{1}{2}(x_1^2 + 2 x_1 y_1 + y_1^2),$$
$$xy = -\sqrt{\tfrac{1}{2}}\,(x_1 + y_1) - \tfrac{1}{2}(x_1^2 - y_1^2),$$
$$-6xy = 6\sqrt{\tfrac{1}{2}}\,(x_1 + y_1) + 3(x_1^2 - y_1^2),$$
$$y^2 = [-1 - \sqrt{\tfrac{1}{2}}\,(x_1 - y_1)]^2$$
$$= 1 + 2\sqrt{\tfrac{1}{2}}\,(x_1 - y_1) + \tfrac{1}{2}(x_1^2 - 2 x_1 y_1 + y_1^2),$$

$$-6x = -6\sqrt{\tfrac{1}{2}}\,(x_1 + y_1),$$
$$2y = -2 - 2\sqrt{\tfrac{1}{2}}\,(x_1 - y_1);$$

and the given equation becomes, by omitting the numbers below the letters (§ 93),

$$\tfrac{1}{2}(x^2 + 2xy + y^2) + 6\sqrt{\tfrac{1}{2}}\,(x + y) + 3(x^2 - y^2)$$
$$+ 1 + 2\sqrt{\tfrac{1}{2}}\,(x - y) + \tfrac{1}{2}(x^2 - 2xy + y^2)$$
$$- 6\sqrt{\tfrac{1}{2}}\,(x + y) - 2 - 2\sqrt{\tfrac{1}{2}}\,(x - y) + 5 = 0;$$

which becomes, by reduction,

$$4x^2 - 2y^2 + 4 = 0.$$

Dividing by 4, and transposing, we have

$$\frac{y^2}{2} - \frac{x^2}{1} = 1;$$

which is the required equation.

2. Transform the equation of Ex. 2, § 87, to a new system of rectangular coördinates, such that $x^\circ = \frac{3}{4}$, $y^\circ = -1$, $\frac{x_1}{x} = 0$.

Ans. $x^2 + y^2 = \frac{33}{16}$.

3. Transform the equation of Ex. 3, § 87, to a new system of rectangular coördinates, such that $x^\circ = \frac{1}{2}$, $y^\circ = -\frac{1}{2}$, $\tan 2\left(\frac{x_1}{x}\right) = -1$.

Ans. (Tr. § 51.) $\frac{2(2 - \sqrt{2})}{9}x^2 + \frac{2(2 + \sqrt{2})}{9}y^2 = 1.$

4. Transform the equation of Ex. 4, § 87, to a system of oblique coördinates for which $x^\circ = 0$, $y^\circ = 0$, $\tan\frac{x_1}{x} = \frac{3}{2}$, $\tan\frac{y_1}{x} = -\frac{3}{2}$.

Ans. $\frac{2x^2}{15} + \frac{2y^2}{15} = 1.$

5. Transform the equation of Ex. 5, § 87, to a system of rectangular coördinates for which $x^\circ = 2$, $y^\circ = 1$, $\tan\frac{x_1}{x} = \frac{1}{2}$.

Ans. $y^2 = \sqrt{\tfrac{4}{5}}\,.\,x.$

Examples.

6. Transform the equation of Ex. 6, § 87, to a system of rectangular coördinates for which

$$x^\circ = \tfrac{3}{2},\quad y^\circ = \tfrac{1}{2},\quad \tan \frac{x_1}{x} = \left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{13}.\right)$$

Ans. $$\frac{(2\sqrt{13} + 2)\,x^2}{27} - \frac{(2\sqrt{13} - 2)\,y^2}{27} = 1.$$

7. Transform the equation of Ex. 7, § 87, to a system of rectangular coördinates for which $x^\circ = 0$, $y^\circ = 0$, $\frac{x_1}{x} = \frac{1}{4}\pi$.

Ans. $x^2 - y^2 = 32.$

8. Transform the equation of Ex. 8, § 87, to a system of rectangular coördinates for which $x^\circ = a$, $y^\circ = b$, $\frac{x_1}{x} = 0$.

An $$\frac{x^2}{2} + \frac{y^2}{1} = 0.$$

9. Transform the equation of Ex. 10, § 87, to a system of rectangular coördinates for which $x^\circ = 0$, $y^\circ = -\frac{5}{32}$, $\tan \frac{x_1}{x} = \frac{1}{2}$.

Ans. $y = \pm \frac{1}{16}\sqrt{5}.$

10. Transform the equation of Ex. 11, § 87, to a system of rectangular coördinates for which

$$x^\circ = \tfrac{2}{3},\quad y^\circ = -\tfrac{1}{3},\quad \tan \frac{x_1}{x} = 3 + \sqrt{10}.$$

Ans. $(1 - \sqrt{10})\,x^2 = (1 + \sqrt{10})\,y^2.$

8

CHAPTER V.

EQUATIONS OF LOCI.

107. WE may now use the methods of the last chapter for the purpose for which they were invented, — the application of Algebra to those problems of Geometry which relate to the forms of lines. The object of this chapter is to begin the study of some of the more important lines by finding their equations, referred to various systems of coördinates; and some immediate consequences and applications of these equations will be pointed out in passing. The equation of each locus will first be found in terms of some polar system, and then transformed to such other systems of coördinates as chance, in the case of that locus, to have a special interest.

I.

THE LOCUS OF A POINT WHICH MOVES IN A GIVEN DIRECTION, AND SO AS TO PASS THROUGH A GIVEN POSITION.

108. *Scholium.* This locus is, by Geom. § 12, the *straight line* which is drawn through the given position in the given direction.

109. *Polar Equation.* Let A (Fig. 45) be the given position, and let it be the origin of a system of polar coördinates of which

the axis is AX. Let α denote the given direction. By the definition of the locus, either α or $-\alpha$ is always the direction of the moving point from the origin. Then, for any position, as M, which the point assumes after it has passed through A,

$$\frac{r}{\varrho} = \frac{\alpha}{\varrho}.$$

If we neglect the rule of § 52, this equation is also true of any position, as N, which precedes A; but if we exclude negative values of r, we have for N (§ 20. g, h)

$$\frac{r}{\varrho} = \frac{-\alpha}{\varrho} = \frac{\alpha}{\varrho} + \frac{-\alpha}{\alpha} = \pi + \frac{\alpha}{\varrho}.$$

Since, however (Tr. § 65),

$$\tan\left(\pi + \frac{\alpha}{\varrho}\right) = \tan\frac{\alpha}{\varrho},$$

while no other angle between 0 and 2π has the same tangent as $\frac{\alpha}{\varrho}$, the two equations for M and N may be combined into one, as follows: —

$$\tan\frac{r}{\varrho} = \tan\frac{\alpha}{\varrho}; \qquad (28.)$$

so that (28) is the polar equation of the straight line which has the direction α and passes through the origin.

110. *Corollary.* The origin A may evidently be at any point of the line, since the value of r does not enter into (28). By giving different values to $\frac{\alpha}{\varrho}$, the axis may be drawn in any required direction. Thus, if it has the same direction as the line, (Tr. §§ 55, 56)

$$\frac{\alpha}{\varrho} = 0,$$

$$\tan\frac{r}{\varrho} = \tan\frac{\alpha}{\varrho} = 0,$$

$$\frac{r}{\varrho} = 0, \text{ or } = \pi.$$

111. *Rectangular Equation.* Equation (28) may be transformed to rectangular coördinates. Suppose the new origin to be in the polar axis, as at A_1 (Fig. 45), and the axis of x to have the same direction as the polar axis. Then

$$\frac{\alpha}{\varrho} = \frac{\alpha}{x},$$

and (11) gives

$$\frac{y}{x + r^\circ} = \tan \frac{r}{\varrho} = \tan \frac{\alpha}{\varrho} = \tan \frac{\alpha}{x},$$

$$y = x \tan \frac{\alpha}{x} + r^\circ \tan \frac{\alpha}{x}. \qquad (29.)$$

Dividing by $\tan \frac{\alpha}{x}$ and transposing, we have (Tr. § 10)

$$x = y \cot \frac{\alpha}{x} - r^\circ;$$

or, if we take

$$x' = A_1 A = - A A_1 = - r^\circ,$$

$$x = y \cot \frac{\alpha}{x} + x'. \qquad (30.)$$

Or (29) may be reduced as follows. Let

$$A_1 H = y',$$

then

$$\tan \frac{\alpha}{x} = \frac{y'}{r^\circ},$$

$$r^\circ \tan \frac{\alpha}{x} = y',$$

and (29) becomes

$$y = x \tan \frac{\alpha}{x} + y'. \qquad (31.)$$

Thus, either (30) or (31) may be used as the rectangular equation of the straight line, α denoting its direction, x' the abscissa of the point at which it cuts the axis of x, and y' the ordinate of the point at which it cuts the axis of y.

112. *Scholium.* It appears from § 110 that the polar system of § 109 may be so taken that the origin shall be at the intersec-

tion of the straight line and the axis of x in any rectangular system whatever, and the axis have the same direction as the axis of x. Thus the system to which (28) is transformed in § 111 represents any rectangular system which we may choose to adopt. Even if the proposed axis of x is parallel to the line, they may be conceived to meet at an infinite distance from the rectangular origin; and this infinitely distant point of intersection will be the origin of the polar system from which the transformation of § 111 is effected.

113. *Corollary.* In equations (30) and (31), x', y', and ${}^{\alpha}_{x}$ evidently denote *constants;* but since they have different values for different lines, they are called *arbitrary* constants; and as no two lines can have the same values for more than one of these quantities, a straight line is determined by the values of its arbitrary constants. Thus (Fig. 46), KH and $K''H$ have the same value of y', but different values of x' and ${}^{\alpha}_{x}$; KH and KH'' have the same value of x', but different values of y' and ${}^{\alpha}_{x}$; while KH'' and $K''H$, though they have the same value of ${}^{\alpha}_{x}$, have different values of x' and y'.

114. *Corollary.* The equations of KH, $K'H$, $K'H'$, and KH' (Fig. 46), which have the same absolute values of x', y' and $\tan {}^{\alpha}_{\rho}$, are distinguished by the difference in the signs of these quantities.

For KH,	x' is negative,	y' is positive,	$\tan {}^{\alpha}_{x}$ is positive;
$K'H$,	" positive,	" "	" negative;
$K'H'$,	" "	" negative,	" positive;
KH',	" negative,	" "	" negative.

115. *Corollary.* For a straight line which passes through the origin,

$$x' = 0, \qquad y' = 0,$$

and (30) and (31) become

$$x = y \cot \genfrac{}{}{0pt}{}{\alpha}{x}, \qquad y = x \tan \genfrac{}{}{0pt}{}{\alpha}{x}. \tag{32.}$$

For a straight line parallel to the axis of x (Tr. § 55),

$$\genfrac{}{}{0pt}{}{\alpha}{x} = 0, \qquad \tan \genfrac{}{}{0pt}{}{\alpha}{x} = 0,$$

and (31) becomes

$$y = y'; \tag{33.}$$

and, since (33) is evidently (compare § 80) the equation of every point of a straight line parallel to the axis of x, and at a distance from it equal to y', the remark of § 112 is confirmed, and the case in which the axis of x is parallel to the straight line is included in the general case of § 111.

If the line also passes through the origin, (33) becomes

$$y = 0. \tag{34.}$$

For a straight line parallel to the axis of y (Tr. § 55),

$$\genfrac{}{}{0pt}{}{\alpha}{x} = \tfrac{1}{2}\pi, \qquad \cot \genfrac{}{}{0pt}{}{\alpha}{x} = 0,$$

and (30) becomes

$$x = x'. \tag{35.}$$

If this line passes through the origin, (35) becomes

$$x = 0. \tag{36.}$$

116. *Theorem.* The locus of every equation of the form (31) is a straight line having the direction α, and cutting the axis of y at a distance y' from the origin.

Proof. Refer the given equation to a new rectangular system, of which the axis of abscissas has the direction α, and the origin is in the axis of y, at a distance y' from the origin. We have then, in § 98,

$$\genfrac{}{}{0pt}{}{x_1}{x} = \genfrac{}{}{0pt}{}{\alpha}{x}, \qquad x^\circ = 0, \qquad y^\circ = y';$$

and (15) and (16) become

$$x = x_1 \cos {}^{\alpha}_{x} - y_1 \sin {}^{\alpha}_{x},$$

$$y = y' + x_1 \sin {}^{\alpha}_{x} + y_1 \cos {}^{\alpha}_{x}.$$

Substituting these values in (31), we have (Tr. § 11)

$$y' + x_1 \sin {}^{\alpha}_{x} + y_1 \cos {}^{\alpha}_{x} = \left(x_1 \cos {}^{\alpha}_{x} - y_1 \sin {}^{\alpha}_{x}\right) \tan {}^{\alpha}_{x} + y'$$

$$= x_1 \cos {}^{\alpha}_{x} \tan {}^{\alpha}_{x} - y_1 \sin {}^{\alpha}_{x} \tan {}^{\alpha}_{x} + y'$$

$$= x_1 \sin {}^{\alpha}_{x} - y_1 \frac{\sin^2 {}^{\alpha}_{x}}{\cos {}^{\alpha}_{x}} + y'.$$

Transposing and cancelling,

$$y_1 \left(\cos {}^{\alpha}_{x} + \frac{\sin^2 {}^{\alpha}_{x}}{\cos {}^{\alpha}_{x}}\right) = 0,$$

$$y_1 \left(\cos^2 {}^{\alpha}_{x} + \sin^2 {}^{\alpha}_{x}\right) = 0,$$

which gives (Tr. § 13)

$$y_1 = 0.$$

Now, it is evident, from the principles of rectangular coördinates, that the locus of this last equation is necessarily the axis of x_1; and since it can have no effect on the character of a line to change the system of coördinates to which it is referred, the locus of the given equation must also be the axis of x_1, i. e. a straight line which has the direction α, and cuts the axis of y at a distance y' from the origin.

117. *Scholium.* The force of § 116 is to show that not only may every straight line be represented by an equation of the form (31), but every locus represented by such an equation is a straight line. But since any positive or negative value is possible for $\tan {}^{\alpha}_{x}$ (which is not restricted, like the sine and cosine, to

values between -1 and 1, or, like the secant and cosecant, to values greater than 1, or less than -1), and also for y', an equation of the form

$$y = a\,x + b,$$

in which a and b denote any constant numbers, is properly constructed by a straight line, a being constructed as the tangent of its inclination to the axis of x, and b as the ordinate of the point at which it cuts the axis of y.

118. *Problem.* To construct an equation of the form (31).

Solution. Find in the axis of y a point whose ordinate is equal to the constant term; and, through it, by § 14, *Sol.* 2, draw a straight line inclined to the axis of x by an angle whose tangent is the coefficient of x. It will, by § 116, be the locus required.

119. *Scholium.* A theorem corresponding to § 116 may be proved, in like manner, for an equation of the form (30); and such an equation may be constructed by a method similar to that of § 118.

120. EXAMPLES.

Construct the following equations: —

1. $$y = -3\,x + 2.$$

Solution. In Fig. 47, we have $AR = 2 = y'$; RB, parallel to AX, of any length; and CB, parallel to the axis of y, $= 3\,RB$; so that (§ 20. k; Tr. §§ 61, 64) $\tan BRD = \tan(\pi - CRB) = \tan(-CRB) = \tan BRC = \dfrac{BC}{RB} = -3 = \tan \dfrac{a}{x}$.

Hence, CR, supposed to be produced to infinity in both directions, is the required locus.

2. $$y = 4\,x + 1.$$

3. $$y = \tfrac{1}{2}\,x - 3.$$

Polar Equation.

4. $y = -x - 5.$

5. $y = x.$

6. $y = -2.$

7. $x = -2y + 3.$

8. $x = -3y.$

9. $x = 1.$

10. $x = 0.$

II.

THE LOCUS OF EVERY POINT IN A GIVEN PLANE WHICH IS AT A GIVEN DISTANCE FROM A GIVEN POINT IN THAT PLANE.

121. *Scholium.* This locus is, by Geom. § 85, the *circumference of a circle*, described about the given point as a centre with a radius equal to the given distance.

122. *Polar Equation.* Let the given point A (Fig. 48) be taken as the origin of a system of polar coördinates; and let R denote the given distance. Then the definition of the locus gives for each point of it, as M,

$$AM = R,$$

or

$$r = R; \qquad (37.)$$

so that (37) is the polar equation of the circumference (or, as we commonly say, of the circle) whose radius is R, and centre at the origin.

123. *Corollary.* Since (37) does not involve the value of $\frac{r}{\varrho}$, or in any way depend on it, it is the equation of this locus in any

polar system which has its origin at the centre, whatever be the direction of the axis.

124. *Rectangular Equation, referred to the Centre.* Equation (37) may be transformed, by § 96, to any rectangular system, as that of AX and AY (Fig. 48), which has its origin at the centre. Substituting (12) in (37), we have

$$r = \sqrt{(x^2 + y^2)} = R,$$

or

$$x^2 + y^2 = R^2; \qquad (38.)$$

and (38) is the rectangular equation of the circle whose radius is R, and centre at the origin.

125. *Corollary.* The points at which the circumference cuts either axis may now be found, since the coördinates of such points must satisfy both (38) and the equation of that axis. Thus, for the point at which the circumference cuts the axis of x, we have

$$y = 0,$$

and (38) becomes

$$x^2 = R^2,$$

$$x = \pm R;$$

giving two points, one on each side of the origin, and at a distance from it equal to the radius. For the point at which the circumference cuts the axis of y, we have

$$x = 0,$$

$$y = \pm R;$$

giving two points, one above and one below the origin, and at a distance from it equal to the radius.

(38) gives, by transposition,

$$x^2 = R^2 - y^2,$$

$$x = \pm \sqrt{(R^2 - y^2)}; \qquad (39.)$$

so that, for every value of y, we have two (absolutely) equal -

values of x, with opposite signs, and therefore two points of the locus, equally distant from the axis of y, and on opposite sides of it. Hence the curve is symmetrical with respect to the axis of y. In like mannner, (38) gives

$$y = \pm \sqrt{(R^2 - x^2)}\,; \tag{40.}$$

so that the curve is symmetrical with respect to the axis of x.

If, in (39), we first make $y = 0$, and then increase its absolute value, i. e. if we trace the curve from the axis of x either upwards or downwards, and on either side of the axis of y, the absolute value of x in (39) becomes smaller and smaller, and both portions of the curve gradually approach the axis of y, and therefore each other; till, when

$$y = R, \quad \text{or} \quad = - R,$$

(39) becomes

$$x = \pm\ (R^2 - R^2) = \pm\ 0\,;$$

so that the two portions meet, and the curve closes up, at the distance R from the origin both above and below. If y is taken absolutely greater than R, the quantity under the radical sign in (39) becomes negative, and x imaginary; so that no points of the locus correspond to such values of y.

In the same way it may be shown from (40) that, as we trace either the upper or the lower portion of the curve from the axis of y in either direction, it approaches the axis of x, till, on both sides of the former axis, and at distances from it equal to R, it intersects the axis of x and returns into itself, and that no points of the locus lie beyond these intersections. Hence the curve is an oval, of one branch, and wholly included within a square whose sides are parallel to the axes and at the distance R from them.

This corollary shows how the general properties of the form of a line may be developed by the *discussion* of its equation.

126. *General Rectangular Equation.* Equation (38) may be

transformed by § 100 to a system of parallel axes, having their origin at any desired point; and since the axes of (38) may have any direction without disturbing the equation, these new axes may have any direction, and therefore the new system will represent rectangular systems in general. Let the new axes be $A_1 X_1$ and $A_1 Y_1$ (Fig 48), and let the coördinates of the old origin referred to the new system be $x^\circ_1 = A_1 A'$, and $y^\circ_1 = A' A$. Then § 98 gives

$$x^\circ = A'A_1 = -A_1A' = -x^\circ_1,$$
$$y^\circ = AA' = -A'A = -y^\circ_1;$$

and (19) becomes

$$x = x_1 - x^\circ_1, \qquad y = y_1 - y^\circ_1;$$

which, substituted in (38), give

$$(x_1 - x^\circ_1)^2 + (y_1 - y^\circ_1)^2 = R^2;$$

or, since the numbers under the letters may, by § 93, be omitted,

$$(x - x^\circ)^2 + (y - y^\circ)^2 = R^2; \qquad (41.)$$

which is the equation of the circle, referred to any system of rectangular coördinates, R being the radius, and the point x°, y° being the centre.

127. *Corollary.* Equation (41) is perfectly general, including even (38). For, if A_1 is at the centre,

$$x^\circ = 0, \qquad y^\circ = 0,$$

and (41) becomes

$$x^2 + y^2 = R^2;$$

which is identical with (38).

128. *Corollary.* If the new origin is at A_2, for which

$$x^\circ = R, \qquad y^\circ = 0,$$

(41) becomes

$$(x - R)^2 + y^2 = R^2,$$
$$x^2 - 2\,R\,x + R^2 + y^2 = R^2,$$
$$y^2 = 2\,R\,x - x^2; \qquad (42.)$$

which is the rectangular equation of the circle whose radius is R, the origin being on the circumference, and the axis of x being a diameter drawn from it.

129. *Corollary.* Equation (42) gives

$$y^2 = (2R - x)\,x,$$
$$x : y = y : 2R - x;$$

thus affording an algebraic proof of Geom. § 186.

130. *Theorem.* The locus of every equation of the form (41) is a circle whose radius is R, and centre at the point x°, y°.

Proof. The distance of any point x, y from x°, y° is, by § 75, equal to

$$\sqrt{[(x - x^\circ)^2 + (y - y^\circ)^2]}.$$

But if x, y is a point of the locus of (41), its coördinates must satisfy (41), so that

$$\sqrt{[(x - x^\circ)^2 + (y - y^\circ)^2]} = R;$$

that is, the distance of any point of the locus of (41) from x°, y° is equal to R; and, therefore, the locus is the circumference described about x°, y°, with R for a radius.

131. *Problem.* To construct an equation of the form (41).

Solution. Find the point x°, y°, and about it as a centre, with a radius equal to R, describe a circumference, which will be, by § 130, the required locus.

132. Examples.

Find the loci of the following equations, and their points of intersection with the axes.

1. $$(x + 4)^2 + (y - 1)^2 = 4.$$

Solution. $x^\circ = -4$, $y^\circ = 1$, $R = \sqrt{4} = 2$; and the circle of Fig. 49 is the one required.

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For the points at which the locus cuts the axis of x, $y = 0$; and, substituting this value in the given equation, we have

$$(x + 4)^2 + (-1)^2 = 4,$$

$$(x + 4)^2 = 3,$$

$$x = -4 \pm \sqrt{3} = -2.27 \text{ or } = -5.73.$$

For the points at which the locus cuts the axis of y, $x = 0$; and, substituting this value in the given equation, we have

$$y = 1 \pm \sqrt{-12};$$

and, this value being imaginary, the locus does not cut the axis of y.

2. $$x^2 + y^2 = 9.$$

Ans. If $y = 0$, $x = \pm 3$; if $x = 0$, $y = \pm 3$.

3. $$(x - 2)^2 + (y + 4)^2 = 36.$$

Ans. If $y = 0$, $x = 2 \pm 2\sqrt{5}$; if $x = 0$, $y = -4 \pm 4\sqrt{2}$.

4. $$(x + 1)^2 + (y + 2)^2 = 4.$$

Ans. If $y = 0$, $x = -1$; if $x = 0$, $y = -2 \pm \sqrt{3}$.

5. $$(x + 5)^2 + (y - 4)^2 = 9.$$

Ans. There are no points of intersection.

6. $$(x - 3)^2 + (y - 4)^2 = 25.$$

Ans. If $y = 0$, $x = 6$ or 0; if $x = 0$, $y = 8$ or 0.

7. $$x^2 + (y + 6)^2 = 42\tfrac{1}{4}.$$

Ans. If $y = 0$, $x = \pm 2\tfrac{1}{2}$; if $x = 0$, $y = -12\tfrac{1}{2}$ or $= \tfrac{1}{2}$.

III.

THE LOCUS OF EVERY POINT IN A PLANE, SUCH THAT THE SUM OF ITS DISTANCES FROM TWO GIVEN POINTS IN THAT PLANE IS EQUAL TO A GIVEN LENGTH.

133. *Scholium.* This locus is called the *ellipse*, the

given points are called its *foci*, and the middle of the line which joins them its *centre*.

An ellipse may be described as follows: — Drive two pins into the paper at the points which you mean to make the foci, as at F and F' (Fig. 50); tie together the ends of a thread, put it over the pins, and, placing a pencil-point inside the thread, as at M, move it about F and F', keeping the thread tight. Here the given length is $FM + MF'$, and it is evident (Geom. § 18) that

$$FM + MF' > FF'.$$

The following is another way of drawing an ellipse: — Let AB (Fig. 51) be the given length, and F and F' the foci. From F as a centre, with any part AD of AB as a radius, describe an arc; from F' as a centre, with a radius equal to DB, describe an arc cutting the former arc at M', which will be a point of the ellipse. In like manner, find, conveniently near together, other points, such that the sum of the distances of each from F and F' shall be equal to AB; and the curve drawn through them will be the required ellipse. It will be observed that each set of radii, as AD and DB, determines four points.

In treating of the ellipse, the following notation will be used: —

$$c = \text{half the distance between the foci,}$$
$$A = \text{half the given length,}$$
$$B = \sqrt{(A^2 - c^2)};$$

and since

$$2c < 2A, \qquad \text{or} \quad c < A,$$

B is always real.

134. *Polar Equation, referred to Focus.* Let F (Fig. 50) be the origin, and FF' the axis, of a system of polar coördinates. The coördinates of F' are

$$r = 2c, \qquad {}^{r}_{\varrho} = 0;$$

so that (2) gives for the distance of any point M from F'

$$MF' = \sqrt{\left(4c^2 + r^2 - 4cr\cos\frac{r}{\varrho}\right)}.$$

But if M is a point of the locus, we have, by the definition

$$FM + MF' = 2A = r + \sqrt{\left(4c^2 + r^2 - 4cr\cos\frac{r}{\varrho}\right)},$$

$$\sqrt{\left(4c^2 + r^2 - 4cr\cos\frac{r}{\varrho}\right)} = 2A - r,$$

$$4c^2 + r^2 - 4cr\cos\frac{r}{\varrho} = 4A^2 - 4Ar + r^2,$$

$$4Ar - 4cr\cos\frac{r}{\varrho} = 4A^2 - 4c^2,$$

$$\left(A - c\cos\frac{r}{\varrho}\right)r = A^2 - c^2,$$

$$r = \frac{A^2 - c^2}{A - c\cos\frac{r}{\varrho}}; \qquad (43.)$$

which is the polar equation of the ellipse, the origin being at one focus, and the axis directed towards the other focus.

135. *Corollary.* Since $A > c$, the numerator of the second member of (43) is positive; and, since there is no angle which has a cosine greater than 1 (Tr. §§ 5, 55, 61, 65), $A > c\cos\frac{r}{\varrho}$, so that the denominator is also positive; and therefore r is positive, and, by § 52, admits of construction, for any assumed value of $\frac{r}{\varrho}$. Hence, the curve extends all around the origin.

We may conceive this locus to be described by the extremity of a radius vector which revolves about the origin and at the same time changes its length so as to satisfy (43). When it has made a complete revolution, that is, when $\cos\frac{r}{\varrho}$ has passed through the series of values corresponding to all the values of $\frac{r}{\varrho}$ between

0 and 2 π, this same series of values will be repeated (Tr. § 69) and the extremity of the radius vector will return into its former path. Hence, the curve is an oval.

At the beginning of this revolution, that is, when $\frac{r}{\rho} = 0$,

$$\cos \frac{r}{\rho} = 1,$$

and (43) gives

$$r = A + c.$$

As $\frac{r}{\rho}$ increases from 0 to π, its cosine decreases (Tr. §§ 62, 70, 71), and therefore the denominator of (43) increases, and the value of r decreases, that is, the curve approaches the origin; till, when $\frac{r}{\rho} = \pi$,

$$\cos \frac{r}{\rho} = -1,$$

and (43) gives

$$r = A - c.$$

As $\frac{r}{\rho}$ passes from π to 2 π, its cosine goes through the same series of values, but in reverse order, so that the curve continually recedes again from the origin. $A + c$ and $A - c$ are, therefore, respectively the *maximum* and *minimum* distances of the curve from the focus F.

136. *Corollary.* If we suppose the curve to begin at C', instead of C, that is, if we take for the polar axis FC', which we will call ρ_1, we have (§ 20. g; Tr. §§ 61, 64)

$$\frac{r}{\rho_1} = \frac{\rho}{\rho_1} + \frac{r}{\rho},$$

$$\cos \frac{r}{\rho} = \cos \left(\frac{r}{\rho_1} - \frac{\rho}{\rho_1}\right) = \cos \left(\frac{r}{\rho_1} - \pi\right) = -\cos \frac{r}{\rho_1};$$

and (43) gives

$$r = \frac{A^2 - c^2}{A + c \cos \frac{r}{\rho_1}}. \qquad (44.)$$

137. *Corollary.* Since the ellipse is evidently, by the definition, situated in the same way with regard to both foci, (43) will be its equation, if F' is made the polar origin and $F'C'$ the axis; and (44) will be its equation, if F' is the origin and $F'C$ the axis.

138. *Corollary.* For an ellipse, we may take c of any length less than A. If $c = 0$, F and F' coincide, and both (43) and (44) become

$$r = \frac{A^2}{A} = A;$$

which is the equation of a *circle* of which the centre is at the origin and the radius is A. Hence, the circle may be considered as an ellipse in which both the foci are situated at the centre; and it is, indeed, evident that it satisfies the definition of the ellipse.

139. *Equation in Terms of the Eccentricity.* It is sometimes convenient to introduce into the equation of the ellipse a quantity called its *eccentricity,* such that, if it be denoted by e,

$$e = \frac{2\,c}{2\,A} = \frac{c}{A}.$$

We have

$$c = A\,e,$$

which, substituted in (43), gives

$$r = \frac{A^2 - A^2 e^2}{A - A\,e\cos\frac{r}{\varrho}} = \frac{A^2\,(1 - e^2)}{A\left(1 - e\cos\frac{r}{\varrho}\right)} = \frac{A\,(1 - e^2)}{1 - e\cos\frac{r}{\varrho}}$$
$$= \frac{A\,(1 - e)\,(1 + e)}{1 - e\cos\frac{r}{\varrho}}.$$

And, if we take

$$p = CF = F'C = A - c = A\,(1 - e),$$

$$r = \frac{p\,(1 + e)}{1 - e \cos \frac{r}{\varrho}}; \qquad (45.)$$

which is another form of (43).
In like manner,

$$r = \frac{p\,(1 + e)}{1 + e \cos \frac{r}{\varrho_1}} \qquad (46.)$$

is another form of (44).

140. *Corollary.* Since $A > c$

$$e < 1.$$

When the ellipse is a circle,

$$e = \frac{0}{A} = 0;$$

so that the circle has no eccentricity.

141. *Definitions.* A *diameter* of an ellipse is a line drawn through the centre and terminated at each end by the curve; and, since the curve is an oval and surrounds the centre, a diameter may be drawn in any direction.

The diameter which passes through the foci of an ellipse is called the *transverse axis.* The diameter perendicular to the transverse axis is called the *conjugate xis.* These are also called the *principal diameters.*

The extremities of a diameter are called its *vertices.* iose of the transverse axis are called the *vertices of the pse.*

42. *Equation, referred to Principal Diameters.* (43) may, 95, be transformed to a system of rectangular coördinates, in h the origin is at the centre and the axes of x and y have rections respectively of the transverse axis and of the con- axis of the curve. We have

$$r^{\circ} = FA = c,$$

and (11) gives

$$r = \sqrt{(x^2 + y^2 + c^2 + 2\,c\,x)},$$

$$r \cos \frac{r}{\varrho} = x + c.$$

When (43) is freed from fractions, it becomes

$$A\,r - c\,r\cos\frac{r}{\varrho} = A^2 - c^2.$$

Substituting the above values of r and $r \cos \frac{r}{\varrho}$, we have

$$A\sqrt{(x^2 + y^2 + c^2 + 2\,c\,x)} - c\,x - c^2 = A^2 - c^2.$$

Transposing, squaring, and reducing,

$$A\sqrt{(x^2 + y^2 + c^2 + 2\,c\,x)} = A^2 + c\,x,$$

$$A^2 x^2 + A^2 y^2 + A^2 c^2 + 2\,A^2 c\,x = A^4 + 2\,A^2 c\,x + c^2 x^2,$$

$$A^2 x^2 + A^2 y^2 + A^2 c^2 = A^4 + c^2 x^2,$$

$$(A^2 - c^2)\,x^2 + A^2 y^2 = A^2 (A^2 - c^2),$$

$$B^2 x^2 + A^2 y^2 = A^2 B^2, \qquad (47.)$$

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1; \qquad (48.)$$

and either (47) or (48) is the rectangular equation of the ellipse, referred to its transverse and conjugate axes.

143. *Corollary.* In the case of the circle, we have

$$B^2 = A^2 - c^2 = A^2,$$

and (48) becomes

$$\frac{x^2}{A^2} + \frac{y^2}{A^2} = 1,$$

or

$$x^2 + y^2 = A^2;$$

which is of the same form as (38).

144. *Corollary.* If the conjugate axis of the ellipse be taken for the axis of abscissas, we have in § 99

$$\frac{x_1}{x} = \tfrac{1}{2}\,\pi,$$

so that (17) and (18) become (Tr. § 55)

$$x = -y_1, \quad y = x_1.$$

Substituting these values in (48), and removing the numbers below the letters, we have

$$\frac{x^2}{B^2} + \frac{y^2}{A^2} = 1; \tag{49.}$$

which is therefore the equation of an ellipse which has its foci in the axis of y.

145. *Rectangular Equation, referred to Vertex.* (47) may be transformed by § 100 to a system of rectangular coördinates in which the origin is at the left-hand vertex, and the transverse axis is the axis of x. In this case,

$$x^\circ = -A, \qquad y^\circ = 0,$$

and (19) becomes

$$x = x_1 - A, \qquad y = y_1.$$

Substituting these values in (47), and removing the numbers under the letters, we have

$$B^2 x^2 - 2\,AB^2 x + A^2B^2 + A^2 y^2 = A^2B^2,$$

$$y^2 = \frac{2\,B^2}{A} x - \frac{B^2}{A^2} x^2. \tag{50.}$$

146. *Polar Equation, referred to Centre.* (47) may be transformed to a polar system in which the origin is at the centre, and the polar axis coincides with the axis of x. In this case, § 97 gives

$$x^\circ = 0, \;\; y^\circ = 0, \;\; {}^{\varrho}_{x} = 0,$$

and (13) and (14) become

$$x = r \cos {}^{r}_{\varrho}, \quad y = r \sin {}^{r}_{\varrho},$$

which, substituted in (47), give

$$r^2 \left(B^2 \cos^2 {}^{r}_{\varrho} + A^2 \sin^2 {}^{r}_{\varrho}\right) = A^2B^2,$$

$$r = \frac{AB}{\sqrt{\left(B^2 \cos^2 {}^{r}_{\varrho} + A^2 \sin^2 {}^{r}_{\varrho}\right)}}. \qquad (51.)$$

147. *Corollary.* For two oppositely directed radii vectores, r' and r'', we shall have (Tr. § 65)

$$ {}^{r''}_{\varrho} = \pi + {}^{r'}_{\varrho}, \quad \cos {}^{r''}_{\varrho} = -\cos {}^{r'}_{\varrho}, \quad \sin {}^{r''}_{\varrho} = -\sin {}^{r'}_{\varrho},$$

$$\cos^2 {}^{r''}_{\varrho} = \cos^2 {}^{r'}_{\varrho}, \quad \sin^2 {}^{r''}_{\varrho} = \sin^2 {}^{r'}_{\varrho};$$

so that (51) will give equal values for r' and for r''; that is,

Every diameter of the ellipse is bisected at the centre, and (51) gives the length of the semidiameter which has the direction r.

If ${}^{r}_{\varrho} = 0$, (51) becomes (Tr. § 55)

$$r = \frac{AB}{\sqrt{B^2}} = A.$$

If ${}^{r}_{\varrho} = \frac{1}{2}\pi$, (51) becomes (Ibid.)

$$r = \frac{AB}{\sqrt{A^2}} = B.$$

Hence, *A is equal to the semi-transverse axis of the ellipse, and B to the semi-conjugate axis.*

148. *Definition.* If α denotes the direction of the transverse axis, and α_1 and β_1 those of two diameters which satisfy the equation

$$\tan {}^{\alpha_1}_{\alpha} \tan {}^{\beta_1}_{\alpha} = -\frac{B^2}{A^2}, \qquad (52.)$$

these diameters are said to be *conjugate to each other.*

149. *Corollary.* Since the second member of (52), being the negative of the quotient of two squares, is necessarily (Alg. art. 194) negative, $\tan {}^{\alpha_1}_{\alpha}$ and $\tan {}^{\beta_1}_{\alpha}$ must have opposite signs; so

that of two conjugate diameters, one must make an acute angle with the transverse axis, and the other an obtuse angle.

If α_1 is taken in the direction of the transverse axis, we have (Tr. §§ 55, 57)

$$\genfrac{}{}{0pt}{}{\alpha_1}{\alpha} = 0, \quad \tan \genfrac{}{}{0pt}{}{\alpha_1}{\alpha} = \pm\, 0,$$

$$\pm\, 0 \times \tan \genfrac{}{}{0pt}{}{\beta_1}{\alpha} = -\frac{B^2}{A^2},$$

$$\tan \genfrac{}{}{0pt}{}{\beta_1}{\alpha} = \pm \frac{B^2}{A^2 \times 0} = \pm\, \infty,$$

$$\genfrac{}{}{0pt}{}{\beta_1}{\alpha} = \tfrac{1}{2}\pi, \quad \text{or} \quad = \tfrac{3}{2}\pi;$$

and, therefore, *the principal diameters of an ellipse are conjugate to each other.*

150. *Corollary.* For the circle, $B = A$, so that (52) gives (Tr. § 63)

$$\tan \genfrac{}{}{0pt}{}{\alpha_1}{\alpha} \tan \genfrac{}{}{0pt}{}{\beta_1}{\alpha} = -\frac{A^2}{A^2} = -1,$$

$$\tan \genfrac{}{}{0pt}{}{\beta_1}{\alpha} = -\operatorname{cotan} \genfrac{}{}{0pt}{}{\alpha_1}{\alpha} = \tan \left(\tfrac{1}{2}\pi + \genfrac{}{}{0pt}{}{\alpha_1}{\alpha}\right),$$

$$\genfrac{}{}{0pt}{}{\beta_1}{\alpha} = \tfrac{1}{2}\pi + \genfrac{}{}{0pt}{}{\alpha_1}{\alpha};$$

that is, *any two conjugate diameters of a circle are perpendicular to each other.*

151. *Equation, referred to Conjugate Diameters.* (47) may be referred to conjugate diameters by § 102, if we make $x^\circ = 0$, $y^\circ = 0$. The substitution of (24) and (25) in (47) gives, by reduction,

$$B^2 \left(x_1 \cos \genfrac{}{}{0pt}{}{x_1}{x} + y_1 \cos \genfrac{}{}{0pt}{}{y_1}{x}\right)^2 + A^2 \left(x_1 \sin \genfrac{}{}{0pt}{}{x_1}{x} + y_1 \sin \genfrac{}{}{0pt}{}{y_1}{x}\right)^2$$

$$= B^2 \left(x_1^2 \cos^2 \genfrac{}{}{0pt}{}{x_1}{x} + 2\, x_1\, y_1 \cos \genfrac{}{}{0pt}{}{x_1}{x} \cos \genfrac{}{}{0pt}{}{y_1}{x} + y_1^2 \cos^2 \genfrac{}{}{0pt}{}{y_1}{x}\right)$$

$$+ A^2 \left(x_1^2 \sin^2 \genfrac{}{}{0pt}{}{x_1}{x} + 2\, x_1\, y_1 \sin \genfrac{}{}{0pt}{}{x_1}{x} \sin \genfrac{}{}{0pt}{}{y_1}{x} + y_1^2 \sin^2 \genfrac{}{}{0pt}{}{y_1}{x}\right)$$

$$= \left(B^2 \cos^2 \frac{x_1}{x} + A^2 \sin^2 \frac{x_1}{x}\right) x_1^2 + \left(B^2 \cos^2 \frac{y_1}{x} + A^2 \sin^2 \frac{y_1}{x}\right) y_1^2$$

$$+ 2\left(B^2 \cos \frac{x_1}{x} \cos \frac{y_1}{x} + A^2 \sin \frac{x_1}{x} \sin \frac{y_1}{x}\right) x_1 y_1 = A^2 B^2. \quad (53.)$$

If the semidiameters which have the directions of x_1 and y_1 are denoted respectively by A_1 and B_1, (51) becomes

$$A_1 = \frac{AB}{\sqrt{\left(B^2 \cos^2 \frac{x_1}{x} + A^2 \sin^2 \frac{x_1}{x}\right)}},$$

$$B_1 = \frac{AB}{\sqrt{\left(B^2 \cos^2 \frac{y_1}{x} + A^2 \sin^2 \frac{y_1}{x}\right)}};$$

or

$$B^2 \cos^2 \frac{x_1}{x} + A^2 \sin^2 \frac{x_1}{x} = \frac{A^2 B^2}{A_1^2}, \quad (54.)$$

$$B^2 \cos^2 \frac{y_1}{x} + A^2 \sin^2 \frac{y_1}{x} = \frac{A^2 B^2}{B_1^2}. \quad (55.)$$

(52) gives (Tr. § 11)

$$A^2 \tan \frac{x_1}{x} \tan \frac{y_1}{x} = - B^2,$$

$$A^2 \sin \frac{x_1}{x} \sin \frac{y_1}{x} = - B^2 \cos \frac{x_1}{x} \cos \frac{y_1}{x},$$

$$B^2 \cos \frac{x_1}{x} \cos \frac{y_1}{x} + A^2 \sin \frac{x_1}{x} \sin \frac{y_1}{x} = 0. \quad (56.)$$

Substituting (54), (55), and (56) in (53), we have

$$\frac{A^2 B^2 x_1^2}{A_1^2} = \frac{A^2 B^2 y_1^2}{B_1^2} = A^2 B^2,$$

or, dividing by A^2B^2, and omitting the numbers under the coordinates,

$$\frac{x^2}{A_1^2} + \frac{y^2}{B_1^2} = 1; \quad (57.)$$

which is of the same form as (48), and, indeed, since the principal diameters afford a case of conjugate diameters, includes it.

152. *Corollary.* (57) gives, by reduction,

$$y = \pm \frac{B_1}{A_1}\sqrt{(A_1^2 - x^2)}; \tag{58.}$$

so that each value of x gives two values of y, which differ only in their signs; and hence the curve consists of two portions which are symmetrically situated with respect to the axis of x.

If $x = 0$, the above value of y becomes

$$y = \pm \frac{B_1}{A_1}\sqrt{A_1^2} = \pm B_1.$$

This is the maximum value of y; for, the more x differs from 0, the greater is the value of x^2, and the less the absolute value of y; that is, both portions of the curve approach the axis of x, as they recede either to the right or to the left from the axis of y, till, when $x = \pm A_1$,

$$y = \pm \frac{B_1}{A_1}\sqrt{(A_1^2 - A_1^2)} = \pm 0,$$

and the portions meet at the extremities of the diameter $2A_1$. If x is made absolutely greater than A_1, y becomes imaginary.

Hence, if lines be drawn through the extremities of each of two conjugate diameters, parallel to the other, the parallelogram so formed will include the whole ellipse. If the conjugate diameters taken are the principal ones, this parallelogram is a rectangle.

153. *Theorem.* The locus of every rectangular equation of the form (48), in which $A > B$, is an ellipse which has its centre at the origin and its foci in the axis of x, and for which c, A, and B have the same meanings as in § 133.

Proof. We are to prove that $2A$ is equal to the sum of the distances of any point of the locus of the given equation from two points of which the coördinates are respectively

$$x = c, \quad y = 0; \qquad x = -c, \quad y = 0.$$

The distance r_1 of any point x, y from the former of these points is, by (7), Locus of (48), necessarily an Ellipse.

$$r_1 = \sqrt{[(x-c)^2 + y^2]} = \sqrt{(x^2 - 2cx + c^2 + y^2)}.$$

If x, y is a point of the locus of (48), its coördinates satisfy (48), or

$$y^2 = B^2 - \frac{B^2 x^2}{A^2}.$$

Substituting this value of y^2 and also

$$c^2 = A^2 - B^2,$$

we have

$$\begin{aligned} r_1 &= \sqrt{\left(x^2 - 2cx + A^2 - B^2 + B^2 - \frac{B^2 x^2}{A^2}\right)} \\ &= \sqrt{\left(\frac{A^2 - B^2}{A^2}\, x^2 - 2cx + A^2\right)} \\ &= \sqrt{\left(\frac{c^2 x^2}{A^2} - 2cx + A^2\right)} \\ &= \pm\left(\frac{cx}{A} - A\right). \end{aligned}$$

The distance r_2 of the same point of the locus from the other supposed focus is by (7) and a similar reduction

$$\begin{aligned} r_2 &= \sqrt{[(x+c)^2 + y^2]} \\ &= \sqrt{\left(\frac{c^2 x^2}{A^2} + 2cx + A^2\right)} \\ &= \pm\left(\frac{cx}{A} + A\right). \end{aligned}$$

Now, since the distance between two points is properly denoted by a positive quantity, we must interpret the double signs in such a way as to make r_1 and r_2 positive. But we have

$$x^2 = A^2 - \frac{A^2 y^2}{B^2}, \qquad c^2 = A^2 - B^2,$$

i. e. $\quad x^2 < A^2, \qquad c^2 < A^2,$

i. e. $\quad x < A, \qquad c < A,$

whence

$$c\,x < A^2, \qquad \frac{c\,x}{A} < A;$$

in which x has its absolute value, so that $\frac{c\,x}{A} - A$ is negative, even when x is positive, and $\frac{c\,x}{A} + A$ is positive, even when x is negative. Hence

$$r_1 = -\left(\frac{c\,x}{A} - A\right) = A - \frac{c\,x}{A},$$

$$r_2 = +\left(\frac{c\,x}{A} + A\right) = A + \frac{c\,x}{A},$$

$$r_1 + r_2 = 2\,A.$$

154. *Corollary.* By interchanging A and B together and x and y together in the above proof, it may be shown that the locus of every rectangular equation of the form (48), in which $A < B$, is an ellipse which has its semi-transverse axis equal to B, its semi-conjugate axis equal to A, its centre at the origin, and its foci in the axis of y, at a distance from the origin, on either side of it, equal to $\sqrt{(B^2 - A^2)}$. Such an equation may also be described as an equation of the form (49), in which $A > B$.

155. *Corollary.* Any equation of the form (43) in which $A > c$, will give, by transformation, an equation of the form (48) in which $A > B$; and since, by § 153, the locus of the latter equation is always an ellipse, the locus of the former equation is always an ellipse.

156. *Problem.* To construct an equation of the form (48).

Solution. Draw an ellipse, by one of the methods of § 133, which, if $A > B$, shall have its transverse axis (or given length) equal to $2\,A$, and its foci at the points for which

$$x = \sqrt{(A^2 - B^2)},\ y = 0; \quad x = -\sqrt{(A^2 - B^2)},\ y = 0;$$

but if $A < B$, take the transverse axis equal to $2\,B$ and the foci at the points for which

Construction.

$x = 0, \ y = \sqrt{(B^2 - A^2)}; \quad x = 0, \ y = -\sqrt{(B^2 - A^2)}.$

It must, by §§ 153, 154, be the required locus.

157. EXAMPLES.

1. Construct the equation

$$4x^2 + \frac{y^2}{2} = 1.$$

Solution. If we write the equation

$$\frac{x^2}{\frac{1}{4}} + \frac{y^2}{2} = 1,$$

it becomes of the form (48), so that

$$A = \sqrt{\tfrac{1}{4}} = \tfrac{1}{2},$$
$$B = \sqrt{2} = 1.414;$$

so that $A < B$.

In Fig. 52, we have

$$C'A = AC = \tfrac{1}{2},$$
$$B'A = AB = CF = CF' = \sqrt{2},$$
$$AF \text{ or } AF' = \pm\sqrt{(B^2 - A^2)} = c;$$

and F and F' are the foci of the ellipse.

2. Construct the equation

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$

3. Construct the equation

$$\frac{x^2}{3} + y^2 = 1.$$

4. Construct the equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

5. Construct the equation

$$\frac{x^2}{25} + \frac{y^2}{25} = 1.$$

IV.

THE LOCUS OF EVERY POINT IN A PLANE SO SITUATED THAT THE DIFFERENCE OF ITS DISTANCES FROM TWO GIVEN POINTS IN THAT PLANE IS EQUAL TO A GIVEN LENGTH.

158. *Scholium.* This locus is called the ***hyperbola***, the given points are called its *foci*, and the middle of the line which joins them its *centre*.

An hyperbola may be described as follows: — Let AB (Fig. 53) be the given length, and F and F' the foci. Produce AB and take on it a point D such that $AD > AB$ and $AD + BD > FF'$. Find (Geom. § 128) a point M' at the distances AD from F, and BD from F'; also a point M'' at the distances AD from F', and BD from F. In like manner, find other points, near together, such that the difference of the distances of each from F and F' shall be equal to AB; and the curve drawn through them will be the required hyperbola.

It is evident, from Geom. § 130, that the construction is impossible when

$$FF' < AB.$$

In treating of the hyperbola, the following notation will be used: —

$$c = \text{half the distance between the foci,}$$

$$A = \text{half the given length,}$$

$$B = \sqrt{(c^2 - A^2)};$$

and we have

$$2\,c > 2\,A, \quad \text{or} \quad c > A,$$

which makes B always real.

159. *Polar Equation, referred to Focus.* Let the focus F (Fig. 54) be the origin, and FF' the axis, of a system of polar coördinates. The coördinates of the focus F' are

$$r = 2\,c, \quad \frac{r}{\varrho} = 0\,;$$

so that (2) gives, for the distance of any point M from F',

$$F'M = \sqrt{\left(4\,c^2 + r^2 - 4\,c\,r\,\cos\frac{r}{\varrho}\right)}.$$

If M is a point of the locus, we may suppose either that $FM > F'M$, or that $F'M > FM$. In the former case, we have

$$FM - F'M = 2\,A = r - \sqrt{\left(4\,c^2 + r^2 - 4\,c\,r\,\cos\frac{r}{\varrho}\right)},$$

$$\sqrt{\left(4\,c^2 + r^2 - 4\,c\,r\,\cos\frac{r}{\varrho}\right)} = r - 2\,A,$$

$$4\,c^2 + r^2 - 4\,c\,r\,\cos\frac{r}{\varrho} = r^2 - 4\,A\,r + 4\,A^2,$$

$$4\,A\,r - 4\,c\,r\,\cos\frac{r}{\varrho} = 4\,A^2 - 4\,c^2,$$

$$\left(A - c\,\cos\frac{r}{\varrho}\right) r = A^2 - c^2,$$

$$r = \frac{A^2 - c^2}{A - c\,\cos\frac{r}{\varrho}} = \frac{c^2 - A^2}{c\,\cos\frac{r}{\varrho} - A}. \qquad (59.)$$

If, however, $F'M > FM$,

$$F'M - FM = 2\,A = \sqrt{\left(4\,c^2 + r^2 - 4\,c\,r\,\cos\frac{r}{\varrho}\right)} - r,$$

$$\sqrt{\left(4\,c^2 + r^2 - 4\,c\,r\,\cos\frac{r}{\varrho}\right)} = r + 2\,A,$$

$$4\,c^2 + r^2 - 4\,c\,r\,\cos\frac{r}{\varrho} = r^2 + 4\,A\,r + 4\,A^2,$$

$$r = \frac{c^2 - A^2}{c \cos \frac{r}{\varrho} + A}. \qquad (60.)$$

(59) and (60), in combination, are, therefore, the polar equations of the hyperbola, the origin being at one focus, and the axis directed towards the other focus.

160. *Corollary.* Since, in the hyperbola, $A < c$, the numerator of the last member of (59) is always positive, so that r will admit of construction for all values of $\frac{r}{\varrho}$ which make the denominator positive.

If

$$\frac{r}{\varrho} = 0, \qquad \cos \frac{r}{\varrho} = 1,$$

so that

$$r = \frac{c^2 - A^2}{c - A} = c + A = FC.$$

As $\frac{r}{\varrho}$ increases, its cosine decreases (Tr. §§ 70, 71), and therefore the denominator of the last member of (59) decreases, and the value of r increases; that is, the curve recedes from the origin, till, when the radius vector assumes a direction r' for which

$$\cos \frac{r'}{\varrho} = \frac{A}{c}$$

(which is always a possible direction, since $\frac{A}{c} < 1$), we have

$$c \cos \frac{r}{\varrho} = \frac{c\,A}{c} = A,$$

$$r = \frac{c^2 - A^2}{A - A} = \frac{c^2 - A^2}{0} = \infty.$$

If we take $\frac{r}{\varrho} > \frac{r'}{\varrho}$ we have

$$\cos \frac{r}{\varrho} < \frac{A}{c},$$

$$c \cos \frac{r}{\varrho} < A,$$

and the denominator of the last member of (59), and therefore the value of r, become negative; so that, by § 52, the curve has no corresponding points. But when $\frac{r}{\varrho}$ passes the value π, $\cos \frac{r}{\varrho}$ begins to increase, till, when r takes a direction τ'' such that

$$\frac{\tau''}{\varrho} = 2\pi - \frac{\tau'}{\varrho},$$

we have (Tr. § 68)

$$\cos \frac{r}{\varrho} = \cos \left(2\pi - \frac{\tau'}{\varrho}\right) = \cos \frac{\tau'}{\varrho} = \frac{A}{c},$$

$$r = \infty;$$

and then, as r turns from τ'' to ϱ, $\cos \frac{r}{\varrho}$ goes through the same values, but in reverse order, as when r turned from ϱ to τ'; so that, when $\frac{r}{\varrho} = 2\pi$, we have again $r = c + A$.

Thus, the locus of (59) consists of one infinite branch which is limited by the lines τ'' and τ'.

161. *Corollary.* Again, since the numerator of (60) is positive, r admits of construction when the denominator is positive.

If

$$\frac{r}{\varrho} = 0, \qquad \cos \frac{r}{\varrho} = 1,$$

and we have

$$r = \frac{c^2 - A^2}{c + A} = c - A = FO.$$

As $\frac{r}{\varrho}$ increases, the denominator of (60) decreases, and the curve recedes from the origin, till, when r assumes the direction $-\tau''$,

$$\cos \frac{r}{\varrho} = \cos \left(\pi - \frac{\tau'}{\varrho}\right) = -\cos \frac{\tau'}{\varrho} = -\frac{A}{c},$$

$$c \cos \frac{r}{\varrho} = -\frac{c\,A}{c} = -A$$

$$r = \frac{c^2 - A^2}{-A + A} = \frac{c^2 - A^2}{0} = \infty.$$

If $\frac{r}{\varrho}$ is made still larger,

$$\cos \frac{r}{\varrho} < -\frac{A}{c},$$

$$c \cos \frac{r}{\varrho} < -A,$$

and r becomes negative and cannot be constructed; till, when r takes the direction $-\,\tau'$,

$$\cos \frac{r}{\varrho} = \cos\left(\pi + \frac{\tau'}{\varrho}\right) = -\cos\frac{\tau'}{\varrho} = -\frac{A}{c},$$

$$r = \infty;$$

and then, as r turns from $-\,\tau'$ to ϱ, $\cos \frac{r}{\varrho}$ goes through the same series of values, in reverse order, as when r turned from ϱ to $-\,\tau''$, till, when $\frac{r}{\varrho} = 2\,\pi$, $\quad r = c - A$ again.

The locus of (60) consists, therefore, of a single infinite branch, no part of which is contained between the lines $-\,\tau''$ and $-\,\tau'$; and the hyperbola is a curve of two infinite branches.

162. *Corollary.* Let us see what will result from considering the negative values of r in (59). When r takes the direction τ',

$$\cos \frac{r}{\varrho} = \cos \frac{\tau'}{\varrho} = \frac{A}{c},$$

$$r = \frac{c^2 - A^2}{A - A} = \frac{c^2 - A^2}{\pm\, 0} = \pm\,\infty;$$

which gives two points, in the directions τ' and $-\,\tau'$, infinitely distant from the origin.

If we take $\genfrac{}{}{0pt}{}{r}{\rho} > \genfrac{}{}{0pt}{}{r'}{\rho}$, the last member of (59) becomes negative; and the corresponding absolute value of r, if laid off in the direction opposite to that indicated by the value of $\genfrac{}{}{0pt}{}{r}{\rho}$, determines a certain point M'. Now, in the common system, the polar angle of M' would be (§ 20. g, h)

$$C'FM' = \genfrac{}{}{0pt}{}{-r}{\rho} = \genfrac{}{}{0pt}{}{r}{\rho} + \genfrac{}{}{0pt}{}{-r}{r} = \pi + \genfrac{}{}{0pt}{}{r}{\rho},$$

and (59) gives (Tr. §§ 61, 64)

$$\cos \genfrac{}{}{0pt}{}{r}{\rho} = \cos\left(\pi - \genfrac{}{}{0pt}{}{-r}{\rho}\right) = -\cos \genfrac{}{}{0pt}{}{-r}{r},$$

$$r = \frac{c^2 - A^2}{-c\cos \genfrac{}{}{0pt}{}{-r}{\rho} - A} = -\frac{c^2 - A^2}{c\cos \genfrac{}{}{0pt}{}{-r}{\rho} + A}, \qquad (61.)$$

which is the same absolute value of r as that which (60) gives, if we take $\genfrac{}{}{0pt}{}{r}{\rho}$ in (60) equal to $\genfrac{}{}{0pt}{}{-r}{\rho}$ in (61), that is, equal to $F'FM'$; and the negative sign in (61) means that the point is to be taken in the direction FM' from the origin, which is indicated in (60) by the positive sign. Hence the same point M' is determined by (60) and by (61); so that the series of points determined by the negative values of r in (59) is the same as that determined by the positive values of r in (60); and, if the rule of § 52 is disregarded, (59) may be taken for the equation of the whole hyperbola.

Suppose, for example, we take $\genfrac{}{}{0pt}{}{r}{\rho} = \pi$, (59) becomes

$$r = \frac{c^2 - A^2}{c\cos\pi - A} = \frac{c^2 - A^2}{-c - A} = -\frac{c^2 - A^2}{c + A} = -(c - A),$$

which gives the point C'.

In like manner, it may be shown that, if the rule of § 52 is disregarded, (60) is the equation of the whole hyperbola.

If (59) be compared with (43), it will be seen to be identical with it, and to answer to the analogous case, that, namely, in which the curve is supposed to begin at C. The absolute value of r in (60) is also the same as that in (44); and here again, for each equation, we suppose the curve to begin at C'. In the ellipse, however, we have $A > c$, and in the hyperbola, $A < c$.

163. *Corollary.* It is evident, as in § 137, that, if F' is made the origin, (59) is the equation of the left branch of the hyperbola, or (in the system of § 162) of the whole hyperbola, supposed to begin at C', and (60) is the equation of the right branch, or of the whole hyperbola, supposed to begin at C.

164. *Equation in Terms of the Eccentricity.* The *eccentricity* of the hyperbola is

$$e = \frac{2\,c}{2\,A} = \frac{c}{A} = \sec\frac{\tau'}{\varrho} = \sec\frac{\tau''}{\varrho}.$$

Then, if we take

$$p = FC' = CF' = c - A = A\,e - A = A\,(e - 1),$$

(59) becomes

$$r = \frac{A^2\,e^2 - A^2}{A\,e\cos\frac{r}{\varrho} - A} = \frac{A^2\,(e^2 - 1)}{A\,(e\cos\frac{r}{\varrho} - 1)}$$
$$= \frac{A\,(e - 1)\,(e + 1)}{e\cos\frac{r}{\varrho} - 1} = \frac{p\,(1 + e)}{e\cos\frac{r}{\varrho} - 1}. \qquad (62.)$$

In like manner, (60) becomes

$$r = \frac{p\,(1 + e)}{1 + e\cos\frac{r}{\varrho}}. \qquad (63.)$$

165. *Corollary.* Since $A < c$, $e > 1$.

166. *Definitions.* That part of the line joining the foci

of the hyperbola which is contained between its branches is called the *transverse axis.*

The extremities of the transverse axis are called the *vertices* of the hyperbola.

The *conjugate axis* is a line which passes through the centre of the hyperbola, at right angles with the transverse axis, and has on each side of it the length B.

Thus $C'C$ is the transverse axis of the hyperbola of Fig. 54, and $B'B$ its conjugate axis.

167. *Equation, referred to Principal Diameters.* The hyperbola may be referred, by § 95, to a system of rectangular coördinates in which the origin is at the centre and the axes have the directions of the transverse and conjugate axes. In this case, we have, as in § 142,

$$r^{\circ} = c,$$

$$r = \sqrt{(x^2 + y^2 + c^2 + 2\,c\,x)},$$

$$r \cos \frac{r}{\varrho} = x + c.$$

Then, since (59) is of the same form with (43), the substitution of the above values must lead in both cases to the same result, namely,

$$(A^2 - c^2)\,x^2 + A^2\,y^2 = A^2\,(A^2 - c^2). \qquad (64.)$$

Freeing (60) from fractions, and substituting, we have

$$c\,r \cos \frac{r}{\varrho} + A\,r = c^2 - A^2,$$

$$c\,x + c^2 + A\sqrt{(x^2 + y^2 + c^2 + 2\,c\,x)} = c^2 - A^2,$$

$$A\sqrt{(x^2 + y^2 + c^2 + 2\,c\,x)} = -(A^2 + c\,x);$$

which, by squaring and reducing, also gives (64). If we substitute B^2 in (64), it becomes

$$-B^2\,x^2 + A^2\,y^2 = -A^2\,B^2, \qquad (65.)$$

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1; \tag{66.}$$

so that either (65) or (66) is the rectangular equation of the hyperbola, referred to its transverse and conjugate axes.

168. *Corollary.* If it be remembered that B^2 in the hyperbola is the negative of B^2 in the ellipse, it will be seen that (65) and (66) are in truth the same as (47) and (48); but the B which is real in the ellipse is imaginary in the hyperbola.

169. *Equilateral Hyperbola.* If $B = A$, (66) gives

$$\frac{x^2}{A^2} - \frac{y^2}{A^2} = 1,$$

$$x^2 - y^2 = A^2. \tag{67.}$$

Such an hyperbola is called *equilateral.*

170. *Conjugate Hyperbolas.* If an hyperbola be drawn having the same foci as that of Fig. 54, but having $2B$ for the given length, then, since

$$B = \sqrt{(c^2 - A^2)},$$

$$A^2 + B^2 = c^2,$$

$$A = \sqrt{(c^2 - B^2)};$$

so that B takes the place of A, and A the place of B; and (66) becomes

$$\frac{x^2}{B^2} - \frac{y^2}{A^2} = 1.$$

Now, if this new hyperbola be turned about its centre till its foci come into the axis of y, or, what comes to the same thing, if its conjugate axis be taken for a new axis of x, we have in § 99

$$\frac{x_1}{x} = \tfrac{1}{2}\pi,$$

so that (17) and (18) become

$$x = -y_1, \quad y = x_1.$$

Substituting these values in the preceding equation, and removing the numbers below the letters, we have

$$\frac{y^2}{B^2} - \frac{x^2}{A^2} = 1. \qquad (68.)$$

The loci of (66) and (68), if constructed for the same coördinate axes, are said to be *conjugate to each other.* Fig. 55 affords an instance of such hyperbolas.

171. *Rectangular Equation, referred to Vertex.* (65) may be transformed, by § 100, to a system of rectangular coördinates in which the origin is at the right-hand vertex, and the axis of x on the transverse axis. In this case,

$$x^\circ = A, \qquad y^\circ = 0,$$

and (19) becomes

$$x = x_1 + A, \qquad y = y_1.$$

Substituting these values in (65), and removing the numbers below the letters, we have

$$-B^2 x^2 - 2AB^2 x - A^2 B^2 + A^2 y^2 = -A^2 B^2,$$

$$y^2 = \frac{2B^2}{A} x + \frac{B^2}{A^2} x^2. \qquad (69.)$$

172. *Definitions.* A line drawn through the centre of an hyperbola, and terminated by the curve, is called a *diameter* either of that hyperbola or of its conjugate. Thus, $B_1'B_1$ is a diameter of either of the hyperbolas of Fig. 55.

The extremities of a diameter are called its *vertices.*

173. *Polar Equation, referred to Centre.* The hyperbola may be referred to a polar system in which the origin is at the centre, and the polar axis coincides with the axis of x. In this case, we have, as in § 146,

$$x = r \cos {}^{r}_{\varrho}, \quad y = r \sin {}^{r}_{\varrho},$$

which, substituted in (65), give

$$r^2 \left(- B^2 \cos^2 \genfrac{}{}{0pt}{}{r}{\varrho} + A^2 \sin^2 \genfrac{}{}{0pt}{}{r}{\varrho}\right) = - A^2 B^2,$$

$$r^2 = \frac{AB}{\sqrt{\left(B^2 \cos^2 \genfrac{}{}{0pt}{}{r}{\varrho} - A^2 \sin^2 \genfrac{}{}{0pt}{}{r}{\varrho}\right)}}. \tag{70.}$$

In the same way, (68) gives

$$r^2 \left(B^2 \cos^2 \genfrac{}{}{0pt}{}{r}{\varrho} - A^2 \sin^2 \genfrac{}{}{0pt}{}{r}{\varrho}\right) = - A^2 B^2,$$

$$r = \frac{AB}{\sqrt{\left(A^2 \sin^2 \genfrac{}{}{0pt}{}{r}{\varrho} - B^2 \cos^2 \genfrac{}{}{0pt}{}{r}{\varrho}\right)}}. \tag{71.}$$

174. *Corollary*. The value of r in (70) is real when the dènominator of the second member is real; that is, when

$$A^2 \sin^2 \genfrac{}{}{0pt}{}{r}{\varrho} < B^2 \cos^2 \genfrac{}{}{0pt}{}{r}{\varrho},$$

$$\left(\frac{\sin^2 \genfrac{}{}{0pt}{}{r}{\varrho}}{\cos^2 \genfrac{}{}{0pt}{}{r}{\varrho}} = \tan^2 \genfrac{}{}{0pt}{}{r}{\varrho}\right) < \frac{B^2}{A^2}.$$

On the other hand, r is real in (71) when

$$A^2 \sin^2 \genfrac{}{}{0pt}{}{r}{\varrho} > B^2 \cos^2 \genfrac{}{}{0pt}{}{r}{\varrho}$$

$$\tan^2 \genfrac{}{}{0pt}{}{r}{\varrho} > \frac{B^2}{A^2}.$$

Now, if through the centre of the hyperbola two lines, τ' and τ'', be drawn, such that

$$\tan \genfrac{}{}{0pt}{}{\tau'}{\varrho} = \frac{B}{A}, \quad \tan \genfrac{}{}{0pt}{}{\tau''}{\varrho} = - \frac{B}{A}, \tag{72.}$$

or

$$\tan^2 \genfrac{}{}{0pt}{}{\tau'}{\varrho} = \tan^2 \genfrac{}{}{0pt}{}{\tau''}{\varrho} = \frac{B^2}{A^2},$$

then, for any direction of r between τ'' and τ', or between $-\tau''$ and $-\tau'$, $\tan^2 \genfrac{}{}{0pt}{}{r}{\varrho} < \frac{B^2}{A^2}$; and for any direction of r between τ' and $-\tau''$, or between $-\tau'$ and τ'', $\tan^2 \genfrac{}{}{0pt}{}{r}{\varrho} > \frac{B^2}{A^2}$; while, if r has the direction of ι', ι'', $-\iota'$, or $-\iota$ $\tan^2 \frac{r}{\varrho} = \frac{B^2}{A^2}$, and r becomes ∞ both in (70) and (71). Hence, the whole of the first hyperbola is contained between the lines ι'' and ι', $-\iota''$ and $-\iota'$, and meets them at an infinite distance from the centre; and the whole of the conjugate hyperbola is contained between ι' and $-\iota''$, $-\iota'$ and ι'', and likewise meets them at an infinite distance from the centre.

175. *Corollary.* It may be shown, by the same reasoning as in § 147, that *every diameter of the hyperbola is bisected at the centre;* so that half the length of a diameter included between ι'' and ι' is given by (70), and half that of a diameter included between ι' and $-\iota''$ by (71).

For the semi-transverse axis of the first hyperbola, $\frac{r}{\varrho} = 0$, and (70) becomes (Tr. § 55)

$$r = \frac{AB}{\sqrt{B^2}} = A.$$

For the semi-transverse axis of the conjugate hyperbola, $\frac{r}{\varrho} = \frac{1}{2}\pi$, and (71) becomes (*Ibid.*)

$$r = \frac{AB}{\sqrt{A^2}} = B,$$

which is the semi-conjugate axis of the first hyperbola; so that,

If two hyperbolas are conjugate to each other, the transverse axis of the one is the conjugate axis of the other.

The transverse and conjugate axes of an hyperbola are called its *principal diameters.*

176. *Definition.* The lines ι' and ι'', determined by (72), are called the *asymptotes* of either of the two hyperbolas which they limit.

177. *Corollary.* The asymptotes of an hyperbola may be drawn as follows: — At C (Fig. 55) erect a perpendicular, $E'CE$, taking

$$CE \text{ or } CE' = \pm B;$$

connect AE and AE'. Then

$$\tan CAE = \frac{B}{A},$$

$$\tan CAE' = -\frac{B}{A};$$

so that AE and AE', indefinitely produced, are the required asymptotes.

178. *Corollary.* The triangles ACE and ACE' give

$$AE = AE' = \sqrt{(A^2 + B^2)} = c,$$

so that

$$\cos CAE = \cos CAE' = \frac{A}{c}.$$

Hence $\mathfrak{r}'$ and $\mathfrak{r}''$ in § 176 denote the same directions as in § 160. Also

$$\sec CAE = \sec CAE' = \frac{c}{A} = e;$$

so that the greater the angle between the asymptotes, the greater the eccentricity (Tr. § 70).

179. *Corollary.* For the equilateral hyperbola, $B = A$, so that (Tr. §§ 59, 64)

$$\tan \frac{\mathfrak{r}'}{x} = \frac{A}{A} = 1,$$

$$\frac{\mathfrak{r}'}{x} = \tfrac{1}{4}\pi;$$

$$\tan \frac{\mathfrak{r}''}{x} = -1,$$

$$\frac{\mathfrak{r}''}{x} = -\tfrac{1}{4}\pi;$$

$$\frac{\mathfrak{r}'}{\mathfrak{r}''} = \frac{x}{\mathfrak{r}''} + \frac{\mathfrak{r}'}{x} = -\frac{\mathfrak{r}''}{x} + \frac{\mathfrak{r}'}{x} = \tfrac{1}{4}\pi + \tfrac{1}{4}\pi = \tfrac{1}{2}\pi.$$

180. *Equation, referred to the Asymptotes.* (65) may be transformed by § 102 to a system of coördinates in which $\mathfrak{r}''$ is the axis of abscissas, and $\mathfrak{r}'$ that of ordinates. In that case,

$$x^\circ = 0, \quad y^\circ = 0,$$

$$\cos\frac{x_1}{x} = \frac{A}{c}, \quad \sin\frac{x_1}{x} = -\frac{B}{c}, \quad \cos\frac{y_1}{x} = \frac{A}{c}, \quad \sin\frac{y_1}{x} = \frac{B}{c},$$

and (24) and (25) become

$$x = \frac{A}{c}(x_1 + y_1), \quad y = \frac{B}{c}(y_1 - x_1),$$

which, substituted in (65), give

$$\frac{A^2B^2}{c^2}[-(x_1 + y_1)^2 + (y_1 - x_1)^2] = -A^2B^2,$$

$$(x_1 + y_1)^2 - (y_1 - x_1)^2 = 4\,x_1\,y_1 = c^2,$$

or, by dividing and removing the numbers,

$$x\,y = \frac{c^2}{4}. \qquad (73.)$$

In like manner, by substituting the above values of x and y in (68), we should have, for the equation of the conjugate hyperbola,

$$x\,y = -\frac{c^2}{4}. \qquad (74.)$$

181. *Definition.* If α denotes the direction of the transverse axis and α_1 and β_1 those of two diameters which satisfy the equation

$$\tan\frac{\alpha_1}{\alpha}\tan\frac{\beta_1}{\alpha} = \frac{B^2}{A^2}. \qquad (75.)$$

these diameters are said to be *conjugate to each other.*

Thus, $C_1'C_1$ and $B_1'B_1$ (Fig. 55) are conjugate to each other.

182. *Corollary.* Since the B^2 of the hyperbola is the negative of the B^2 of the ellipse, the condition of (75) is the same as that of (52).

183. *Corollary.* Since the second member of (75) is necessarily positive, the two factors of the first member must have the

same sign; so that the angles made by two conjugate diameters with the transverse axis must be both acute or both obtuse.

If α_1 is taken in the direction of the transverse axis, we have (Tr. §§ 55, 57)

$$\overset{\alpha_1}{\underset{\alpha}{}} = 0, \quad \tan \overset{\alpha_1}{\underset{\alpha}{}} = \pm 0,$$

$$\tan \overset{\beta_1}{\underset{\alpha}{}} = \pm \frac{B^2}{A^2 \times 0} = \pm \infty,$$

$$\overset{\beta_1}{\underset{\alpha}{}} = \tfrac{1}{2}\pi, \quad \text{or} \quad = \tfrac{3}{2}\pi;$$

and, therefore, *the principal diameters of an hyperbola are conjugate to each other.*

184. If

$$\overset{\alpha_1}{\underset{\alpha}{}} < \overset{\tau'}{\underset{\alpha}{}},$$

then (Tr. § 70)

$$\tan \overset{\alpha_1}{\underset{\alpha}{}} < \tan \overset{\tau'}{\underset{\alpha}{}},$$

$$\frac{\tan \overset{\alpha_1}{\underset{\alpha}{}} \tan \overset{\beta_1}{\underset{\alpha}{}}}{\tan \overset{\alpha_1}{\underset{\alpha}{}}} > \frac{\tan \overset{\alpha_1}{\underset{\alpha}{}} \tan \overset{\beta_1}{\underset{\alpha}{}}}{\tan \overset{\tau'}{\underset{\alpha}{}}},$$

$$\tan \overset{\beta_1}{\underset{\alpha}{}} > \frac{B^2}{A^2} \div \frac{B}{A},$$

$$\tan \overset{\beta_1}{\underset{\alpha}{}} > \frac{B}{A},$$

and, since $\overset{\beta_1}{\underset{\alpha}{}}$ is acute, by the last section,

$$\overset{\beta_1}{\underset{\alpha}{}} > \overset{\tau'}{\underset{\alpha}{}}.$$

If $\overset{\alpha_1}{\underset{\alpha}{}}$ is obtuse, and if

$$\overset{\alpha_1}{\underset{\alpha}{}} < \overset{-\tau''}{\underset{\alpha}{}},$$

then (Tr. §§ 70, 71)

$$\tan \genfrac{}{}{0pt}{}{\alpha_1}{\alpha} < \left(\tan \genfrac{}{}{0pt}{}{-\tau''}{\alpha} = -\frac{B}{A}\right),$$

$$\tan \genfrac{}{}{0pt}{}{\beta_1}{\alpha} > \frac{B^2}{A^2} \div \left(-\frac{B}{A}\right),$$

$$\tan \genfrac{}{}{0pt}{}{\beta_1}{\alpha} > -\frac{B}{A},$$

and, since $\genfrac{}{}{0pt}{}{\beta_1}{\alpha}$ is obtuse, by the last section,

$$\genfrac{}{}{0pt}{}{\beta_1}{\alpha} > \genfrac{}{}{0pt}{}{-\tau''}{\alpha}.$$

Hence, *of two conjugate diameters of an hyperbola, one terminates in that hyperbola, the other in its conjugate hyperbola.*

If

$$\genfrac{}{}{0pt}{}{\alpha_1}{\alpha} = \genfrac{}{}{0pt}{}{\tau'}{\alpha}, \quad \text{or} \quad = \genfrac{}{}{0pt}{}{-\tau''}{\alpha},$$

$$\tan \genfrac{}{}{0pt}{}{\beta_1}{\alpha} = \frac{\tan \genfrac{}{}{0pt}{}{\alpha_1}{\alpha} \tan \genfrac{}{}{0pt}{}{\beta_1}{\alpha}}{\tan \genfrac{}{}{0pt}{}{\tau'}{\alpha}} = \frac{B}{A} = \tan \genfrac{}{}{0pt}{}{\tau'}{\alpha},$$

or

$$= \frac{\tan \genfrac{}{}{0pt}{}{\alpha_1}{\alpha} \tan \genfrac{}{}{0pt}{}{\beta_1}{\alpha}}{\tan \genfrac{}{}{0pt}{}{-\tau''}{\alpha}} = -\frac{B}{A} = \tan \genfrac{}{}{0pt}{}{-\tau''}{\alpha},$$

$$\genfrac{}{}{0pt}{}{\beta_1}{\alpha} = \genfrac{}{}{0pt}{}{\tau'}{\alpha}, \quad \text{or} \quad = \genfrac{}{}{0pt}{}{-\tau''}{\alpha};$$

that is, *if one of two conjugate diameters coincides with an asymptote, the other coincides with the same asymptote.*

185. *Corollary.* In the equilateral hyperbola, $B = A$, so that, if we denote the direction of the conjugate axis by β, (75) gives (Tr. §§ 8, 10)

$$\tan {}^{\alpha_1}_{\alpha} \tan {}^{\beta_1}_{\alpha} = 1,$$

$$\tan {}^{\alpha_1}_{\alpha} = \cot {}^{\beta_1}_{\alpha} = \tan \left(\tfrac{1}{2}\pi - {}^{\beta_1}_{\alpha}\right),$$

$${}^{\alpha_1}_{\alpha} = \tfrac{1}{2}\pi - {}^{\beta_1}_{\alpha} = {}^{\beta}_{\alpha} - {}^{\beta_1}_{\alpha} = {}^{\alpha}_{\beta_1} + {}^{\beta}_{\alpha} = {}^{\beta}_{\beta_1};$$

so that *two conjugate diameters of an equilateral hyperbola make equal angles respectively with the principal diameters.*

186. *Equation, referred to Conjugate Diameters.* (65) may be referred to conjugate diameters by § 102, if we take $x^\circ = 0$, $y^\circ = 0$. (65) will, evidently, give an equation differing from (53) only in the sign of B^2, namely,

$$\left(A^2 \sin^2 {}^{x_1}_{x} - B^2 \cos^2 {}^{x_1}_{x}\right) x_1^2 + \left(A^2 \sin^2 {}^{y_1}_{x} - B^2 \cos^2 {}^{y_1}_{x}\right) y_1^2$$
$$+ 2\left(A^2 \sin {}^{x_1}_{x} \sin {}^{y_1}_{x} - B^2 \cos {}^{x_1}_{x} \cos {}^{y_1}_{y}\right) x_1 y_1 = - A^2 B^2. \quad (76.)$$

If the semidiameters which have the directions of x_1 and y_1 are denoted respectively by A_1 and B_1, and if the axis of x_1 is taken on that diameter which cuts the hyperbola, (70) and (71) give, by § 175,

$$A_1 = \frac{AB}{\sqrt{\left(B^2 \cos^2 {}^{x_1}_{x} - A^2 \sin^2 {}^{x_1}_{x}\right)}},$$

$$B_1 = \frac{AB}{\sqrt{\left(A^2 \sin^2 {}^{y_1}_{x} - B^2 \cos^2 {}^{y_1}_{x}\right)}};$$

or

$$A^2 \sin^2 {}^{x_1}_{x} - B^2 \cos^2 {}^{x_1}_{x} = - \frac{A^2 B^2}{A_1^2}, \quad (77.)$$

$$A^2 \sin^2 {}^{y_1}_{x} - B^2 \cos^2 {}^{y_1}_{x} = \frac{A^2 B^2}{B_1^2}. \quad (78.)$$

(75) gives, as in § 151,

$$A^2 \sin\frac{x_1}{x} \sin\frac{y_1}{x} - B^2 \cos\frac{x_1}{x} \cos\frac{y_1}{y} = 0. \qquad (79.)$$

Substituting (77), (78), and (79) in (76), we have

$$-\frac{A^2B^2x_1^2}{A_1^2} + \frac{A^2B^2y_1^2}{B_1^2} = -A^2B^2,$$

or, dividing by $-A^2B^2$, and omitting the numbers under the coördinate letters,

$$\frac{x^2}{A_1^2} - \frac{y^2}{B_1^2} = 1; \qquad (80.)$$

which, since the principal diameters are conjugate to each other, includes (66).

187. *Corollary.* In like manner,

$$\frac{y^2}{B_1^2} - \frac{x^2}{A_1^2} = 1, \qquad (81.)$$

is the equation of the hyperbola conjugate to that of (80), referred to the same axes.

188. *Corollary.* (80) gives, by reduction,

$$x = \pm\frac{A_1}{B_1}\sqrt{(y^2 + B_1^2)}, \qquad (82.)$$

$$y = \pm\frac{B_1}{A_1}\sqrt{(x^2 - A_1^2)}; \qquad (83.)$$

so that the curve is symmetrical with respect to either axis.

Any value of y in (82) makes x real; but y is imaginary in (83), if, in absolute value,

$$x < A_1.$$

Hence the curve extends from $-\infty$ to $+\infty$ in the direction of the axis of y; but no part of it is included between two lines drawn parallel to the axis of y and cutting the axis of x at the distances A_1 and $-A_1$ from the origin.

189. *Theorem.* The locus of every rectangular equation of the form (66) is an hyperbola which has its centre at the origin and its foci in the axis of x, and for which c, A, and B have the same meanings as in § 158.

Proof. We are to prove that $2\,A$ is equal to the difference of the distances of any point of the locus of the equation from two points of which the coördinates are respectively

$$x = c, \quad y = 0; \qquad x = -c, \quad y = 0.$$

The distance of r_1 of any point x, y from the former of these two points is, by (7),

$$r_1 = \sqrt{[(x-c)^2 + y^2]} = \sqrt{(x^2 - 2\,c\,x + c^2 + y^2)}.$$

If x, y is a point of the locus of (66), that equation gives

$$y^2 = \frac{B^2\,x^2}{A^2} - B^2.$$

The substitution of this value of y^2, and also of

$$c^2 = A^2 + B^2,$$

gives, by a reduction like that of § 153,

$$r_1 = \sqrt{\left(\frac{A^2 + B^2}{A^2}x^2 - 2\,c\,x + A^2\right)} = \pm\left(\frac{c\,x}{A} - A\right).$$

Again, the distance r_2 of the same point from the other supposed focus is

$$r_2 = \sqrt{[(x+c)^2 + y^2]} = \pm\left(\frac{c\,x}{A} + A\right).$$

The double signs are to be interpreted in such a way as to make r_1 and r_2 positive. Now,

$$x^2 = A^2 + \frac{A^2\,y^2}{B^2}, \qquad c^2 = A^2 + B^2,$$

$$\text{i. e.} \quad x^2 > A^2, \qquad c^2 > A^2,$$

so that, if we attend only to the absolute value of x,

$$x > A, \qquad c > A,$$

$$cx > A^2, \qquad \frac{cx}{A} > A.$$

Hence, the positive values of r_1 and r_2 are, if x is positive,

$$r_1 = +\left(\frac{cx}{A} - A\right) = \frac{cx}{A} - A.$$

$$r_2 = +\left(\frac{cx}{A} + A\right) = \frac{cx}{A} + A;$$

but, if x is negative,

$$r_1 = -\left(\frac{cx}{A} - A\right) = -\frac{cx}{A} + A,$$

$$r_2 = -\left(\frac{cx}{A} + A\right) = -\frac{cx}{A} - A.$$

In the former case, then,

$$r_2 - r_1 = 2A;$$

and, in the latter case,

$$r_1 - r_2 = 2A.$$

190. *Corollary.* By interchanging x and y together and A and B together in the above proof, it may be shown that the locus of any equation of the form (68) is the conjugate to the hyperbola of (66).

191. *Corollary.* Any equation of the form (59) or (60), in which $A < c$, will give, by transformation, an equation of the form (66); and since, by § 189, the locus of the latter equation is always an hyperbola, the locus of either of the former equations is always one branch, or, if taken as in § 162, both branches, of an hyperbola.

192. *Problem.* To construct an equation of the form (66) or (68).

Solution. In the first case, draw an hyperbola, by § 158, having its transverse axis (or given length) equal to $2A$, and its foci at the points for which

$$x = \sqrt{(A^2 + B^2)}, \quad y = 0; \quad x = -\sqrt{(A^2 + B^2)}, \quad y = 0.$$

It must, by § 189, be the locus required.

By interchanging x and y together, and A and B together, in this solution, it applies to an equation of the form (68).

193. Examples.

1. Find the locus of $\frac{x^2}{4} - \frac{y^2}{2} = 1$, and draw its asymptotes.

Solution. The equation is of the form (66), and

$$A = \sqrt{4} = 2,$$
$$B = \sqrt{2} = 1.414.$$

In Fig. 56, take

$$C'A = AC = 2,$$
$$E'C = CE = 1.414,$$
$$FA = AF' = AE = \sqrt{(A^2 + B^2)} = c;$$

and F and F' are the foci of the hyperbola, and AE and AE' its asymptotes.

2. Find the locus of $\frac{x^2}{16} - 9y^2 = 1$, and draw its asymptotes.

3. Find the locus of $\frac{y^2}{4} - \frac{x^2}{9} = 1$, and draw its asymptotes.

4. Find the loci of $x^2 - y^2 = 4$, and of $y^2 - x^2 = 4$, and draw their asymptotes.

V.

THE LOCUS OF EVERY POINT IN A PLANE WHICH IS EQUALLY DISTANT FROM A GIVEN POINT AND A GIVEN STRAIGHT LINE IN THAT PLANE.

194. *Scholium.* This locus is called the *parabola*, the given point its *focus*, the given line its *directrix*, and the

line drawn through the focus, perpendicular to the directrix, its *axis*.

The parabola may readily be drawn from its definition.

In the treatment of the parabola, the following notation will be used: —

p = half the distance of the focus from the directrix.

195. *Polar Equation.* Let F (Fig. 57) be the focus, and DE the directrix, of a parabola. Let F be the origin of a system of polar coördinates in which the axis has the same direction, FP, as the axis of the parabola. If M is a point of the locus, the definition gives (Tr. § 32)

$$FP = r \cos {}^{r}_{\rho},$$

$$r = FM = QM = DP = DF + FP = 2p + r \cos {}^{r}_{\rho},$$

$$r\left(1 - \cos {}^{r}_{\rho}\right) = 2p,$$

$$r = \frac{2p}{1 - \cos {}^{r}_{\rho}}; \tag{84.}$$

which is, therefore, the polar equation of the parabola, the origin being at the focus, and the polar axis being the axis of the parabola.

196. *Corollary.* As the numerator of (84) is positive, r will be positive when the denominator is positive, and this will be the case for all directions of r, since the cosine of an angle cannot exceed 1, in absolute value. The curve, therefore, extends on all sides of the origin.

If $\quad {}^{r}_{\rho} = 0, \quad \cos {}^{r}_{\rho} = 1,$

so that $\quad r = \frac{2p}{0} = \infty.$

At the moment, however, that the rotating radius vector begins to turn from the direction of the axis, $\cos {}^{r}_{\varrho}$ becomes less than 1, and r, in (84), becomes finite. As ${}^{r}_{\varrho}$ increases, $\cos {}^{r}_{\varrho}$ decreases, r, in (84), decreases, and the curve approaches the origin; till, when ${}^{r}_{\varrho} = \pi$, $\cos {}^{r}_{\varrho} = -1$, and

$$r = \frac{2p}{1+1} = p.$$

As ${}^{r}_{\varrho}$ passes from π to 2π, $\cos {}^{r}_{\varrho}$ goes through the same series of values, but in reverse order, as when ${}^{r}_{\varrho}$ passed from 0 to π, so that the curve recedes from the origin at the same rate at which it approached it; till, when ${}^{r}_{\varrho} = 2\pi$, we have again $\cos {}^{r}_{\varrho} = 1$, $r = \infty$; and then, for any greater increase of ${}^{r}_{\varrho}$, $\cos {}^{r}_{\varrho}$ repeats the same series of values as before, that is, the curve is an *oval.* Since, however, it is infinitely long, it does not practically return into itself.

197. *Corollary.* If the curve is conceived to begin at C', that is, if we take for the polar axis ϱ_1, in the direction opposite to ϱ, we shall have (Tr. § 65)

$${}^{r}_{\varrho} = {}^{\varrho_1}_{\varrho} + {}^{r}_{\varrho_1} = \pi + {}^{r}_{\varrho_1},$$

$$\cos {}^{r}_{\varrho} = -\cos {}^{r}_{\varrho_1},$$

and (84) becomes

$$r = \frac{2p}{1 + \cos {}^{r}_{\varrho_1}}. \qquad (85.)$$

198. *Corollary.* If, in (45) and (46), we take $e = 1$, which is the maximum value of e for the ellipse, those equations become identical with (84) and (85). If, again, in (62) and (63),

we take $e = 1$, which is the minimum value of e for the hyperbola, these equations give the same (absolute) values of r as (84) and (85).

199. *Definition.* The point at which the parabola cuts its axis, which is at the distance p from the focus and from the directrix, is called the *vertex* of the axis or of the parabola.

200. *Rectangular Equation.* The parabola may, by § 94, be referred to a rectangular system in which the origin is at the vertex, and the axis of x is the axis of the parabola. In this case

$$r^\circ = p, \quad {r^\circ \atop \rho} = \pi, \quad {x \atop \rho} = 0,$$

so that

$$\sin\left({r^\circ \atop \rho} - {x \atop \rho}\right) = 0, \cos\left({r^\circ \atop \rho} - {x \atop \rho}\right) = -1, \cos\left({r \atop \rho} - {x \atop \rho}\right) = \cos {r \atop \rho};$$

and (8) and (9) become

$$r \cos {r \atop \rho} = x - r^\circ = x - p,$$

$$r = \sqrt{(x^2 + y^2 + p^2 - 2\,p\,x)}.$$

(84), freed from fractions, gives by the substitution of the above values,

$$r - r\cos {r \atop \rho} = \sqrt{(y^2 + x^2 - 2\,p\,x + p^2)} - x + p = 2\,p,$$

$$\sqrt{(y^2 + x^2 - 2\,p\,x + p^2)} = x + p,$$

$$y^2 + x^2 - 2\,p\,x + p^2 = x^2 + 2\,p\,x + p^2,$$

$$y^2 = 4\,p\,x. \qquad (86.)$$

(86) is, therefore, the rectangular equation of the parabola, the origin being at the vertex, and the axis of x on the axis of the curve.

201. *Corollary.* In the above section, it is supposed that the positive direction of the axis of x is the same as ρ, that is, as the direction *from* the vertex *to* the focus. If the directions of the axes be reversed, all the values of x and y are thereby changed in sign, but not otherwise affected; so that (86) will become

$$y^2 = -4\,p\,x.$$

But this form is included in (86), if p be taken to denote the distance of the focus *from* the vertex; for then p becomes negative, when the focus is in the negative direction from the vertex. (86), therefore, applies to a parabola which curves towards the left.

202. *Corollary.* If the axis of the parabola be taken for that of ordinates, we have, in § 99, (Tr. §§ 55, 64,)

$$\frac{x_1}{x} = -\tfrac{1}{2}\pi, \quad \sin\frac{x_1}{x} = -1, \quad \cos\frac{x_1}{x} = 0;$$

and (17) and (18) become

$$x = y_1, \quad y = -x_1,$$

which, substituted in (86), give, after the omission of the subjacent numbers,

$$x^2 = 4\,p\,y; \tag{87.}$$

which is, therefore, the rectangular equation of the parabola, the origin being at the vertex, and the axis of y on the axis of the curve.

203. *Definitions.* Any line, as $C_1\,X_1$ (Fig. 57), which is drawn perpendicular to the directrix from a point of the parabola, is called a *diameter* of the parabola.

The point at which it cuts the curve, as C_1, is called the *vertex* of the diameter.

The axis of a parabola is its *principal diameter.*

204. *Corollary.* If p_1 denotes the distance of the vertex of

any diameter from the directrix, and if x° and y° are the coördinates of the vertex,

$$p_1 = p + x^\circ. \qquad (88.)$$

205. *Definition.* If a line is drawn from the vertex of a parabola, so that, if its direction is denoted by β_1, and that of the axis by α, and if y° is used as above,

$$\tan \frac{\beta_1}{\alpha} = \frac{2p}{y^\circ}, \qquad (89.)$$

it is called the *conjugate* of that diameter.

206. *Corollary.* If the diameter considered is the axis, $y^\circ = 0$, and (89) gives (Tr. § 55)

$$\tan \frac{\beta_1}{\alpha} = \frac{2p}{0} = \infty,$$

$$\frac{\beta_1}{\alpha} = \tfrac{1}{2}\pi;$$

so that *the conjugate of the axis is perpendicular to it.*

207. *Parabola referred to Conjugate Axes.* (86) may be transformed, by § 102, to a system of coördinates in which the axis of x is on any diameter, and the axis of y on its conjugate. In this case,

$$\frac{x_1}{x} = 0, \quad \sin \frac{x_1}{x} = 0, \quad \cos \frac{x_1}{x} = 1,$$

and (24) and (25) become

$$x = x^\circ + x_1 + y_1 \cos \frac{y_1}{x},$$

$$y = y^\circ + y_1 \sin \frac{y}{x};$$

which, substituted in (86), give

$$y_1^2 \sin^2 \frac{y_1}{x} + 2\, y^\circ y_1 \sin \frac{y_1}{x} + y^{\circ 2} = 4\, p\, x_1 + 4\, p\, y_1 \cos \frac{y_1}{x} + 4\, p\, x^\circ,$$

$$y_1^2 \sin^2 \frac{y_1}{x} + 2 \left(y^\circ \sin \frac{y_1}{x} - 2\, p \cos \frac{y_1}{x}\right) y_1 + y^{\circ 2} - 4\, p\, x^\circ = 4\, p\, x_1. \quad (90.)$$

But (89) gives

$$\tan\frac{y_1}{x} = \frac{\sin\frac{y_1}{x}}{\cos\frac{y_1}{x}} = \frac{2p}{y^\circ},$$

$$y^\circ \sin\frac{y_1}{x} = 2p\cos\frac{y_1}{x},$$

$$y^\circ \sin\frac{y_1}{x} - 2p\cos\frac{y_1}{x} = 0. \qquad (91.)$$

Again, since the origin is in the parabola, its coördinates must satisfy the equation of the parabola; so that

$$y^{\circ 2} = 4\,p\,x^\circ,$$

$$y^{\circ 2} - 4\,p\,x^\circ = 0. \qquad (92.)$$

The substitution of (91) and (92) in (90) gives

$$y_1^2 \sin^2\frac{y_1}{x} = 4\,p\,x_1,$$

$$y_1^2 = 4\,p\,x_1 \operatorname{cosec}^2\frac{y_1}{x}. \qquad (93.)$$

Tr. §§ 10, 12, and 14 give, together with (88) and (89),

$$\operatorname{cosec}^2\frac{y_1}{x} = \sec^2\frac{y_1}{x}\cos^2\frac{y_1}{x}\operatorname{cosec}^2\frac{y_1}{x} = \sec^2\frac{y_1}{x}\,\frac{\cos^2\frac{y_1}{x}}{\sin^2\frac{y_1}{x}} = \sec^2\frac{y_1}{x}\cot^2\frac{y_1}{x}$$

$$= \frac{\sec^2\frac{y_1}{x}}{\tan^2\frac{y_1}{x}} = \frac{\tan^2\frac{y_1}{x}+1}{\tan^2\frac{y_1}{x}} = \frac{4p^2+y^{\circ 2}}{4p^2} = \frac{4p^2+4p\,x^\circ}{4p^2} = \frac{p+x^\circ}{p},$$

$$p\operatorname{cosec}^2\frac{y_1}{x} = p + x^\circ = p_1,$$

and (93) becomes, by the omission of the numbers below the coördinate letters,

$$y^2 = 4\,p_1\,x; \qquad (94.)$$

which, by § 206, includes (86).

208. *Corollary.* For all real values of y, y^2, and therefore

$4p_1 x$, must be positive; so that x must have the same sign as p_1, that is, *the whole curve is on the same side of the axis of y.*

For each value of x, (94) gives two values of y, viz.:

$$y = \pm 2 \sqrt{(p_1 x)},$$

which only differ in respect to their signs. *The parabola is therefore symmetrical with respect to the axis of y.*

But for each value of y, (94) gives one value of x, viz.:

$$x = \frac{y^2}{4p_1}.$$

Hence, any straight line which is perpendicular to the directrix of the parabola will cut the curve in one point, and only one.

If $x = 0$, $y = \pm 0$; and, as x increases, y increases; till, when $x = \infty$, $y^2 = \infty$, $y = \sqrt{\infty} = \pm \infty$; so that, *as the curve recedes from the vertex, it constantly recedes from the axis.* This result is in conflict with that of § 196. It arises from the infinite length of the oval, which even goes to an infinite distance from the vertex before it reaches its maximum point, at which it turns back towards the axis. The discrepancy will be more fully explained hereafter.

209. *Theorem.* The locus of every rectangular equation of the form (86) is a parabola which has its vertex at the origin, its axis on the axis of x, and its focus at the distance p from the origin.

Proof. We are to prove that the distance of every point of the locus from the point

$$x = p, \quad y = 0,$$

is equal to its distance from the straight line drawn parallel to the axis of y, at a distance $-p$ from it.

The distance of any point x, y from the axis of y is x; so that its distance r_1 from the supposed directrix is

$$r_1 = x + p.$$

Its distance r_2 from the supposed focus is, by (7),

$$r_2 = \sqrt{[(x-p)^2 + y^2]}.$$

If the coördinates of the point satisfy (86),

$$r_2 = \sqrt{[(x-p)^2 + 4\,p\,x]} = \sqrt{(x^2 - 2\,p\,x + p^2 + 4\,p\,x)}$$
$$= \sqrt{(x^2 + 2\,p\,x + p^2)} = x + p = r_1.$$

210. *Corollary.* By interchanging the letters x and y in the statement and proof of the above theorem, it applies to any rectangular equation of the form (87).

211. *Corollary.* Any equation of the form (84) will give, by transformation, an equation of the form (86); and since, by § 209, the locus of the latter equation is always a parabola, that of the former equation is always a parabola.

212. *Problem.* To construct an equation of the form (86) or (87).

Solution. Draw a parabola, having its focus and vertex situated as in § 209 or § 210. It must, by those sections, be the required locus.

A simple method of drawing the parabola is given in the solution of the first example of the next section.

213. Examples.

1. Construct the rectangular equation $x^2 = -2\,y$.

Solution. This is an equation of the form (87) in which $p = -\frac{1}{2}$.

In Fig. 58, in which $X'CX$ and $Y'CY$ are the axes, take $GC = 4\,p = -2$. From a point on the axis of y as a centre, describe such a circumference as will cut the axis of y at G and any other point R, and the axis of x at any two points P. The point M, of which the coördinates are $x' = CP$, $y' = CR$, is a point of the required locus; for by Geom. § 186

$$2\,p : x' = x' : y',$$

$$x'^2 = 4\,p\,y' = -\,2\,y'.$$

By continuing this process, other points of the curve may be found.

2. Construct the equation $y^2 = 12\,x$.

3. Construct the equation $y^2 = -\,3\,x$.

4. Construct the equation $x^2 = y$.

5. Construct the equation $5\,x^2 = -\,4\,y$.

VI.

SIMILAR LOCI.

214. *Definitions.* Two loci are said to be *similar*, if they can be regarded as loci of the same equation constructed on different scales, that is, with different values of the unit length.

The *corresponding points* of similar curves are those for which the ratio of the abscissas and that of the ordinates are each equal to that of the units of length.

Thus, if the equation of § 87, Ex. 1, be constructed on a scale of half an inch, its locus will be similar to the curve of Fig. 34, in which the unit of length is a quarter of an inch; and the corresponding points will be those whose abscissas and ordinates have the same *numerical* value.

215. *Corollary.* The abscissa of any point of a curve is to its ordinate as the abscissa of the corresponding point of a similar curve is to its ordinate.

216. *Corollary.* If two ellipses or two hyperbolas are

similar, the values of A, B, and c are proportional. For, by § 214, the units of length may be so taken that A, B, and c shall have the same numerical values for the two curves, and then they will be to each other in the common ratio of the units.

217. *Corollary.* Since, for the asymptotes of the hyperbola

$$\tan \frac{\imath'}{x} = \frac{B}{A}, \quad \tan \frac{\imath''}{x} = -\frac{B}{A},$$

these angles are equal for two similar hyperbolas; so that, if the hyperbolas are drawn, as in Fig. 67, with their centres at the same point and their transverse axes on the same line, they will have the same asymptotes.

218. *Theorem.* If the semi-axes of two ellipses or of two hyperbolas are proportional, the curves are similar.

Proof. Let the values of A and B for the two curves be

$$A' \text{ and } B', \qquad A'' \text{ and } B''.$$

Express A' and B' in numbers on a scale in which o' is the unit of length, and A'' and B'' on a scale in which o'' is the unit, o' and o'' being so taken that

$$o' : o'' = A' : A'' = B' : B''.$$

Then

$$A' : o' = A'' : o'',$$
$$B' : o' = B'' : o''.$$

But the ratio of a quantity to its unit is its numerical value. Hence the numerical values of A' and A'' are the same; and also those of B' and B''; and the substitution of A' and B' in (48) or (66) gives the same equation as the substitution of A'' and B''. The curves, therefore, come under the definition of § 214.

219. *Theorem.* All parabolas are similar.

Proof. Let the values of p for two parabolas be p' and p''. Express p' and p'' in terms of the units of length o' and o'', so taken that

$$o' : o'' = p' : p''.$$

Then

$$p' : o' = p'' : o''.$$

Hence the numerical values of p' and p'' become the same; and the equations of the parabolas of the form (86) become identical.

CHAPTER VI.

THE ELLIPSE, THE PARABOLA, AND THE HYPERBOLA: RELATED CASES UNDER THE SAME GENERAL LAW OF FORM.

220. It has been seen, in the last chapter, that the general characteristics of the form of a curve may be deduced from its equation. Thus, we have found that the circle and the ellipse are finite ovals, that the hyperbola is an infinite curve of two branches, that the parabola consists of a single branch which may be regarded as an infinitely long oval. It is, indeed, evident that, since a curve may be completely drawn from its equation alone, that equation must determine it in every respect. It ought, therefore, to lead by algebraic processes to the development of the figure and of all the properties of the locus; so that, if two equations are identical, their loci must have precisely the same properties, and, if constructed for the same origin and axes, they will coincide.

221. In order to the identity of two equations, they must agree in two respects: (1.) in their algebraic *form* (that is, they must express the same relation between their constant and variable quantities); (2.) in the *values of the constants* which enter into them. But of these conditions the first alone is necessary to make the loci of two equations curves of the same kind. It has been shown, for instance, in §§ 153, 154, that any equation of

the form (48) is an ellipse, whatever be the values of A and B. Are we, then, to conclude that, in all cases, the loci of two equations which have the same algebraic form are, however unlike in apparent figure, nothing but different examples of the same curve?

222. An equation is a sentence, and, as such, the expression of a thought. The equation of a locus affirms through its algebraic form a certain relation between the arbitrary constants and the coördinates, which vary for different points of the locus; and this relation constitutes an *algebraic idea*, just as the definition of the curve constitutes a *geometric* one. If, then, two equations are the same in form, their loci, though they may differ widely in outward characteristics, are the same in idea, at least in algebraic idea, and their particulars of resemblance and of dissimilarity can be brought under one law, — a law which is expressed in the common form of their equations, and which may be called the Law of their Form. In answering the question of the last section in the affirmative, the mathematician, therefore, will only follow the example of the naturalist, who classifies animals and plants according to the intellectual principles of their structure, not to their superficial resemblances.

223. It is the object of this chapter to deduce from the equations of the ellipse, the parabola, and the hyperbola, the result that these curves may be regarded as different species belonging to the same class; to show that they arrange themselves in an unbroken series which comprises all the possible cases under their general law of form; and to trace this series of curves from beginning to end, so as to unfold their relations to each other.

224. *Scholium.* In the use of polar coördinates in this chapter, the rule of § 52 will be so far disregarded as to allow the construction of those negative values of the radius vector which arise from particular values given to the constants in a general algebraic expression for r. Such values of r are to be constructed in the direction opposite to that which is indicated by the corresponding value of the polar angle.

225. *Discussion of* (43). Let us begin by tracing the series of forms which the locus of (43) assumes, as A and c pass continuously through all possible modifications of their value. It is, however, necessary to consider only changes in the *relative* value of these quantities; for as long as their ratio remains the same, nothing is altered but the scale on which the curve is drawn (§§ 214, &c.). We may, then, take A as fixed, and suppose c alone to vary. Let the length of A be $AC = C'A$ (Fig. 59). Since c, being an absolute length, cannot be negative, we have only to suppose it to pass continuously from the value 0 to the value ∞.

If
$$c = 0,$$
$$e = \frac{c}{A} = 0, \quad B = \sqrt{(A^2 - c^2)} = A,$$
and the locus is a circle, as shown in § 138.

As c increases from 0, the foci separate and move away from the origin, and the curve becomes an ellipse; e increases, B diminishes, and the ellipse gradually flattens; till, when
$$c = A,$$
the foci fall on C and C', and
$$e = 1, \qquad B = 0,$$
$$r = \frac{A^2 - A^2}{A - A\cos\frac{r}{\varrho}} = \frac{A - A}{1 - \cos\frac{r}{\varrho}} = \frac{0}{1 - \cos\frac{r}{\varrho}};$$
so that r is zero, and determines only the origin, unless it has such a direction that the denominator of the above fraction becomes zero; that is, unless

$$\cos\frac{r}{\varrho} = 1, \quad \frac{r}{\varrho} = 0,$$

$$r = \frac{\pm 0}{\pm 0},$$

in which case, r is indeterminate, both in amount and in sign, and corresponds to any point in the line of the transverse axis.

If the foci are conceived to move beyond the extremities of the transverse axis,

$$c > A, \quad e > 1,$$

B of § 133 becomes imaginary, and B of § 158 becomes real, and the locus is, by § 191, an hyperbola. As c increases, e and B increase, and the angle between the asymptotes, determined by

$$\cos\frac{\tau'}{\varrho} = \cos\frac{\tau''}{\varrho} = \frac{A}{c},$$

also increases, so that the hyperbola gradually opens; till, when,

$$c = \infty,$$

$$e = \infty, \quad B = \surd(\infty - A^2) = \infty, \quad \cos\frac{\tau'}{\varrho} = \cos\frac{\tau''}{\varrho} = 0,$$

so that (Tr. § 55) the asymptotes become perpendicular to the transverse axis. In this case, the two branches of the hyperbola become straight lines parallel to the asymptotes. For the vertices of an hyperbola are its points of greatest distance from the asymptotes, and of least distance from the conjugate axis. Hence, when the asymptotes and the conjugate axis coincide, no points of the hyperbola can be farther from this line than are the vertices, and none can be nearer to it; that is, all the points are equidistant from it.

Thus we are led to regard the ellipse and the hyperbola as only different parts of a continuous series of forms, which variously embody one law; the ellipse gradually approaching the hyperbola, as it flattens, and at last changing into it, so to speak, by momentarily becoming the straight line.

226. *Corollary.* When the ellipse receives its utmost flattening, at the moment when $c = A$, we should expect that it would coincide with its own transverse axis, and that the locus of (43) would become the finite straight line $C'C$. If, on the other hand, we should trace the series in reverse order, from $c = \infty$ to $c = A$, we should begin with the hyperbola, each branch of which would gradually close up, till, when $c = A$, the two branches would come to coincide with those parts of the straight line which are outside the extremities of the transverse axis. But the true locus of (43), in this case, is the whole straight line, which may, therefore, be regarded as made up of an ellipse and the two branches of an hyperbola. This result may lead us to conceive that (43) has, in all cases, for its complete locus the combination of an ellipse and an hyperbola; but that, if $c < A$, the part which is an hyperbola is impossible, or imaginary; and if $c > A$, the ellipse is imaginary, so that it is only when $c = A$, that both parts of the ideal locus can actually coexist.

This view is supported by the equations of the ellipse and the hyperbola referred to their principal diameters. For if, in (47), we take values of x absolutely greater than A, the values of y are imaginary, and may be conceived to correspond to an imaginary curve. Moreover, as x increases, y^2 so varies, that if, by undergoing a change of sign, it were made positive, y would give an hyperbola, and therefore the imaginary curve will be an hyperbola. In like manner, if, in (65), we take values of x absolutely less than A, the values of y are imaginary, and may be conceived to correspond to an imaginary ellipse. If, however, $c = A$, $B = 0$, both for the ellipse and for the hyperbola, and the substitution of this value in either (47) or (65) gives

$$A^2 y^2 = 0,$$

$$y = \pm \frac{0}{A} = 0;$$

which is the equation of the axis of x, indefinitely produced.

227. *Discussion of* (45). The relation between the ellipse and the hyperbola is presented in a somewhat different light by the discussion of the polar equation of the ellipse in the form (45).

Instead of supposing the transverse axis to be fixed and the foci to change their places, as in § 225, we will now suppose that one focus and its nearest vertex are fixed, thereby making p, their distance apart, constant, and that the other focus and vertex are shifted along the transverse axis, their distance being the constant p, the form of the curve always accommodating itself to their new position, and its centre being the middle point of the transverse axis.

a. Suppose that the left-hand focus and vertex of an ellipse are fixed at F and C' (Fig. 60), and, in the first place, let F' coincide with F, in which case,

$$c = 0, \quad e = \frac{c}{A} = 0,$$

$$r = \frac{p\,(1 + e)}{1 - e \cos {}^{r}_{\varrho}} = p;$$

which is the equation of a circle of which p is the radius, and the centre at the origin F.

As F' moves towards the right, c and A increase both by the same amount; so that, as c is smaller than A, *its ratio to A*, that is, e, increases (Alg. art. 135); or, to obtain the same result in a different form,

$$c = A - p,$$

$$e = \frac{c}{A} = \frac{A - p}{A} = \frac{A}{A} - \frac{p}{A} = 1 - \frac{p}{A}; \qquad (95.)$$

so that, as A increases (p being constant), $\frac{p}{A}$ diminishes, e gradually approaches unity, and we shall have a succession of ellipses of greater and greater eccentricity.

When F' has moved off to infinity,

The Parabola, the Ellipse of Maximum Eccentricity.

$$c = \infty, \qquad A = \infty,$$

$$e = 1 - \frac{p}{A} = 1 - \frac{p}{\infty} = 1 - 0 = 1;$$

in other words, the difference between c and A, though a finite quantity, p, is now *infinitely small in comparison with either of them* (Geom. § 205), and their ratio, e, becomes the same as that of two equal quantities. The substitution of this value of e in (45) gives (84); so that the locus becomes a parabola; and the parabola may, therefore, be regarded as an ellipse of an infinitely long transverse axis, and with its foci infinitely far from the centre.

b. Suppose, now, that the right-hand focus and vertex of an hyperbola are fixed at F and C'. For the ellipse, $p = A - c$, and for the hyperbola, $p = c - A$ (§§ 139, 164); and, as these values are negatives of each other, the second members of (45) and (62), though in form they have opposite signs, are really the same fraction; for p in (62) has the same value as in (45), if its sign is reversed, and this operation makes the two equations identical. Hence, and by § 163, (45) is the equation of both branches of the hyperbola, if the origin is at F and the initial point at C, the vertex farthest from F. Again,

$$c = A + p,$$

$$e = \frac{c}{A} = \frac{A + p}{A} = \frac{A}{A} + \frac{p}{A} = 1 + \frac{p}{A},$$

which is the same value as that given above, since p is the negative of the p in the ellipse. Let us suppose C and F to move towards the left from their extreme position on the right.

C cannot be farther to the right than C'. When C coincides with C', (Tr. § 55,)

$$A = 0, \qquad c = p.$$

$$e = 1 + \frac{p}{0} = 1 + \infty = \infty = \sec \frac{\tau'}{\alpha},$$

$$\underset{\alpha}{\tau'} = \tfrac{1}{2}\,\pi, \qquad \underset{\alpha}{\tau''} = -\tfrac{1}{2}\,\pi.$$

Now, no point of the hyperbola can be farther than its vertices from the asymptotes. But in this case the vertices are at the centre, and therefore on the asymptotes. Hence the two branches of the hyperbola and its two asymptotes all coincide in a straight line drawn perpendicular to the transverse axis through the centre.

As C and F' move towards the left, A increases, and $\frac{p}{A}$, e, and $\underset{\alpha}{\tau'}$ (Tr. § 70) decrease (p being constant); that is, we have a succession of hyperbolas of less and less eccentricity.

When C and F' have moved to infinity,

$$A = \infty,$$

$$e = 1 + \frac{p}{\infty} = 1 + 0 = 1 = \cos \underset{\alpha}{\tau'},$$

$$\underset{\alpha}{\tau'} = \underset{\alpha}{\tau''} = 0;$$

and the substitution of this value of e in (45) gives, as before, (84), the equation of the parabola, which may, therefore, be regarded as an hyperbola of an infinitely long transverse axis, and with its asymptotes parallel to the transverse axis.

228. *Corollary.* In the last step of the above discussion, there is a difficulty which deserves a moment's notice. The polar axis was taken to be directed from F towards C, that is, towards the left; but as the polar axis of (84) is directed towards the right, it would seem as if the vertex of the parabola obtained from the hyperbola by moving C off to infinity on the left, must be on the right of the origin. But the p of (45) being for the hyperbola negative, all the values of r, which are positive for the absolute value of p, become negative, and each is to be laid off in the direction opposite to that indicated by the corresponding value of the polar angle. This reverses the position of the pa-

rabola, and makes it identical with the curve obtained by lengthening the transverse axis of the ellipse to infinity.

229. *Corollary.* We have for the ellipse, by (95) and § 133,

$$A = \frac{p}{1-e},$$

$$B^2 = A^2 - c^2 = (A-c)(A+c) = Ap(1+e) = p^2\frac{1+e}{1-e};$$

and the square of the ratio of the semi-axes is

$$\frac{B^2}{A^2} = p^2\frac{1+e}{1-e} : p^2\frac{1}{(1-e)^2} = (1+e)(1-e) = 1-e^2.$$

When $e = 0$, which is the case of the circle, the above equations give

$$A = p, \qquad B = p,$$

$$\frac{B}{A} = \sqrt{(1-e^2)} = \sqrt{1} = 1.$$

As e increases (p remaining the same), A increases, because its denominator is diminished; B^2 (and therefore B) increases, because its numerator is enlarged and its denominator diminished; and $\frac{B}{A} = \sqrt{(1-e^2)}$ decreases. In other words, both axes increase, but the conjugate axis more slowly than the transverse.

When $e = 1$, which is the case of the parabola,

$$A = \frac{p}{0} = \infty,$$

$$B^2 = p^2\frac{2}{0} = \infty, \qquad B = \sqrt{\infty} = \infty,$$

$$\frac{B}{A} = \sqrt{(1-e^2)} = \sqrt{0} = 0;$$

that is, both axes become infinitely long, but the transverse axis *infinitely longer than the conjugate.*

When $e > 1$, the denominator of A becomes negative; but as its numerator changes its sign simultaneously, A remains positive,* B^2 becomes negative, and B therefore imaginary, and $\frac{B}{A}$ also imaginary.

But if for p and B^2 we substitute $-p$ and $-B^2$, these letters will represent the p and B^2 of the hyperbola, and the above equations become

$$A = \frac{-p}{1-e} = \frac{p}{e-1},$$

$$-B^2 = (A-c)(A+c) = p^2\frac{1+e}{1-e},$$

$$B^2 = (c-A)(A+c) = p(A+c) = p^2\frac{e+1}{e-1},$$

$$-\frac{B^2}{A^2} = 1-e^2, \qquad \frac{B}{A} = \sqrt{(e^2-1)}.$$

When $e = 1$, we have, as before,

$$A = \infty, \qquad B = \infty, \qquad \frac{B}{A} = 0.$$

As e increases (p being constant), the denominator of A is enlarged, and therefore A is lessened; c likewise decreases, since $c = A + p$, so that B^2, and therefore B, decreases; and the ratio of the conjugate axis to the transverse increases.

230. *Scholium.* We have before (Geom. §§ 203 – 205) met with quantities which are not only infinitely small, but which are so *in comparison with other infinitesimals*, so that the latter infinitesimals are *infinitely large* in comparison with the former. We are now led to remark, that there are, on the other hand, quantities which are infinitely large even *in comparison with other infinite quan-*

* This change is, however, regarded in a different light in § 249.

tities, so that these latter are *infinitely small* in comparison with the former.

That the existence of such quantities is possible, however hard to conceive, may be shown by a simple geometrical example. The rectangles AD and AF (Fig. 61), having the same altitude, are to each other as their bases (Geom. § 244); so that, if AB, the base of AD, is made infinitely long, the rectangle AD will be infinitely larger than AF; and this is, obviously, true, *whatever be the common altitude of the two rectangles.* It is true, therefore, if the altitude is taken infinitely large; but, in this case, the rectangle AF contains an infinite amount of surface; so that the surface of AD will be infinitely large in comparison with another infinite surface.

Again, algebraically, if any quantity is multiplied by infinity, it is thereby infinitely increased; so that *infinity itself* is infinitely increased by being *squared.* Indeed

$$\infty^2 : \infty = \infty : 1;$$

so that we have infinite quantities of a higher or lower *order*, according as they involve as a factor a higher or lower *power* of infinity. Thus, the area of the larger of the above rectangles is of the second order of infinity, being the product of its base and its altitude, which are each infinite; and it is therefore infinitely large in comparison with the smaller rectangle, the area of which is the product of the infinite altitude and the finite base, and therefore of the first order of infinity. So, in the parabola, which this scholium is meant to explain, B^2 is of a lower order of infinity than A^2, and therefore infinitely small in comparison with it.

231. *Corollary.* By § 177, the asymptotes of the parabola, when regarded as an hyperbola, are drawn from the centre of the hyperbola, the distance of which from the vertex, on the left of it, is

$$A = \infty,$$

so as to cut off, on a line perpendicular to the transverse axis at the vertex on either side of it, the length

$$B = \infty.$$

Then

$$\tan \underset{\alpha}{\tau'} = \frac{B}{A} = 0,$$

$$\underset{\alpha}{\tau'} = 0;$$

as has been shown in § 227. Therefore, though the asymptotes of the parabola diverge from the transverse axis, at the centre, at an infinitely small angle, yet, when they are drawn to the infinite distance A, they become separated from the axis by the infinite distance B (which is however infinitely smaller than A), and so run outside of the parabola, parallel to the axis.

232. *Corollary.* It results from the discussion of §§ 227–231, that, as e in (45) is supposed to change continuously from the value 0, the lower limit of its possible values, up to ∞, the higher limit, we may conceive that the locus of that equation, without deviating from its law, passes gradually through a series of forms; beginning with the circle, or the ellipse of equal axes; going on, through ellipses in which the transverse axis becomes greater and greater in proportion to the conjugate, to the parabola; changing, through the parabola, into the hyperbola; then proceeding through hyperbolas in which the asymptotes diverge more and more from each other, up to that extreme form in which they take precisely opposite directions. We must conclude, then, in accordance with what has been said at the beginning of this chapter, that the ellipse, the parabola, and the hyperbola, though they have been taken up as distinct curves, variously defined, and though they differ widely in apparent

figure, are, in this view, to be regarded as only the several embodiments of one thought, the several manifestations of one essential form; that they occur, not at random, but in a fixed order in an unbroken series, the parabola being intermediate between the other two,—the single form, incapable of variety (except that which arises from changing the scale on which it is drawn), which the locus momentarily assumes in passing from the series of ellipses on the one hand to the series of hyperbolas on the other; and, moreover, since all values of e are contained between 0 and ∞, that no form is included under the law of (45) except one of these three, in their various modifications.

233. *Scholium.* While the above series of forms has been spoken of as unbroken, because it answers to a gradual alteration in the value of e, there is in the geometrical representation of the change of the locus, in Fig. 60, an apparent interruption of continuity; for, as e increases in the ellipse, the points F' and C move off on the right till they reach infinity, and then, as e still increases and the locus becomes an hyperbola, they suddenly appear on the left, moving up from infinity towards the origin. But it has been shown, on page 42, that this abrupt change in the position of a moving point from infinite remoteness on one side to infinite remoteness on the other, without going through any intermediate positions, is not to be regarded, at least in the algebraic treatment of Geometry, as a breach of the continuity of its motion.

It may be worth while to insert an ingenious geometrical fiction which has been invented to explain this difficulty; and the student may attach to it whatever value he thinks proper. Suppose that the apparent plane in which the figure is drawn, instead of being really a plane, is the surface of a sphere of infinite radius, and that the so-called straight lines in the plane are not straight lines, but arcs of great circles of the sphere. Then any finite portion of the surface, being infinitely small in comparison with the whole surface, will be, in fact, a plane; and any finite arc of a great circle, being an infinitesimal arc of the circumference, will be a straight line; so that on this supposition the figure, as far as it can actually be drawn, is unaltered. Now if two points be conceived to move in the line $C'F$ from F as an origin, in opposite directions, they will, as long as they keep within a finite distance from F, move in a straight line, and become farther and farther apart. But if both go half-way round the circumference of which $C'F$ is an arc, they meet and coincide; and if then they pass each other and still keep on in their respective directions, they will at length reappear, the point which went off on the right coming in on the left, and that which went off on the left coming in on the right, and both will have changed sides by retiring to infinity, that is, by making half the circuit of a circumference of infinite radius. This fanciful hypothesis is useful as enabling us to *represent to ourselves* how that can be which finite minds cannot really *comprehend*. It may also show us that there is nothing in the mere incomprehensibility to us of what relates to infinity, or in its apparent inconsistency with former knowledge, to hinder the perfect harmonizing of all these truths to a higher intelligence, who sees the whole; since even to us, by modes within the scope of our invention, these seeming absurdities, though still inconceivable, may be represented as logically possible.

234. *Corollary.* If the locus of (45) be referred to a rectangular system in which C' (Fig. 60) is the origin, and $C'F$ the axis of x, its equation is, as in § 145,

Rectangular Equation, referred to Vertex.

$$y^2 = \frac{2\,B^2}{A}\,x - \frac{B^2}{A^2}\,x^2. \qquad (50.)$$

Now, if the locus undergoes the changes supposed in § 227, then, as in §§ 227, 229, A and B increase, and $\frac{B^2}{A^2}$ decreases. Also, by § 229,

$$\frac{2\,B^2}{A} = \frac{2\,A\,p\,(1+e)}{A} = 2\,p\,(1+e);$$

so that, as e increases, this fraction also becomes greater. Hence, for any given value of x, the positive term of the above value of y^2 increases, and the negative term decreases (in absolute value), so that y^2, and therefore the absolute value of y, increases.

When $e = 1$,

$$\frac{B^2}{A^2} = 0, \qquad \frac{2\,B^2}{A} = 2\,p\,(1+e) = 4\,p;$$

and (50) becomes

$$y^2 = 4\,p\,x,$$

which is the equation of the parabola, referred to its vertex as origin, and to its axis as the axis of x.

When e becomes greater than 1, A and B^2 pass through infinity and become negative; so that the vertex C begins to come in on the left of C'. If, now, we take A'' and B''^2 equal to the absolute values of A and B^2,

$$A'' = -A,$$
$$A''^2 = A^2,$$
$$B''^2 = -B^2;$$

so that A'' and B''^2 are positive, and (50) becomes

$$y^2 = \frac{2\,B''^2}{A''}\,x + \frac{B''^2}{A''^2}\,x^2;$$

which is, by § 171, the rectangular equation of the hyperbola whose semi-axes are A'' and B'', the origin being at the right-hand vertex, and the axis of x being on the transverse axis.

235. *Corollary.* An explanation can now be given of the inconsistency which has been pointed out in § 208 between the results of (84) and (86). The full form of (86) is given in (50), in which the last term is dropped, because for the parabola $\frac{B^2}{A^2} = 0$. But this does not cause the term to vanish, when $x = \infty$; for then

$$\frac{B^2}{A^2}x^2 = 0 \times \infty = \frac{0}{0},$$

which is indeterminate. If, in (50), $x = A$, then (whatever be the value of A),

$$y = \pm B;$$

if $x > A$, $y^2 < B^2$; if $x = 2A$, $y = \pm 0$; if $x > 2A$, y is imaginary.

236. *Corollary.* In § 234, when $e > 1$, we suppose A to become negative; while, in the same case, in § 229, A was supposed to remain positive and p to become negative. This is accounted for well enough by observing that, in polar coördinates, lines are not negative merely because they point to the left, but, on the contrary, they become so only when the negative sign belongs to their algebraic values (as, for example, $p = A - c$ becomes negative, when $c < A$); while, in rectangular coördinates, a line is necessarily negative, if its direction is opposite to that of either axis.

237. *Corollary.* If the parabola is referred to its centre and principal diameters, its equation, according as the curve is regarded as an ellipse or an hyperbola, is

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1, \quad \text{or} \quad \frac{x^2}{A^2} - \frac{y^2}{B^2} = 1;$$

in which $A = \infty$, $B = \infty$, $\frac{B^2}{A^2} = 0$,

238. *Scholium.* If the ellipse, the parabola, and the hyperbola are only different forms of one and the same curve, it ought to be possible to bring them all under one definition. It will now be shown that such a definition can, in fact, be given.

239. *Theorem.* The locus of (45) may be defined as a curve every point of which is equally distant from one of its foci and the circumference of a circle described about the other focus, with a radius equal to the transverse axis (the latter distance being measured on the line of a radius, since, by Geom. §§ 119, 120, a radius meets the circumference perpendicularly).

Proof. a. For the Ellipse. Let Fig. 62 represent an ellipse, and a circle described about F' as a centre, with a radius equal to $2\,A$, the transverse axis. Since

$$C''F' = C'C,$$
$$C''C' = F'C = p.$$

Let M be a point within the circle, and let it satisfy the conditions. It is impossible that $MM'' = MF$; for

$$FF' + F'M > FM,$$
$$M''F' > FF';$$
$$M''F' + F'M > FM.$$

If

$$MM' = MF,$$
$$F'M + MF = F'M + MM' = F'M' = 2\,A;$$

so that any point within the circle, if it satisfies the conditions, satisfies the definition of the ellipse.

No point without the circle, such as B, can satisfy the conditions; for neither BB' nor BB'' can be equal to BF, since

$$F'F + FB > F'B$$
$$F'B' > F'F$$
$$FB > F'B - F'B, \text{ or } B'B;$$

and

$$B'F' > FF',$$
$$B''B, \text{ or } B''F' + F'B > FF' + F'B > FB.$$

Any point, therefore, which satisfies the conditions, is, when $FF' < 2\,A$, a point of the ellipse described about F and F' as foci, with $2\,A$ for a transverse axis; and, conversely, any point of this ellipse satisfies the conditions; for, if M is such a point, the definition of the ellipse gives

$$F'M + MF = 2\,A = F'M' = F'M + MM',$$
$$MM' = MF.$$

b. For the Hyperbola. Let Fig. 63 represent an hyperbola, and a circle described about F' as a centre, with a radius equal to $2\,A$, the transverse axis. Since

$$F'C'' = CC',$$
$$C''C' = F'C = p.$$

Let M be a point within the circle, such as to satisfy the conditions. It is impossible that $MM' = MF$; for

$$F'M + MG > F'G,$$
$$MG > F'G - F'M,$$
$$F'G = F'M',$$
$$MF > MG > MM'.$$

If $MM'' = MF$,

$$MF - MF' = MM'' - MF' = F'M'' = 2\,A;$$

and M satisfies the definition of the left branch of the hyperbola.

For any point N outside the circle, such that

$$NN'' = NF,$$
$$NF - NF' = NN'' - NF' = F'N'' = 2\,A;$$

so that N must be in the left branch of the hyperbola.

For a point P outside the circle, so placed that

$$P'P = FP,$$

$$F'P - FP = F'P - P'P = F'P' = 2A;$$

so that P must be in the right branch of the hyperbola.

Any point, therefore, which satisfies the conditions, is, when $F'F > 2A$, a point of the hyperbola described about F and F' as foci, with a transverse axis equal to $2A$; and the converse of this is easily shown.

c. For the Parabola. When the ellipse becomes a parabola, F' retires to an infinite distance on the right; and when the hyperbola becomes a parabola, F' is removed to an infinite distance on the left. Therefore, the circumference becomes infinitely great, since its radius becomes so, and any finite part of it, being an infinitely small arc in comparison with the circumference, is a straight line, which, since $C''C' = p$, coincides with the directrix of the parabola. Hence, the theorem, as stated for the parabola, becomes identical with the common definition of that curve.

240. *Corollary.* The above definition of the ellipse, the parabola, and the hyperbola may be used to illustrate the relation between them. As, in Fig. 60, e increases, p being constant, F' moves off to the right, and the circumference gradually curves less and less, because its radius $2A$ increases; till at last, when the locus becomes a parabola, it has no curvature, and becomes a straight line. Then, as the locus passes through the form of the parabola into that of the hyperbola, the circumference bends over towards the left, and, afterwards, it curves more and more on that side, as e increases, and F' approaches from the left. In this change of curvature, it will be seen, there is perfect continuity.

CHAPTER VII.

THE ELLIPSE, THE PARABOLA, AND THE HYPERBOLA IN ANCIENT GEOMETRY AND IN MODERN PHYSICS.

241. It may be well to point out a few particulars in the history of the study of the ellipse, the parabola, and the hyperbola, in order to show why a very important place should be given them among other curves. We have, indeed, seen that they have simple and symmetrical figures, and that they are naturally united, by a curious relationship, into a series, which includes even the circle, the figure which, from its simplicity and frequent occurrence in the forms of bodies around us, deserves our study next after the straight line. Still, this relationship seems to have a merely speculative existence; and it is troubled with difficulties the solution of which requires much refining, and, sometimes, is beyond our comprehension. Some may feel that the truth which the mind beholds by turning in on itself has a value of its own, and that, if the study of these curves leads to a deeper insight into geometric truth, it needs no further justification. But to all it will be acceptable to know that the objects of their study have, for ages, and in many forms, presented themselves to the observation as well as the thought of men; that, far from having originated in human ingenuity, or from having an accidental existence in nature, they are, by the necessity of physical law,

embodied and illustrated in the outward creation, so that, in studying them, we enter into the thought which is there revealed.

I.

THE CONIC SECTIONS.

242. *Definition.* If the indefinite line BB' (Fig. 65), which crosses OO' at A, revolve about OO' as an axis, the whole surface generated, indefinitely produced in both directions, is called, in the higher Geometry, *the surface of a cone.*

The two parts into which the surface is divided at the vertex A (and which would be regarded, in Elementary Geometry, as the surfaces of two distinct cones) are sometimes called the *nappes* of the surface.

243. *Sections of the Cone.* Suppose the cone of Fig. 65 to be cut by a plane which is perpendicular to that of BAD, and which enters the cone at C'.

If the plane has a greater inclination to the axis OO' than DD', the side of the cone, has, it must cut DD' at some point below A, as C_2. In this case, the intersection of the plane with the surface of the cone is, evidently, an oval; and it is, in fact, an *ellipse*, of which $C'C_2$ is the transverse axis. If the plane is perpendicular to the axis, this ellipse is (Geom. § 378) a *circle.*

If the plane has the same inclination to OO' as DD', it is parallel to DD' and does not cut it; so that the curve is, evidently, a curve of one branch, which goes off to infinity without coming round into itself. It is, in fact, a *parabola*, of which the axis takes the direction $C'E$.

If the plane has a less inclination to OO' than DD' has, it will cut DD' above A; as at C_5; and the curve consists, evidently, of two infinite branches, in neither of which the two portions ever cease to recede from each other. In fact, the curve is an *hyperbola*, of which $C'C_5$ is the transverse axis.

All sections of the cone made by parallel planes are *similar curves.*

244. *Definition.* The ellipse, the parabola, and the hyperbola are often included under the generic name of the *conic section.*

245. *Corollary.* If the cutting plane be supposed to turn from the position in which it is perpendicular to the axis, in the direction of positive rotation, it will give a succession of ellipses of greater and greater eccentricity; till, when it becomes parallel to the side of the cone, the section becomes a parabola; and, if it turns still farther, the section is an hyperbola. The relation between the three curves is here exhibited in a form analogous to that of Fig. 60.

246. *Corollary.* If the cutting plane be supposed to turn from the position in which it is perpendicular to the axis of the cone, in the direction of *negative* rotation, the section increases in eccentricity, and the ellipse is gradually flattened. When the plane is turned so far as to coincide with $B'B$, it becomes tangent to the cone along that line, and the straight line, $B'B$, is, in this case, the section. If the plane turns still farther, the section becomes an hyperbola. Here the section undergoes a series of changes analogous to those of Fig. 59.

247. *Corollary.* It appears from §§ 245, 246, that the same relation which has been previously established between the ellipse, the parabola, and the hyperbola is found to exist when those curves are treated as the sections of a cone; but here this relation is presented from a wholly new point of view, as a mere matter of fact.

II.

LAWS OF MOTION AND OF FORCE.

248. It is as the conic sections that the ellipse, the parabola, and the hyperbola enter into ancient Geometry. As the conic sections, they were first conceived by Plato or his immediate disciples; and it was not till within two centuries of the present time that their study was taken up apart from that of the cone. These curves formed a favorite subject of speculation with the geometers of antiquity. Apollonius, one of the greatest of them, wrote an elaborate treatise on the subject, the first four books of which he devoted to what was before known concerning the conic sections, and the last four he filled with his own discoveries.

249. In modern times, however, the interest which attaches to the ellipse, the parabola, and the hyperbola has ceased to be purely geometrical. Indeed, the ancient mathematicians seem to have nearly exhausted by their discoveries the properties of these curves, which would probably, therefore, have sunk from an unnaturally prominent position if it had not been for a discovery which

has not only given to them a new importance for the future, but has thrown light back on the past, by showing that, while the old geometers were studying these insignificant figures, — so remote from actual experience, so incapable, as it seemed, of leading to a higher knowledge of Nature, — in these very curves the great law of force was manifested, — the stars were tracing them in the heavens; so that the full mastering of their properties was no profitless play of the intellect, but the necessary introduction to the Philosophy of the Physical Universe.

250. *Kepler's Laws of Motion.* Kepler, a German astronomer, who was born in 1571, and died in 1630, was distinguished by a fondness for whimsical speculation about analogies and harmonies in different parts of Nature. This passion led him to the invention of many fanciful theories, and it led him also, early in the seventeenth century, to the detection of three great laws of the planetary motions, the most important of the discoveries which immediately prepared the way for that of the law of gravitation. These laws may be stated as follows: —

I. The radius vector of any planet, drawn from the sun, sweeps over equal areas in equal times, and, in any case, the areas are proportional to the times.

II. Each planet moves in an ellipse, which has the sun at one of the foci.

III. The squares of the periods of revolution of different planets are to each other as the cubes of their mean distances from the sun, that is, of the semi-transverse axes of their orbits.

251. *Corollary.* If the ellipse of Fig. 64 represents the orbit

of a planet, and F the sun, the time in which the planet moves from M' to M'' is to that in which it moves from M'' to M''' as the area of $M'FM''$ is to that of $M''FM'''$.

252. *Newton's Law of Force.* Kepler proposed his laws as mere matters of fact, resting on observation. It was left for Newton to show the principle which connects them. When Newton, half a century after Kepler's discovery, first conceived a force acting according to the law of gravitation, he naturally asked himself what support Kepler's laws gave to such an hypothesis. By a mathematical examination of the first law, he found that it proves the existence of a force which acts along the line joining the planet and the sun, in one direction or the other. He deduced from the second law, that this force acts towards the sun, with an intensity inversely proportional to the square of the distance between the two bodies. The third law shows that this force is the same for all the planets. On this basis, Newton assumed the existence of a force which acts on a body with an intensity proportional to its mass, and inversely proportional to the square of its distance from the centre of action; and he showed that, if the planets are bodies which move under the action of the supposed force, their motions must conform to Kepler's laws, except that their orbits, instead of being necessarily ellipses, may be any one of the conic sections.

253. *Scholium.* The expression for the square of the velocity of a body which moves about the sun under the influence of gravitation is

$$v^2 = m\left(\frac{2}{r} - \frac{1}{A}\right);$$

in which v denotes the velocity, m the measure of the force at the distance of the unit of length from the sun, r the distance of the body from the sun, and A the semi-transverse axis of the orbit, A being, as in § 234, positive for the ellipse and negative for the hyperbola. The greater A is taken, if positive, in the above formula, the greater will be the velocity, for equal values of r, that is, when the bodies compared are at equal distances from the sun. If the orbit is a parabola

$$A = \infty,$$

$$v^2 = \frac{2\,m}{r}.$$

If the orbit is an hyperbola, $-\frac{1}{A}$ is positive, and

$$v^2 > \frac{2\,m}{r};$$

and the less the absolute value of A, the greater the value of v^2.

254. *Scholium.* The planets have a velocity so small that they all move in ellipses. The satellites also move in ellipses, described about the primary as a focus. Of comets, some move in ellipses, some in parabolas, and it is supposed, but not satisfactorily ascertained, that others move in hyperbolic orbits. If a comet returns to the solar system, this fact, of course, shows that it moves in an ellipse. Comets are, however, sometimes said to have parabolic orbits, when, in fact, they move in ellipses of exceedingly great eccentricity.

CHAPTER VIII.

SPECIAL FORMS OF THE CONIC SECTIONS.

255. THE ellipse, the parabola, and the hyperbola assume, for special values of the constants, forms which it will be of interest to discuss in a separate chapter.

256. *The Ellipse.* If, in an equation of the form (48), any real values whatever are given to A and B, the locus is, by §§ 153, 154, an ellipse, which we may suppose to be represented by the largest of those in Fig. 66. Now, if the values of A^2 and B^2 are gradually diminished, but in such a manner that their ratio remains unchanged, we shall, by § 218, have a succession of similar ellipses. If we reduce A^2 to zero, B^2 must also become zero; and (48) gives, if we multiply by B^2,

$$\frac{B^2}{A^2}x^2 + y^2 = B^2 = 0.$$

Both terms of the first member of this equation, being squares, are positive; so that, as their sum is equal to zero, each term must be equal to zero, and therefore, since $\frac{B^2}{A^2}$ is not zero,

$$x = 0, \qquad y = 0;$$

which are the equations of a single point, the origin.

If A^2 is made less than zero, or negative, then, in order that the ratio $\frac{B^2}{A^2}$ may be undisturbed, B^2 must also be made negative; and, if we take

$$A'^2 = -A^2, \qquad B'^2 = -B^2,$$

(48) becomes

$$-\frac{x^2}{A'^2} - \frac{y^2}{B'^2} = 1;$$

in which, since the second member is positive, the first member must also be positive, which can only be the case when either x^2 or y^2 is negative, that is, when either x or y is imaginary. In this case, therefore, the equation has no locus, or its locus may be said to be imaginary.

Hence, the *point* may be regarded as an ellipse, with its axes equal to zero, similar to any given ellipse, and the limit of all real forms of the curve.

257. *Scholium.* If a succession of planes, parallel to that of $C'C_2$, be passed through the cone of Fig. 65, the sections which they make will, by the last paragraph of § 243, be ellipses similar to that of $C'C_2$. These ellipses will be smaller as they approach the vertex of the cone, and the plane which passes through the vertex itself gives only a point, which appears therefore among the conic sections as an ellipse, similar to any given ellipse.

258. *Scholium.* If a plane, parallel to the previous planes, be passed above the vertex of the cone, it cuts the upper nappe and gives an ellipse similar to the former series of ellipses. The ellipse, when considered as a conic section, does not, therefore, pass through the form in which it is a point into its imaginary form.

This last change is, however, exhibited in the case of the elliptic sections of some other surfaces. All the plane sections of the sphere, for instance, are circles. The nearer the cutting plane is to a parallel tangent plane, the smaller will be the circle. If it coincides with the tangent plane, the circle becomes a point. If it passes outside of the tangent plane, it no longer cuts the sphere, and the circle becomes imaginary.

259. *The Parabola.* If, in an equation of the form (86), $p = 0$, the equation is, by § 209, a parabola, similar, by § 219, to any other parabola. But, in this case, (86) becomes

$$y^2 = 0 \times x,$$

which gives, if x is finite,

$$y = \pm\, 0,$$

the equation of the axis of x. But, if $x = \infty$,

$$y^2 = 0 \times \infty = \frac{0}{0},$$

$$y = \pm\, \frac{0}{0},$$

so that y has two indeterminate values.

The parabola becomes, therefore, in this case, the combination of two straight lines which pass through the vertex at an infinitely small angle with the axis, one on one side of it and one on the other, so that they will coincide with the axis for a finite distance from the vertex, and form a single straight line; but when produced to an infinite distance from the vertex, they may become separated from the axis, so as to appear as two straight lines, parallel to it.

The case in which $p = 0$ is intermediate between those in which p is positive and those in which p is negative, so that

The *straight line*, or the combination of *two parallels*, may be regarded as a parabola, intermediate between the parabolas which curve in the positive direction from the vertex and those which curve in the negative direction.

260. *Scholium.* The locus of any equation of the form (48), in which $B : A = 0$, is, by §§ 218, 237, similar to the parabola, and therefore it is a parabola. $B : A = 0$, if B is finite and A infinite; and, in this case, (48) becomes

$$\frac{x^2}{\infty} + \frac{y^2}{B^2} = 1;$$

which gives, if x is finite, so that $\frac{x^2}{\infty} = 0$,

$$\frac{y^2}{B^2} = 1,$$

$$y = \pm B,$$

the equation of two straight lines parallel to the axis of x at the distance B above and below it. If, however, $x = A$ or $= -A$, (48) becomes

$$1 + \frac{y^2}{B^2} = 1,$$

$$\frac{y^2}{B^2} = 0,$$

$$y = 0;$$

so that, at the infinite distance A on either side of the origin, the two parallels meet in the axis of x.

For the ellipse,

$$p = A - c = \frac{A^2 - c^2}{A + c} = \frac{B^2}{A + c};$$

so that, in this case,

$$p = \frac{B^2}{\infty} = 0.$$

Hence, the case of the parabola discussed in this section is the same as that discussed in § 259; but the origin is here placed at the centre, so that the parallels appear separated from each other.

If $B = 0$, the parallels actually coincide throughout.

261. *Scholium.* Any section of the cone made by a plane parallel to that of $C'E$ (Fig. 65) is a parabola. If the plane pass through the vertex, it becomes tangent to the cone along its side

$D'AD$, and the section becomes a single straight line. If the plane be passed above the vertex, it cuts the upper nappe of the cone, and the section is a parabola which curves upwards, instead of downwards.

262. *Scholium.* The case in which the parabola becomes the combination of two parallels does not occur among the sections of the cone proper. It is given, however, by a peculiar form of the cone. Let the circular section $B'D'$ be spoken of, for the moment, as the base of the cone. Suppose that, without any change in this base, the vertex A of the cone is moved farther and farther away from it; the sides will make a more and more acute angle with each other; and, when A is removed to an infinite distance, they become parallel, and the cone becomes a *cylinder*.

Now, if a plane cuts a cylinder, parallel to the side, the section of the convex surface consists of two parallel straight lines; these parallels approach each other if the plane is passed farther from the axis; and if it coincides with the side of the cylinder, the parallels become one straight line. The last case corresponds to that in which $B = 0$.

263. *Corollary.* It will now be seen that there is no essential difference between the series of forms given to the locus of (43), by the discussion of § 225, and that given to the locus of (43) in the form (45) by the discussion of §§ 227 - 232. The locus passes from the ellipse into the hyperbola, in the latter case through the parabola, in the former case through the straight line, which is a special form of the parabola.

264. *The Hyperbola.* If, in an equation of the form (66), we assume any real values whatever for A and B, the locus is, by § 189, an hyperbola, which we may suppose to be represented by that which has $C'C$ for its transverse axis, in Fig. 67. By gradually diminishing the values of A^2 and B^2, but in such a manner that their ratio is unchanged, we shall, by § 218, have a

succession of similar hyperbolas, which, by § 217, will all have the same asymptotes. If A^2 is reduced to zero, B^2 also becomes zero; and (66) gives, if we multiply by B^2,

$$\frac{B^2}{A^2}x^2 - y^2 = B^2 = 0,$$

$$y^2 = \frac{B^2}{A^2}x^2,$$

$$y = \pm\frac{B}{A}x;$$

which is equivalent to a combination of the two equations,

$$y = \frac{B}{A}x,$$

$$y = -\frac{B}{A}x.$$

But these are, by § 116, the equations of two straight lines which pass through the origin at angles with the axis of x of which the tangents are $\frac{B}{A}$ and $-\frac{B}{A}$. But

$$\tan \frac{\tau'}{x} = \frac{B}{A},$$

$$\tan \frac{\tau''}{x} = -\frac{B}{A};$$

so that the above straight lines coincide with the asymptotes of the previous series of hyperbolas.

Next, if A^2 is made less than zero, that is, negative, B^2 also becomes negative; and, if

$$A'^2 = -A^2, \qquad B'^2 = -B^2,$$

(66) becomes

$$-\frac{x^2}{A'^2} + \frac{y^2}{B'^2} = \frac{y^2}{B'^2} - \frac{x^2}{A'^2} = 1;$$

which is, by § 190, the equation of an hyperbola described on the axis of y as a transverse axis, and having the same asymptotes as the previous series of hyperbolas.

Hence, *two straight lines which cross each other* may be regarded in combination as an hyperbola, with its axes equal to zero, similar to either of the sets of hyperbolas of which they are the asymptotes, and intermediate between them.

265. *Corollary.* If the two straight lines of Fig. 67 are regarded as belonging to the set of hyperbolas of which the transverse axes are on the axis of x, its two branches will be $E'AE$ and $E_1'AE_1$. On the other hand, if it is regarded as belonging to the set of which the transverse axes are on the axis of y, its branches will be EAE_1 and $E'AE_1'$.

266. *Scholium.* If a succession of planes, parallel to that of $C'C_5$, be passed through a cone, the sections will, by § 243, be hyperbolas similar to that of $C'C_5$. The axes of these hyperbolas will be smaller, according as they are nearer to the vertex of the cone, and the plane which passes through the vertex gives for its section the combination of two crossing straight lines.

267. *Scholium.* The section made by a plane, parallel to the former planes, and beyond the vertex, will be an hyperbola similar to the previous hyperbolas, and having its transverse axis parallel to $C'C_5$. The hyperbolic section of the cone, therefore, does not pass, through the form in which it is the combination of two straight lines, into a series of hyperbolas conjugate to the previous series. This change does, however, take place in the hyperbolic section of the *hyperboloid of one nappe*, a remarkable surface which has a curious relation to the cone, and of which, indeed, the cone may be regarded as a special case.

CHAPTER IX.

ORDERS OF LOCI.

268, *Theorem.* The degree of the equation of a locus, expressed in terms of rectilinear coördinates, is not changed by the transformation of the equation to any new system of rectilinear coördinates.

Proof. It will be enough to show that the degree of the equation is unchanged, first, if the locus be referred to a new rectilinear system in which the origin is the same as in the original system, while the axes are altered, and, secondly, if the new axes have the same directions as those of the original system, while the origin is otherwise placed. For if the equation of a locus be transformed, first to a system in which the directions of the axes are alone changed, and then to a third system in which the axes have the same directions as in the second system, the result must be the same as if the equation had been transformed immediately from the first system to the third, which may be any rectilinear system whatever; so that if the first two transformations do not affect its degree, their combination cannot affect it.

The proof will refer only to (22) and (23), since these are the general equations, which apply whether the coördinates are rectangular or oblique.

a. If the origins are the same,

$$x^{\circ} = 0, \quad y^{\circ} = 0;$$

so that the values of x and y given in (22) and (23) become, in respect to the variable coördinates x_1 and y_1, homogeneous polynomials of the first degree; and, therefore, the expression for any power, as the nth, of x or y, is (Alg. art. 34) homogeneous of the nth degree. Then, in transforming to the new system, we shall substitute for each term of the equation a homogeneous expression of the same degree. Moreover, it is impossible that all the terms of any given degree in the new equation should cancel each other, so as to leave no terms of that degree; for the aggregate of these terms is equal to the aggregate of the terms of the same degree in the original equation; and, if the latter aggregate is not equal to zero, the former aggregate cannot be equal to zero.

b. If the directions of the axes are the same,

$$\frac{x_1}{x} = 0, \qquad \frac{y_1}{x} = \frac{y}{x};$$

so that (22) and (23) become (Tr. § 55)

$$x = x_1 + x^\circ, \qquad y = y_1 + y^\circ.$$

If, then, the expression for any power, as the nth, of x or y be developed according to the binomial theorem, the first term will be x_1^n or y_1^n, and the remaining terms will be all of a lower degree than the nth, with reference to x_1 and y_1. Suppose, now, that the given equation is of the nth degree, and represent the term which involves x^n by $A\,x^n$. The value of this term, in the new system, is

$$A\,x^n = A\,x_1^n + n\,x^\circ\,A\,x_1^{n-1} + \&c.$$

Now, $A\,x_1^n$ cannot be cancelled by any other term in the equation; for, since no other term in the original equation involves so high a power of x as the nth, no other can give so high a power of x_1 as the nth. In like manner, the term which involves y^n will give a term involving y_1^n, and the only such term; the term which involves $x^{n-1}\,y$ will give a term involving $x_1^{n-1}\,y_1$, and the only such term; and so with all the terms of the nth, or highest, degree. But the degree of an equation is the same as that of the term which, with reference to the variables, has the highest degree (Alg. art. 106); so that, in this case, the new equation is of the same degree as the old.

269. *Scholium.* The proof of § 268. *b* does not apply to terms of a lower degree than the *n*th. The term which involves x^{n-1} gives a term involving x_1^{n-1}, and the term which involves x^n also gives a term involving x_1^{n-1}; and these terms may cancel each other.

270. *Scholium.* The degree of an equation, being independent of the particular system of rectilinear coördinates to which its locus is referred, is naturally taken, in Analytic Geometry, as a basis of classification. A locus is said to be *transcendental*, if its equation is transcendental (Alg. art. 107). All other loci are called *algebraic*, and are divided into *orders*, according to the degree of their equations.

Thus, the straight line is a locus of the first order, because (31) is of the first degree; the circle, the ellipse, the hyperbola, and the parabola are loci of the second order, because (38), (48), (66), and (86) are of the second degree; while the curves of § 87, Ex. 30 and 31, are transcendental loci, because their equations involve trigonometric and logarithmic functions.

271. *Definitions.* Equations of the first degree and their loci are sometimes called *Linear;* those of the second degree, *Quadratic.*

272. *Definition.* *The General Equation of any degree* is one in which all the terms that can enter into an equation of that degree are represented, and the constants are denoted by letters, which can have any positive or negative values, including zero.

Thus,

$$\begin{aligned} &Ax^n + A'x^{n-1}y + A''x^{n-2}y^2 + \&c. \ldots + A^{[n-1]}xy^{n-1} + A^{[n]}y^n \\ &\quad + Bx^{n-1} + B'x^{n-2}y + \&c. \ldots\ldots\ldots + B^{[n-1]}y^{n-1} \\ &\quad + \&c. \ldots\ldots \\ &\quad + M = 0 \end{aligned}$$

is the general equation of the nth degree; and, by giving the proper values to A, A', B, &c., it may be made identical with any given equation of the nth degree.

273. *Scholium.* Every equation which involves two variables may be constructed, as in § 86, by a curve. The general equation of the nth degree may be regarded as the equation of the general locus of the nth order, which will take different actual forms according to the particular values given to the constants. From the discussion of the general equation of any degree we may determine the character of its locus, the special forms which it may assume, and their relations to each other. As a preliminary to this investigation, it is best to reduce the equation to its simplest form by transforming it to a new system of coördinates so taken as to diminish as far as possible the number of the terms. This chapter will contain a general investigation of the loci of the first and second orders, supposed to be referred to rectangular coördinates.

I.

LOCUS OF THE FIRST ORDER.

274. The general form of the equation of the first degree is

$$A\,x + B\,y + M = 0. \qquad (96.)$$

275. *Problem.* To reduce the general rectangular equation of the first degree to its simplest form, and to determine its locus.

Solution. If (96) is referred to a new rectangular system by the substitution for x and y of their values given by (15) and (16), it becomes

$$A\left(x^\circ + x_1 \cos \frac{x_1}{x} - y_1 \sin \frac{x_1}{x}\right) + B\left(y^\circ + x_1 \sin \frac{x_1}{x} + y_1 \cos \frac{x_1}{x}\right)$$
$$+ M = \left(A \cos \frac{x_1}{x} + B \sin \frac{x_1}{x}\right) x_1 + \left(B \cos \frac{x_1}{x} - A \sin \frac{x_1}{x}\right) y_1$$
$$+ A\,x^\circ + B\,y^\circ + M = 0. \tag{97.}$$

Let the new origin be so taken that the constant terms shall cancel each other; that is, that

$$A\,x^\circ + B\,y^\circ + M = 0. \tag{98.}$$

This is always possible; for between two undetermined and unlimited quantities we can, in general, by the principles of Alg. arts. 143 - 145, assume arbitrarily two equations; and, since (98) is of the first degree, real values of x° and y° can be found to satisfy it.

Let the directions of the new axes be so taken as to make the coefficient of x_1 vanish; that is, let

$$A \cos \frac{x_1}{x} + B \sin \frac{x_1}{x} = 0; \tag{99.}$$

which gives

$$\tan \frac{x_1}{x} = -\frac{A}{B}; \tag{100.}$$

which is always possible, since a tangent may have any real value.

(97) becomes, by the substitution of (98) and (99),

$$\left(B \cos \frac{x_1}{x} - A \sin \frac{x_1}{x}\right) y_1 = 0. \tag{101.}$$

If a product is equal to zero, one of its factors must be equal to zero. But, since the coefficient x_1 in (97) is equal to zero, that of y_1 cannot be equal to zero; for, in this case, the transformation of coördinates would cause all the terms which are of the first degree with reference to the variables to disappear; which is, by § 268, impossible. Hence (101) gives

$$y_1 = 0; \tag{102.}$$

which is the equation of the axis of x_1.

The locus of (96) is, therefore, *a straight line*, drawn through any point of which the coördinates, x° and y°, satisfy (98), and at an angle with the axis of x of which the tangent is given by (100).

276. *Corollary.* It is evident from the very form of (98) that the new origin must be a point in the locus of (96); for, since (98) is of the same form as (96), any values of x° and y° which satisfy (98) must be values of x and y which will satisfy (96), and must, therefore, be coördinates of a point in the locus of (96).

277. *Corollary.* If A and B have the same sign, the angle which the line makes with the axis of x is, by (100) and Tr. §§ 62, 67, between $\frac{1}{2}\pi$ and π, or between $\frac{3}{2}\pi$ and 2π.

If A and B have opposite signs, this angle is, by Tr. § 66, between 0 and $\frac{1}{2}\pi$, or between π and $\frac{3}{2}\pi$.

If $A = 0$, the line is, by Tr. §§ 55, 56, parallel to the axis of x.

If $B = 0$, the line is, by Tr. §§ 55, 57, perpendicular to the axis of x.

278. *Corollary.* If

$$y^\circ = 0,$$

(98) gives

$$x^\circ = -\frac{M}{A}; \tag{103.}$$

which is the abscissa of the point at which the line cuts the axis of x.

If, then, A and M have the same sign, this point is on the left of the origin; if they have opposite signs, it is on the right of the origin.

If $A = \pm 0$, $x^{\circ} = \pm \infty$; and this is, by § 277, the case in which the line is parallel to the axis of x.

If $M = 0$, the line passes through the origin.

279. *Corollary.* If

$$x^{\circ} = 0,$$

(98) gives

$$y^{\circ} = -\frac{M}{B}; \qquad (104.)$$

which is the ordinate of the point at which the line cuts the axis of y; and this expression admits of a discussion like that of § 278.

280. *Corollary.* If A, B, and M have the same sign, the general position of the line will be represented by that of KH' (Fig. 46).

If A and B have the same sign, which is opposite to that of M, the general position of the line is represented by that of $K'H$ (Fig. 46).

If A and M have the same sign, which is opposite to that of B, the general position of the line is represented by that of KH (Fig. 46).

If B and M have the same sign, which is opposite to that of A, the general position of the line is represented by that of $K'H'$ (Fig. 46).

281. *Corollary.* By transposition and division, (96) becomes

$$y = -\frac{A}{B}x - \frac{M}{B};$$

which is, by (100) and (104), identical with (31).

282. *Problem.* To construct an equation of the first degree.

Solution 1st. Reduce the equation to the form (96). Denote the coefficient of x by A, that of y by B, and the sum of the constant terms by M. Assume, at pleasure, any convenient value of x° or of y°, and find, by (98), the corresponding value of y° or of x°. Lay down the point x°, y°, and draw through it, by § 14, *Solution 2d,* a straight line, so that the tangent of its angle with the axis of x shall be $-\frac{A}{B}$. It will, by § 275, be the required locus.

Solution 2d. Find, by (103) and (104), the points at which the straight line cuts the axes, and draw a straight line through them.

283. *Corollary.* Since the direction and whole position of a straight line are determined by the values of the arbitrary constants in (96), the line is given, if those constants are known. Practically, however, it is necessary to find only the ratios of any one of the constants to the other two, or the reciprocals of those ratios. If the equation be divided by B, the coefficient of y in the reduced equation is the known quantity, 1, and the only undetermined constants are the ratios, $A : B$ and $M : B$. So, if the equation is divided by either of the other constants.

284. *Problem.* To find the angle of two straight lines in a plane.

Solution. Let the equations of the lines be

$$A' x + B' y + M' = 0, \qquad (105.)$$

$$A'' x + B'' y + M'' = 0; \qquad (106.)$$

and let α' and α'' denote the directions of the lines. Then (§ 20. g; Tr. § 52)

$$\frac{\alpha''}{\alpha'} = \frac{x}{\alpha'} + \frac{\alpha''}{x} = \frac{\alpha''}{x} - \frac{\alpha'}{x}, \qquad (107.)$$

$$\tan {}^{\alpha''}_{\alpha'} = \frac{\tan {}^{\alpha''}_{x} - \tan {}^{\alpha'}_{x}}{1 + \tan {}^{\alpha''}_{x} \tan {}^{\alpha'}_{x}}. \qquad (108.)$$

But (100) gives

$$\tan {}^{\alpha'}_{x} = -\frac{A'}{B'}, \qquad \tan {}^{\alpha''}_{x} = -\frac{A''}{B''};$$

the substitution of which values in (108) gives

$$\tan {}^{\alpha''}_{\alpha'} = \frac{A'B'' - A''B'}{A'A'' + B'B''}. \qquad (109.)$$

285. *Corollary.* If the lines are parallel, then (Tr. § 55)

$${}^{\alpha''}_{\alpha'} = 0,$$

$$\tan {}^{\alpha''}_{\alpha'} = 0;$$

so that either the numerator of (109) is zero, or the denominator infinite. But the denominator, being the sum of two products of finite quantities, cannot be infinite. Hence

$$A'B'' - A''B' = 0, \qquad (110.)$$

or

$$\frac{A'}{B'} = \frac{A''}{B''}; \qquad (111.)$$

either of which equations expresses the condition of the parallelism of two lines.

286. *Corollary.* If the lines are perpendicular, then (Tr. § 55)

$${}^{\alpha''}_{\alpha'} = \tfrac{1}{2}\pi,$$

$$\tan {}^{\alpha''}_{\alpha'} = \infty;$$

so that, since the numerator of (109) cannot be infinite, its denominator must be zero; that is,

$$A'A'' + B'B'' = 0; \qquad (112.)$$

which expresses that two lines are perpendicular to each other.

287. *Problem.* To find the coördinates of the point of intersection of two straight lines in a plane.

Solution. Let x' and y' be the coördinates of the point of intersection. Since this point is in both loci, its coördinates must satisfy the equation of each locus; so that, if the equations are represented by (105) and (106), we shall have

$$A'x' + B'y' + M' = 0,$$
$$A''x' + B''y' + M'' = 0.$$

These two equations are enough to determine the values of the quantities x' and y' in terms of the arbitrary constants. The values so determined are

$$x' = \frac{B'M'' - B''M'}{A'B'' - A''B'}, \qquad (113.)$$

$$y' = \frac{A''M' - A'M''}{A'B'' - A''B'}; \qquad (114.)$$

288. *Corollary.* Since (113) and (114) give only one set of values for x' and y', two straight lines can cross each other in only one point. Thus, this simple property of the straight line appears, in Analytic Geometry, as a consequence of the fact that its equation is of the first degree.

289. *Corollary.* If the lines are parallel, the substitution of (110) in (113) and (114) gives

$$x' = \pm\,\infty, \qquad y' = \pm\,\infty;$$

so that two parallel lines may be conceived, as we have already conceived them, as meeting at an infinite distance from any part at which we choose to consider them.

290. *Corollary.* The last corollary is, however, modified, if the lines are both parallel to either of the coördinate axes; for then, by § 277,

$$A' = A'' = 0,$$

or

$$B' = B'' = 0.$$

The student should examine this case.

291. *Problem.* To find the equation of a straight line which passes through a given point.

Solution. Let the required equation be represented by an equation of the form (96), in which A, B, and M are unknown; let α denote the direction of the line, and let the coördinates of the given point be x' and y'. Since x', y' is a point of the required line, its coördinates must satisfy the equation of the line; so that

$$A x' + B y' + M = 0. \qquad (115.)$$

Subtracting (115) from (96), we have

$$A (x - x') + B (y - y') = 0,$$

$$y - y' = -\frac{A}{B} (x - x'), \qquad (116.)$$

which may also be written

$$y - y' = (x - x') \tan \frac{\alpha}{x}. \qquad (117.)$$

Since $-\frac{A}{B}$, or $\tan \frac{\alpha}{x}$, is undetermined, there is an infinite number of lines which satisfy the condition.

292. *Problem.* To find the equation of a straight line which passes through two given points.

Solution. Let (96) represent the equation of the line; and let the given points be x', y' and x'', y''. The coördinates of each of these points will satisfy the equation of the line, and will give (115) and

$$A x'' + B y'' + M = 0,$$

Subtracting this equation from (115), we have

$$A\,(x' - x'') + B\,(y' - y'') = 0,$$

$$\tan{}^{\alpha}_{x} = -\frac{A}{B} = \frac{y' - y''}{x' - x''}; \qquad (118.)$$

and (116) becomes by substitution

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'); \qquad (119.)$$

which determines the line, since all the constants are known quantities.

293. *Corollary.* If x' and y' are both greater or both less than x'' and y'' respectively, $\tan{}^{\alpha}_{x}$ is positive, and ${}^{\alpha}_{x}$ is between 0 and $\frac{1}{2}\pi$, or between π and $\frac{3}{2}\pi$. If $x' < x''$ while $y' > y''$, or if $x' > x''$, while $y' < y''$, $\tan{}^{\alpha}_{x}$ is negative, and ${}^{\alpha}_{x}$ is between $\frac{1}{2}\pi$ and π, or between $\frac{3}{2}\pi$ and 2π.

If $y' = y''$,

$$\tan{}^{\alpha}_{x} = \frac{y' - y''}{x' - x''} = 0, \qquad {}^{\alpha}_{x} = 0;$$

so that the line is parallel to the axis of x.

If $x' = x''$,

$$\tan{}^{\alpha}_{x} = \frac{y' - y''}{x' - x''} = \infty, \qquad {}^{\alpha}_{x} = \tfrac{1}{2}\pi;$$

so that the line is perpendicular to the axis of x.

The geometric interpretation of these results must be carefully attended to.

294. *Problem.* To find the equation of a straight line which passes through a given point and makes a given angle with a given straight line.

Solution. Let the given point be x', y'; and let the direction of the given straight line be α'.

$${}^{\alpha}_{x} = {}^{\alpha'}_{x} + {}^{\alpha}_{\alpha'};$$

and (117) becomes, by substitution,

$$y - y' = (x - x') \tan \left(\genfrac{}{}{0pt}{}{\alpha'}{x} + \genfrac{}{}{0pt}{}{\alpha}{\alpha'}\right). \qquad (120.)$$

295. *Corollary.* If the required line is to be parallel to the given line

$$\genfrac{}{}{0pt}{}{\alpha}{\alpha'} = 0,$$

and (120) becomes

$$y - y' = (x - x') \tan \genfrac{}{}{0pt}{}{\alpha'}{x}. \qquad (121.)$$

296. *Corollary.* If the required line is to be perpendicular to the given line, then (Tr. § 63)

$$\genfrac{}{}{0pt}{}{\alpha}{\alpha'} = \tfrac{1}{2}\pi,$$

$$y - y' = (x - x') \cot \genfrac{}{}{0pt}{}{\alpha'}{x}. \qquad (122.)$$

297. Examples.

1. Find the loci of the equations,

$$-4y + 11 = 0,$$
$$2x + 3y - 7 = 0;$$

find their inclination to each other, and their point of intersection.

Solution: We have

$$A' = 0, \quad B' = -4, \quad M' = 11;$$
$$A'' = 2, \quad B'' = 3, \quad M'' = -7.$$

The ordinate of the point at which the first line crosses the axis of y is, by (104),

$$-\frac{11}{-4} = 2\tfrac{3}{4};$$

and, by § 277, the line is parallel to the axis of x. Therefore, to construct the first equation, we may take on the axis of y (Fig. 68)

Examples.

$$AR' = 2\tfrac{3}{4},$$

and draw through it a straight line parallel to the axis of y.

The ordinate of the point at which the second line cuts the axis of y is, by (104),

$$-\frac{-7}{3} = 2\tfrac{1}{3};$$

and the abscissa of the point at which it cuts the axis of x is, by (103),

$$-\frac{-7}{2} = 3\tfrac{1}{2}.$$

Take

$$AR'' = 2\tfrac{1}{3}, \qquad AP'' = 3\tfrac{1}{2};$$

and the straight line drawn through P'' and R'' is the locus of the second equation.

The substitution of the above values of the constants in (109) gives

$$\tan {}^{\alpha''}_{\alpha'} = \frac{8}{-12} = -\tfrac{2}{3}.$$

$$\log \tfrac{2}{3} = 9.82391 = \log \tan 146° \, 18' \, 35'',$$

$${}^{\alpha''}_{\alpha'} = 146° \, 18' \, 35'';$$

the obtuse angle being selected because $\tan {}^{\alpha''}_{\alpha'}$ is negative.

(113) and (114) give for the point of intersection

$$x' = \frac{28 - 33}{8} = -\tfrac{5}{8},$$

$$y' = \frac{22}{8} = 2\tfrac{3}{4}.$$

2. Construct the following equations, and find the inclination and the points of intersection of their loci: —

$$x - 3\,y - 2 = 0,$$

$$5\,x + 2\,y = 0.$$

Ans. ${}^{\alpha''}_{\alpha'} = 93 \;\; 21' \, 59''; \;\; x' = \frac{4}{17}; \;\; y' = -\frac{10}{17}.$

Examples.

3. Construct the following equations, and find the inclination and the points of intersection of their loci: —

$$3x - 4y + 6 = 0,$$
$$3x + 2y - 3 = 0.$$

Ans. ${}_{\alpha'}^{\alpha''} = 86° \, 49' \, 13''$; $x' = 0$; $y' = 1\frac{1}{2}$.

4. Find the equation of a straight line which passes through the point of which the coördinates are $x' = 5$, $y' = 2$, and makes the angle $\frac{1}{4}\pi$ with the locus of the equation

$$2x + 3y - 13 = 0.$$

Solution. By (100),

$$\tan {}_{x}^{\alpha'} = -\tfrac{2}{3}.$$

By Tr. §§ 52, 59

$$\tan {}_{\alpha'}^{\alpha} = \tan \tfrac{1}{4}\pi = 1,$$

$$\tan\left({}_{x}^{\alpha'} + {}_{\alpha'}^{\alpha}\right) = \frac{\tan {}_{x}^{\alpha'} + \tan {}_{\alpha'}^{\alpha}}{1 - \tan {}_{x}^{\alpha'} \tan {}_{\alpha'}^{\alpha}} = \frac{1 - \frac{2}{3}}{1 + \frac{2}{3}} = \tfrac{1}{5};$$

which, substituted in (120), gives for the required equation

$$y - 2 = \tfrac{1}{5}(x - 5),$$

or

$$x - 5y + 5 = 0.$$

5. Find the equation of a straight line which passes through the point of which the coördinates are $x' = 1 - \sqrt{3}$, $y' = 1 + \sqrt{3}$, and makes the angle $\frac{1}{3}\pi$ with the straight line of which the equation is

$$2x - 2y + 5 = 0.$$

Ans. $(1 + \sqrt{3})\,x - (1 - \sqrt{3})\,y = 0.$

6. Find the equation of a straight line which passes through the points of which the coördinates are, respectively, $x' = 4$, $y' = 1$, and $x'' = 5$, $y'' = -4$.

Ans. $5x + y - 21 = 0.$

II.

LOCUS OF THE SECOND ORDER.

298. The general equation of the second degree is

$$A\,x^2 + B\,x\,y + C\,y^2 + D\,x + E\,y + M = 0. \quad (123.)$$

299. *Problem.* To reduce the general equation of the first degree to its simplest form.

Solution. Instead of transforming (123) directly to that system of coördinates for which it has the simplest form, it is best, for reasons which will presently appear, to transform it first to a system in which the axes have the desired directions, while the origin is unchanged, and afterwards, without changing the directions of the axes, to a system in which the origin has the desired place; and the result of these two transformations must obviously be the same as if we transformed immediately to the last system.

a. If the origin is unchanged, the equations of transformation are (17) and (18), which, substituted in (123), give

$$A\left(x_1^2 \cos^2 \tfrac{x_1}{x} - 2\,x_1\,y_1 \sin \tfrac{x_1}{x} \cos \tfrac{x_1}{x} + y_1^2 \sin^2 \tfrac{x_1}{x}\right)$$

$$+ B\left(x_1^2 \sin \tfrac{x_1}{x} \cos \tfrac{x_1}{x} + x_1\,y_1\left[\cos^2 \tfrac{x_1}{x} - \sin^2 \tfrac{x_1}{x}\right] - y_1^2 \sin \tfrac{x_1}{x} \cos \tfrac{x_1}{x}\right)$$

$$+ C\left(x_1^2 \sin^2 \tfrac{x_1}{x} + 2\,x_1\,y_1 \sin \tfrac{x_1}{x} \cos \tfrac{x_1}{x} + y_1^2 \cos^2 \tfrac{x_1}{x}\right)$$

$$+ D\left(x_1 \cos \tfrac{x_1}{x} - y_1 \sin \tfrac{x_1}{x}\right)$$

$$+ E\left(x_1 \sin \tfrac{x_1}{x} + y_1 \cos \tfrac{x_1}{x}\right)$$

$$+ M$$

$$= \left(A \cos^2 \frac{x_1}{x} + B \sin \frac{x_1}{x} \cos \frac{x_1}{x} + C \sin^2 \frac{x_1}{x}\right) x_1^2$$
$$+ \left(2\,[C - A] \sin \frac{x_1}{x} \cos \frac{x_1}{x} + B\left[\cos^2 \frac{x_1}{x} - \sin^2 \frac{x_1}{x}\right]\right) x_1\, y_1$$
$$+ \left(A \sin^2 \frac{x_1}{x} - B \sin \frac{x_1}{x} \cos \frac{x_1}{x} + C \cos^2 \frac{x_1}{x}\right) y_1^2$$
$$+ \left(D \cos \frac{x_1}{x} + E \sin \frac{x_1}{x}\right) x_1$$
$$+ \left(E \cos \frac{x_1}{x} - D \sin \frac{x_1}{x}\right) y_1$$
$$+ M = 0. \qquad (124.)$$

Let $\frac{x_1}{x}$ be taken of such a value that the coefficient of $x_1\, y_1$ shall vanish, and denote the coefficients of x_1^2, y_1^2, x_1, and y_1 by A_1, B_1, D_1, and E_1; so that

$$A_1 = A \cos^2 \frac{x_1}{x} + B \sin \frac{x_1}{x} \cos \frac{x_1}{x} + C \sin^2 \frac{x_1}{x}, \qquad (125.)$$

$$B_1 = A \sin^2 \frac{x_1}{x} - B \sin \frac{x_1}{x} \cos \frac{x_1}{x} + C \cos^2 \frac{x_1}{x}, \qquad (126.)$$

$$D_1 = D \cos \frac{x_1}{x} + E \sin \frac{x_1}{x}, \qquad (127.)$$

$$E_1 = - D \sin \frac{x_1}{x} + E \cos \frac{x_1}{x}, \qquad (128.)$$

$$2\,(C - A) \sin \frac{x_1}{x} \cos \frac{x_1}{x} + B\left(\cos^2 \frac{x_1}{x} - \sin^2 \frac{x_1}{x}\right) = 0; \qquad (129.)$$

and (124) becomes

$$A_1\, x_1^2 + B_1\, y_1^2 + D_1\, x_1 + E_1\, y_1 + M = 0. \qquad (130.)$$

b. The origin may now be changed by (19), in which, however, the old coördinates, there denoted by x and y, must be denoted by x_1 and y_1, and the new coördinates, there denoted by x_1 and y_1, must be represented by x_2 and y_2. (130) then becomes, by substitution,

$$A_1 (x_2^2 + 2\, x_1^\circ\, x_2 + x_1^{\circ 2}) + B_1 (y_2^2 + 2\, y_1^\circ\, y_2 + y_1^{\circ 2})$$
$$+ D_1 (x_2 + x_1^\circ) + E_1 (y_2 + y_1^\circ) + M$$
$$= A_1\, x_2^2 + B_1\, y_2^2 + (2\, A_1\, x_1^\circ + D_1)\, x_2 + (2\, B_1\, y_1^\circ + E_1)\, y_2$$
$$+ A_1\, x_1^{\circ 2} + B_1\, y_1^{\circ 2} + D_1\, x_1^\circ + E_1\, y_1^\circ + M = 0. \qquad (131.)$$

If the coördinates of the new origin be taken so as to satisfy the equations

$$2\, A_1\, x_1^\circ + D_1 = 0, \qquad (132.)$$
$$2\, B_1\, y_1^\circ + E_1 = 0, \qquad (133.)$$

and if M_1 denote the sum of the constant terms, so that

$$M_1 = A_1\, x_1^{\circ 2} + B_1\, y_1^{\circ 2} + D_1\, x_1^\circ + E_1 y_1^\circ + M, \quad (134.)$$

(131) becomes

$$A_1\, x_2^2 + B_1\, y_2^2 + M_1 = 0\,; \qquad (135.)$$

which is the simplest form of the equation of the second degree.

300. *Theorem.* The reduction of § 299 is always possible.

Proof. It is to be shown that any given equation of the second degree can be transformed to a new system such that $\frac{x_1}{x}$ shall satisfy (129), and then to a third system in which x_1° and y_1° are determined by (132) and (133), and that none of the coefficients of the equation, thus transformed, become imaginary.

By Tr. § 49,

$$2 \sin \frac{x_1}{x} \cos \frac{x_1}{x} = \sin 2 \left(\frac{x_1}{x}\right), \qquad \cos^2 \frac{x_1}{x} - \sin^2 \frac{x_1}{x} = \cos 2 \left(\frac{x_1}{x}\right);$$

which, substituted in (129), give

$$(C - A) \sin 2 \left(\frac{x_1}{x}\right) + B \cos 2 \left(\frac{x_1}{x}\right) = 0,$$
$$\tan 2 \left(\frac{x_1}{x}\right) = - \frac{B}{C - A}\,; \qquad (136.)$$

the value of which is necessarily real; and, since tangents may have any real values, it is always possible to take $2\left(\frac{x_1}{x}\right)$ so as to satisfy (136), that is, to take $\frac{x_1}{x}$ so as to satisfy (129). Again, since, as has just been shown, $\sin\frac{x_1}{x}$ and $\cos\frac{x_1}{x}$ have real values, the values of A_1, B_1, D_1, and E_1, given by (125) - (128), must be real; and the first transformation is always possible.

Formulæ (132) and (133) give

$$x_1^\circ = -\frac{D_1}{2A_1}, \tag{137.}$$

$$y_1^\circ = -\frac{E_1}{2B_1}; \tag{138.}$$

so that these quantities are necessarily real; and, if they are substituted in (134), they give a real value for M_1. Hence the second transformation is always possible; and every equation of the second degree can be reduced to the form (135).

301. *Corollary.* If $A_1 = 0$, (137) gives

$$x_1^\circ = \pm\infty,$$

except in the special case in which $D_1 = 0$. This introduces two infinite terms into the value of M_1 given by (134). Hence, if $A_1 = 0$, while D_1 is not zero, it will be best to transform the equation to a system of coördinates different from that of (135). None of the coefficients of (130) can be infinite; for, since the values of $\cos\frac{x_1}{x}$ and $\sin\frac{x_1}{x}$ must be less than 1, none of the terms of the second members of (125) - (128) can be infinite. There is, therefore, no difficulty with the transformation of § 299. *a*; and (130) becomes

$$B_1 y_1^2 + D_1 x_1 + E_1 y_1 + M = 0.$$

The condition (133) may be retained; but, instead of (132), we may take x_1° so as to make the constant terms in (131) cancel each other, that is, so as to satisfy

$$B_1 y_1^{\circ 2} + D_1 x_1^\circ + E_1 y_1^\circ + M = 0; \qquad (139.)$$

which is possible, since it gives for x_1° the real and finite value

$$x_1^\circ = \frac{-B_1 y_1^{\circ 2} - E_1 y_1^\circ - M}{D_1}. \qquad (140.)$$

(131) is thus reduced to the form,

$$B_1 y_2^2 + D_1 x_2 = 0. \qquad (141.)$$

302. *Corollary.* If $B_1 = 0$, while E_1 is not zero, x_1° may be taken to satisfy (132), and y_1° to make the sum of the constant terms in (131) equal to zero; in which case, (131) is reduced to the form,

$$A_1 x_2^2 + E_1 y_2 = 0. \qquad (142.)$$

303. *Corollary.* A_1 and B_1 cannot both equal zero; for, by § 268, the transformation cannot cause all the terms of the second degree to vanish.

304. *Problem.* To determine the forms of the locus of the second order.

Solution. Either A_1 and B_1 represent quantities which have the same sign, or they represent quantities which have opposite signs, or one of them is equal to zero. These three cases may be treated separately.

(1.) *If A_1 and B_1 have the same sign,* let

$$A_2 = \sqrt{\left(\pm \frac{M_1}{A_1}\right)}, \qquad B_2 = \sqrt{\left(\pm \frac{M_1}{B_1}\right)}; \qquad (143.)$$

in which that sign is to be taken which will make the quantities under the radical sign positive, that is, which will make A_2 and B_2 real; and this sign must be the same in both cases. Equations (143) give

$$\frac{A_1}{M_1} = \pm \frac{1}{A_2^2}, \qquad \frac{B_1}{M_1} = \pm \frac{1}{B_2^2}; \qquad (144.)$$

so that, dividing (135) by M_1, and substituting the above values, we have,

$$\pm\frac{x_2^2}{A_2^2}\pm\frac{y_2^2}{B_2^2}+1=0. \qquad (145.)$$

a. If M_1 has the sign opposite to that of A_1 and B_1, the lower sign must be used in (143) - (145); and (145) becomes, by transposition,

$$\frac{x_2^2}{A_2^2}+\frac{y_2^2}{B_2^2}=1;$$

the locus of which is, by §§ 153, 154, necessarily *an ellipse* which has its centre at the origin, and its semi-axes equal to A_2 and B_2, and laid off respectively on the axes of x_2 and y_2.

b. If $M_1=0$, equations (143) give

$$A_2=0, \qquad B_2=0;$$

so that, in this case, the ellipse is by § 256 reduced to a single *point*, namely, the origin.

c. If M_1 has the same sign as A_1 and B_1, the upper sign must be used in (143) - (145); and the first member of (145) is the sum of three positive quantities, and cannot be equal to zero, unless either x_2^2 or y_2^2 is negative, that is, except for points for which either x_2 or y_2 is imaginary.

In this case, therefore, the locus of (123) is *imaginary;* in other words, it has *no locus.*

(2.) *If A_1 and B_1 have opposite signs*, let

$$A_2=\sqrt{\left(\pm\frac{M_1}{A_1}\right)}, \qquad B_2=\sqrt{\left(\mp\frac{M_1}{B_1}\right)}; \qquad (146.)$$

in each of which equations that sign is to be taken which will make the quantity under the radical sign positive; so that, obviously, the sign must be *plus* in one case and *minus* in the other. Equations (146) give

$$\frac{A_1}{M_1}=\pm\frac{1}{A_2^2}, \qquad \frac{B_1}{M_1}=\mp\frac{1}{B_2^2}; \qquad (147.)$$

so that, dividing (135) by M_1 and substituting these values, we have

$$\pm \frac{x_2^2}{A_2^2} \mp \frac{y_2^2}{B_2^2} + 1 = 0. \qquad (148.)$$

a. If M_1 has the same sign as B_1, the lower sign must be used in (146) – (148); and (148) becomes, by transposition,

$$\frac{x_2^2}{A_2^2} - \frac{y_2^2}{B_2^2} = 1;$$

the locus of which is, by § 189, *an hyperbola* which has its centre at the origin, its semi-transverse axis equal to A_2 and laid off on the axis of x_2, and its semi-conjugate axis equal to B_2 and laid off on the axis of y_2.

b. If $M_1 = 0$, equations (146) give

$$A_2 = 0, \qquad B_2 = 0;$$

so that, in this case, the hyperbola is, by § 264, reduced to the combination of *two straight lines which cross each other* at the origin and at angles with the axis of x_2 of which the tangents are

$$\pm \frac{B_2}{A_2} = \pm \sqrt{\left[\left(\pm \frac{M_1}{B_1}\right) \div \left(\mp \frac{M_1}{A_1}\right)\right]} = \pm \sqrt{\left(-\frac{A_1}{B_1}\right)}; \qquad (149.)$$

which is real, since A_1 and B_1 have opposite signs.

c. If M_1 has the same sign as A_1, the upper sign must be used in (146) – (148); and (148) becomes, by transposition,

$$\frac{y_2^2}{B_2^2} - \frac{x_2^2}{A_2^2} = 1;$$

the locus of which is, by § 190, necessarily *an hyperbola* which has its centre at the origin, its semi-transverse axis equal to B_2 and laid off on the axis of y, and its semi-conjugate axis equal to A_2 and laid off on the axis of x_2.

(3.) *a. If* $A_1 = 0$, *while* D_1 *is not zero,* let

$$4p = -\frac{D_1}{B_1}; \qquad (150.)$$

and (141) becomes, by division, substitution, and transposition,

$$y_2^2 = 4p\,x_2;$$

the locus of which is, by § 209, necessarily *a parabola* which has its vertex at the origin and its focus in the axis of x_2 at the distance p from the origin; and, by § 201, the locus curves to the left or to the right, according as D_1 and B_1 have the same sign or opposite signs.

In like manner, *if* $B_1 = 0$, *while* E_1 *is not zero,* by taking

$$4p = -\frac{E_1}{A_1}, \qquad (151.)$$

we reduce (142) to the form

$$x_2^2 = 4p\,y_2.$$

b. If $A_1 = 0$, *and* $D_1 = 0$, x_1° becomes indeterminate in (137); and (135) gives

$$B_1\,y_2^2 + M_1 = 0;$$

$$y_2 = \pm\sqrt{\left(-\frac{M_1}{B_1}\right)};$$

the locus of which is, in general, the combination of *two straight lines, parallel to the axis of* x_2, and at distances from it equal to $\pm\sqrt{\left(-\frac{M_1}{B_1}\right)}$. If M_1 is of the same sign as B_1, these lines are imaginary; if $M_1 = 0$, they coincide with the axis x_2, and the given equation of the second degree is only the square of an equation of the first degree, which is the equation of the straight line in question; if M_1 differs in sign from B_1, the parallels are real and separate.

If $B_1 = 0$, *and* $E_1 = 0$, the locus is the combination of two straight lines, parallel to the axis of y_2.

305. *Corollary.* The results of the last section may be exhibited in a tabular form. Since A_1 and B_1 cannot both equal zero, we may suppose $B_1 > 0$; for if it is negative, i. e. less than zero, it can be made positive by changing all the signs of the equation

$$B_1 > 0.$$

$$A_1 > 0;\ \begin{cases} M_1 > 0, & \text{No Locus;} \\ M_1 = 0, & \text{Point;} \\ M_1 < 0, & \text{Ellipse.} \end{cases}$$

$$A_1 = 0;\ \begin{cases} D_1 > 0, & \text{Negative Parabola;} \\ D_1 = 0, & \text{Two Parallels;} \\ D_1 < 0, & \text{Positive Parabola.} \end{cases}$$

$$A_1 < 0;\ \begin{cases} M_1 > 0, & \text{X-Hyperbola;*} \\ M_1 = 0, & \text{Two Crossing Straight Lines;} \\ M_1 < 0, & \text{Y-Hyperbola.} \end{cases}$$

306. *Corollary.* It is evident that the analysis which has been given of the quadratic locus completely exhausts all its possible forms; for since every equation of the second degree can be reduced to one of the forms (135), (141), (142), every such equation can be brought under one of the heads of the above table, and will therefore be constructed by either an ellipse, a parabola, or an hyperbola, or one of their modifications. The general equation of the second degree may, therefore, be regarded as expressing that general law of form (§ 222) which is differently manifested in these several curves. Moreover, the results of the preceding sections conform to and sustain, not only the identity of the law of the

* That is, the hyperbola which has its transverse axis on the axis of x.

ellipse, the parabola, and the hyperbola, but that mutual relation of their special forms which has already been established between them, both by means of the equations drawn from these definitions, and also by taking them as the sections of a cone. For the lines of the second order, when classified, as above, according to the algebraic conditions which affect their coefficients in the equation, arrange themselves in a series, in which the parabola appears as an intermediate form between the ellipse and the hyperbola, and as belonging to the class of ellipses, if A_1 is considered as being equal to $+0$, or infinitesimally larger than zero, or to the class of hyperbolas, if $A_1 = -0$, or is infinitesimally smaller than zero.

307. *Corollary.* In the table of § 305, the special forms of the conic sections arise as in Chapter VIII. The point occurs as the limit of real ellipses; the combination of two parallels, as intermediate between parabolas which curve in the positive direction from the vertex and those which curve in the negative direction from the vertex; and the combination of two straight lines which cross each other, as intermediate between hyperbolas which have their transverse axes on the axis of x and those which have their transverse axes on the axis of y.

308. *Problem.* To find the positions of the origin and the axes of (135), and to compute the values of the constants in that equation.

Solution. We may find $\frac{x_1}{x} = \frac{x_2}{x}$ by (136), and, by substituting its value in (125) - (128), find A_1, B_1, D_1, and E_1, then get $x_1{}^\circ$ and $y_1{}^\circ$ by (137), (138), and M_1 by (134). It is possible, however, by simplifying some of the above equations, to shorten this process somewhat.

a. To find A_1. Adopt the notation

$$L = 2\,A \cos\frac{x_1}{x} + B \sin\frac{x_1}{x}, \qquad (152.)$$

$$L' = 2\,C \sin\frac{x_1}{x} + B \cos\frac{x_1}{x}. \qquad (153.)$$

The double of (125) is

$$\left(2\,A \cos\frac{x_1}{x} + B \sin\frac{x_1}{x}\right)\cos\frac{x_1}{x} + \left(2\,C \sin\frac{x_1}{x} + B \cos\frac{x_1}{x}\right)\sin\frac{x_1}{x}$$

$$= 2\,A_1 = L \cos\frac{x_1}{x} + L' \sin\frac{x_1}{x}. \qquad (154.)$$

Also (129) may be written in the form

$$\left(2\,C \sin\frac{x_1}{x} + B \cos\frac{x_1}{x}\right)\cos\frac{x_1}{x} - \left(2\,A \cos\frac{x_1}{x} + B \sin\frac{x_1}{x}\right)\sin\frac{x_1}{x}$$

$$= 0 = L' \cos\frac{x_1}{x} - L \sin\frac{x_1}{x}. \qquad (155.)$$

If (154) be multiplied by $\cos\frac{x_1}{x}$ and (155) by $\sin\frac{x_1}{x}$, and the second product subtracted from the first, the remainder is (Tr. § 13)

$$2A_1 \cos\frac{x_1}{x} = L\left(\cos^2\frac{x_1}{x} + \sin^2\frac{x_1}{x}\right) + L'\left(\sin\frac{x_1}{x}\cos\frac{x_1}{x} - \sin\frac{x_1}{x}\cos\frac{x_1}{x}\right)$$

$$= L = 2\,A \cos\frac{x_1}{x} + B \sin\frac{x_1}{x}; \qquad (156.)$$

which gives

$$2\,(A_1 - A) \cos\frac{x_1}{x} - B \sin\frac{x_1}{x} = 0. \qquad (157.)$$

So, the product of (154) by $\sin\frac{x_1}{x}$ added to that of (155) by $\cos\frac{x_1}{x}$ gives (Tr. § 13)

$$2\,A_1 \sin\frac{x_1}{x} = L' = 2\,C \sin\frac{x_1}{x} + B \cos\frac{x_1}{x}, \qquad (158.)$$

or

$$2\,(A_1 - C)\sin\frac{x_1}{x} - B\cos\frac{x_1}{x} = 0. \qquad (159.)$$

From (157) and (159), $\sin\frac{x_1}{x}$ may be eliminated, by multiplying (157) by $2\,(A_1 - C)$, its coefficient in (159), and (159) by B, the negative of its coefficient in (157), and adding the products together. The sum, divided by $\cos\frac{x_1}{x}$, is

$$4\,(A_1 - A)\,(A_1 - C) - B^2 = 0. \qquad (160.)$$

If we put X in this equation instead of A_1, we have

$$4\,(X - A)\,(X - C) - B^2 = 0; \qquad (161.)$$

the roots of which are

$$X = \tfrac{1}{2}\,(A + C) \pm \tfrac{1}{2}\sqrt{[B^2 + (A - C)^2]}; \qquad (162.)$$

so that A_1 is equal to either of these values of X.

b. To find B_1. If, in the second member of (125), A is changed to C, C to A, and B to $-B$, that expression becomes identical with the second member of (126), and gives the value of B_1, instead of A_1. If the same changes be made in (152) and (153), and L and L' changed to L_1 and L_1',

$$L_1 = 2\,C\cos\frac{x_1}{x} - B\sin\frac{x_1}{x}, \qquad (163.)$$

$$L_1' = 2\,A\sin\frac{x_1}{x} - B\cos\frac{x_1}{x}; \qquad (164.)$$

so that, instead of (154), we have

$$2\,B_1 = L_1\cos\frac{x_1}{x} + L_1'\sin\frac{x_1}{x}. \qquad (165.)$$

Again, if the same changes be made in (129), they only reverse the sign of the first member, which, being equal to zero, is not thereby affected; so that, instead of (155), we have

$$0 = L_1'\cos\frac{x_1}{x} - L_1\sin\frac{x_1}{x}. \qquad (166.)$$

(165) and (166) are of the same form as (154) and (155). If, therefore, similar operations are performed on the two sets of equations, they will lead to similar results; for the form of an algebraic result depends only on the forms of the expressions from which it springs, and not on the particular meanings which may be attached to the letters. Hence, instead of going through the process of art. *a*, we have only to make the above changes in (160), which gives

$$4\,(B_1 - C)\,(B_1 - A) - B^2 = 0. \qquad (167.)$$

If X is put in this equation instead of B_1, it becomes identical with (161); so that B_1 is equal to either of the values of X given by (162).

Further, A_1 and B_1 are not the same value of X, but its two values. For the sum of the two values of X is $A + C$; and the sum of (125) and (126) is (Tr. § 13)

$$A_1 + B_1 = A\left(\cos^2\frac{x_1}{x} + \sin^2\frac{x_1}{x}\right) + C\left(\sin^2\frac{x_1}{x} + \cos^2\frac{x_1}{x}\right)$$
$$= A + C.$$

The sum of A_1 and B_1 is, therefore, equal to that of the two values of X, and each of these quantities is equal to one of the values of X; so that it must be that A_1 and B_1 are the two roots of (161).

c. To find $\frac{x_1}{x}$. (157) gives, by transposition and division,

$$\tan\frac{x_1}{x} = \frac{2\,(A_1 - A)}{B}. \qquad (168.)$$

This value may be constructed by § 14, *Solution* 2, and will give the positions of the axes of x_1 and y_1.

d. To find D_1 *and* E_1. By Tr. §§ 10, 11, 14,

$$\cos\frac{x_1}{x} = \frac{1}{\sec\frac{x_1}{x}} = \frac{1}{\sqrt{\left(1 + \tan^2\frac{x_1}{x}\right)}}, \qquad (169.)$$

$$\sin\frac{x_1}{x} = \cos\frac{x_1}{x}\tan\frac{x_1}{x}. \qquad (170.)$$

The values of $\cos\frac{x_1}{x}$ and $\sin\frac{x_1}{x}$ may be computed by these formulæ and substituted in (127) and (128).

e. To find $x_1{}^\circ$ *and* $y_1{}^\circ$. Substitute the values of A_1, B_1, D_1, and E_1 in (137) and (138). The new origin may then be found, and the axes of x_2 and y_2 drawn through it parallel to those of x_1 and y_1.

f. To find M_1. (137) and (138) give

$$x_1{}^{\circ 2} = \frac{D_1^2}{4A_1^2}, \qquad y_1{}^{\circ 2} = \frac{E_1^2}{4B_1^2};$$

$$A_1 x_1{}^{\circ 2} = \frac{D_1^2}{4A_1}, \qquad B_1 y_1{}^{\circ 2} = \frac{E_1^2}{4B_1};$$

$$D_1 x_1{}^\circ = -\frac{D_1^2}{2A_1} = -\frac{2D_1^2}{4A_1}, \quad E_1 y_1{}^\circ = -\frac{E_1^2}{2B_1} = -\frac{2E_1^2}{4B_1};$$

and (134) becomes

$$M_1 = \frac{D_1^2}{4A_1} + \frac{E_1^2}{4B_1} - \frac{2D_1^2}{4A_1} - \frac{2E_1^2}{4B_1} + M$$

$$= -\frac{D_1^2}{4A_1} - \frac{E_1^2}{4B_1} + M. \qquad (171.)$$

309. *Problem.* To find the position of the origin and the axes of (141), and to compute the values of the constants.

Solution. The constants $\frac{x_1}{x}$, A_1, B_1, D_1, E_1, and $y_1{}^\circ$ may be found as in § 308; and $x_1{}^\circ$ may be found by (140).

310. *Corollary.* The positions of the origin and axes, and the values of the constants, may be found for (142) in like manner.

311. *Corollary.* For the case in which $A_1 = 0$, while D_1 is not zero, (171) becomes

$$M_1 = -\frac{D_1^2}{\pm 0} - \frac{E_1^2}{4B_1} + M = \mp \infty;$$

in which, since D_1^2 is necessarily positive, the sign of M_1 is opposite to that of A_1.

If the zero value of A_1 is supposed to be of the same sign as B_1, the case comes under § 304. (1.) a; and (143) gives

$$A_2^2 = -\frac{M_1}{A_1} = \frac{\infty}{0} = \infty \times \infty = \infty^2,$$

$$B_2^2 = -\frac{M_1}{B_1} = \frac{\infty}{\pm B_1} = \infty,$$

$$A_2^2 : B_2^2 = \infty : 1 = 1 : 0;$$

so that the locus is an ellipse of which the conjugate axis is infinitely long and the transverse axis infinitely longer; and this ellipse has been shown, in § 229, to be a parabola.

If the sign of A_1 is supposed to be opposite to that of B_1, the case comes under § 304. (2.) a; and (146) gives

$$A_2^2 = -\frac{M_1}{A_1} = \frac{\infty}{0} = \infty^2,$$

$$B_2^2 = \frac{M_1}{B_1} = \frac{\infty}{\pm B_1} = \infty,$$

$$A_2^2 : B_2^2 = \infty : 1 = 1 : 0;$$

so that the locus is an hyperbola of which the conjugate axis is infinitely long and the transverse axis infinitely longer; and this hyperbola has been shown, in § 229, to be a parabola.

312. *Corollary.* If we free (161) from parentheses and divide by the coefficient of X^2, we have

$$X^2 - (A + C)\,X + \tfrac{1}{4}\,(4\,AC - B^2) = 0;$$

in which $\frac{1}{4}\,(4\,AC - B^2)$ is (Alg. art. 275) the product of A_1

and B_1, the roots of the equation. Hence A_1 and B_1 have the same sign, or have opposite signs, or one of them is equal to zero, according as the above product is positive, negative, or equal to zero.

313. *Corollary.* The locus of (123) is

an *ellipse,*	if $4\,AC - B^2 > 0$;
a *parabola,*	if $4\,AC - B^2 = 0$;
an *hyperbola,*	if $4\,AC - B_2 < 0$;

the special forms of these curves, as given in § 305, being included.

314. *Corollary.* By (136) and (162), the values of $\frac{x_1}{x}$ and of A_1 and B_1 depend only on those of A, B, and C. Also, for the ellipse, by (143),

$$\frac{A_2^2}{B_2^2} = \left(\pm \frac{M_1}{A_1}\right) \div \left(\pm \frac{M_1}{B_1}\right) = \frac{B_1}{A_1}; \qquad (172.)$$

and, for the hyperbola, by (146),

$$\frac{A_2^2}{B_2^2} = \left(\pm \frac{M_1}{A_1}\right) \div \left(\mp \frac{M_1}{B_1}\right) = -\frac{B_1}{A_1}. \qquad (173.)$$

Hence, the character of the locus (whether an ellipse, a parabola, or an hyperbola), its form, or amount of eccentricity, and the directions of its principal diameters, are determined wholly by the coefficients of those terms which are of the second degree; and the other constants in the equation affect only the position of the centre, and the scale on which the curve is drawn.

315. *Problem.* To construct a rectangular equation of the second degree.

Solution. Find the values of A_1, B_1, $\frac{x_1}{x}$, D_1, and E_1, by § 308; and draw the axes of x_1 and y_1. Unless $A_1 = 0$, while D_1 is not zero, or $B_1 = 0$, while E_1 is not zero, find x_1°, y_1°, and

M_1 by § 308, lay down the axes of x_2 and y_2, determine the form of the locus by § 304, and draw it by previous methods.

In the excepted case of the parabola, find x_1°, y_1°, and p by §§ 304, 309, 310, lay down the axes of x_2 and y_2, and draw the curve as in § 213, Ex. 1.

316. Examples.

1. Construct the equation of § 87, Ex. 1, by the methods of this chapter.

Solution. The comparison of the given equation with (123) gives

$$A = 1, \quad B = -6, \quad C = 1, \quad D = -6, \quad E = 2, \quad M = 5.$$

$$4\,AC - B^2 = -32 < 0;$$

so that, by § 313, the curve belongs to the hyperbolic class.

The substitution of the above values of A, B, and C in (162) gives

$$x = \tfrac{1}{2} \times 2 \pm \tfrac{1}{2}\sqrt{36} = 1 \pm 3 = 4 \text{ or } -2.$$

A_1 may be taken equal to either of these values, and B_1 to the other. We will take

$$A_1 = 4, \qquad B_1 = -2.$$

(168), (169), (170), (127), (128), (137), (138), and (171) give

$$\tan \frac{x_1}{x} = \frac{6}{-6} = -1;$$

$$\cos \frac{x_1}{x} = \frac{1}{\sqrt{(1+1)}} = \pm \sqrt{\tfrac{1}{2}};$$

$$\sin \frac{x_1}{x} = \mp \sqrt{\tfrac{1}{2}};$$

$$D_1 = \mp 6\sqrt{\tfrac{1}{2}} \mp 2\sqrt{\tfrac{1}{2}} = \mp 8\sqrt{\tfrac{1}{2}};$$

$$E_1 = \mp 6\sqrt{\tfrac{1}{2}} \pm 2\sqrt{\tfrac{1}{2}} = \mp 4\sqrt{\tfrac{1}{2}};$$

$$x_1^\circ = -\frac{\mp 8}{8}\sqrt{\tfrac{1}{2}} = \pm \sqrt{\tfrac{1}{2}};$$

Examples.

$$y_1^\circ = -\frac{\mp 4}{-4}\sqrt{\tfrac{1}{2}} = \mp\sqrt{\tfrac{1}{2}};$$

$$D_1^2 = 32; \qquad E_1^2 = 8;$$

$$M_1 = -\frac{32}{16} + \frac{8}{8} + 5 = 4.$$

The equation comes, therefore, under § 304. (2.) c, and is constructed by an hyperbola which has its foci in the axis of y_2, and of which the semi-transverse and semi-conjugate axes are respectively

$$B_2 = \sqrt{-\frac{4}{-2}} = \sqrt{2}; \quad A_2 = \sqrt{\frac{4}{4}} = 1.$$

Fig. 72 exhibits the required locus.

Scholium. The value of $\tan \frac{x_1}{x}$ gives the axis of x_1 on the line $X_1'A\,X_1$, the positive direction being taken either way (Tr. § 65). This doubt corresponds to the ambiguity in the sign of $\cos \frac{x_1}{x}$. If that sign is taken positive, AX_1 and AY_1 are (Tr. § 64) the positive directions of the axes of x and y; but if that sign is negative, AX_1' and AY_1' are (Tr. § 62) the positive directions. In the former case, x_1° is positive and y_1° negative; in the latter case, the signs of these coördinates are reversed; but, as the directions of the axes are reversed at the same time, the position of the origin is not thereby affected. Hence, the double sign belonging to the square root in (169) may be disregarded; but it will be found a good practical rule to take $\cos \frac{x_1}{x}$ always positive.

If we should take

$$A_1 = -2, \qquad B_1 = 4,$$

this would have no other effect on the above solution than to interchange the axes of x_1 and y_1 together, and those of x_2 and y_2 together, and to bring the equation under § 304. (2.) a; so that neither the form nor the position of the locus would be changed.

2. Construct the equation of § 87, Ex. 2, by the methods of this chapter.

Examples.

$$Ans.\quad A_1 = 2; \qquad B_1 = 2; \qquad \tan\frac{x_1}{x} = \frac{0}{0};$$

$$\left(\text{taking } \frac{x_1}{x} = 0\right) x_1^\circ = \frac{3}{4}; \quad y_1^\circ = -1; \quad M_1 = -\frac{33}{8}.$$

3. Construct the equation of § 87, Ex. 3, by the methods of this chapter.

$$Ans.\ A_1 = 2 - \sqrt{2};\ B_1 = 2 + \sqrt{2};\ \tan\frac{x_1}{x} = 1 + \sqrt{2};$$

$$x_1^\circ = \frac{1-\sqrt{2}}{4\sqrt{(2-\sqrt{2})}};\ y_1^\circ = -\frac{3+2\sqrt{2}}{2\sqrt{(10+7\sqrt{2})}};\ M_1 = -\frac{9}{2}.$$

4. Construct the equation of § 87, Ex. 4, by the methods of this chapter.

$$Ans.\quad A_1 = \tfrac{1}{4}; \qquad B_1 = \tfrac{1}{9}; \qquad \tan\frac{x_1}{x} = 0;$$

$$x_1^\circ = 0; \qquad y_1^\circ = 0; \qquad M_1 = -1.$$

5. Construct the equation of § 87, Ex. 5, by the methods of this chapter.

$$Ans.\quad A_1 = 0; \quad B_1 = 5; \quad \tan\frac{x_1}{x} = \tfrac{1}{2};$$

$$x_1^\circ = \sqrt{5}; \quad y_1^\circ = 0; \quad 4p = \sqrt{\tfrac{4}{5}}.$$

6. Construct the equation of § 87, Ex. 6, by the methods of this chapter.

$$Ans.\quad A_1 = -\tfrac{1}{2}(1+\sqrt{13}); \quad B_1 = -\tfrac{1}{2}(1-\sqrt{13});$$

$$\tan\frac{x_1}{x} = -\tfrac{1}{2}(3+\sqrt{13}); \quad x_1^\circ = \frac{-5+\sqrt{13}}{2\sqrt{(65+17\sqrt{13})}};$$

$$y_1^\circ = \frac{-5-\sqrt{13}}{4\sqrt{(13+2\sqrt{13})}}; \qquad M_1 = \frac{27}{4}.$$

7. Construct the equation of § 87, Ex. 7, by the methods of this chapter.

$$Ans.\quad A_1 = \tfrac{1}{2}; \quad B_1 = -\tfrac{1}{2}; \quad \tan\frac{x_1}{x} = 1;$$

$$x_1^\circ = 0; \quad y_1^\circ = 0; \quad M_1 = -16.$$

8. Construct the equation of § 87, Ex. 8, by the methods of this chapter.

$$Ans.\quad A_1 = 2; \quad B_1 = 1; \quad \tan\frac{x_1}{x} = \infty;$$

$$x_1^\circ = b; \quad y_1^\circ = -a; \quad M_1 = 0.$$

Examples.

9. Construct the equation of § 87, Ex. 9, by the methods of this chapter.

$$\textit{Ans.}\quad A_1 = \tfrac{1}{2}(3 + \sqrt{5});\quad B_1 = \tfrac{1}{2}(3 - \sqrt{5});$$

$$\tan\frac{x_1}{x} = -\tfrac{1}{2}(1 - \sqrt{5});\quad x_1^\circ = \frac{1}{\sqrt{[5(2 - \sqrt{5})]}};$$

$$y_1^\circ = \frac{1 - \sqrt{5}}{\sqrt{[10(11 + 5\sqrt{5})]}};\quad M_1 = 3.$$

10. Construct the equation of § 87, Ex. 10, by the methods of this chapter.

$$\textit{Ans.}\quad A_1 = 10;\quad B_1 = 0;\quad \tan\frac{x_1}{x} = 2;$$

$$x_1^\circ = -\tfrac{1}{4}\sqrt{\tfrac{1}{5}};\quad y_1^\circ = \frac{0}{0};\quad M_1 = -\tfrac{1}{8}.$$

11. Construct the equation of § 87, Ex. 11, by the methods of this chapter.

$$\textit{Ans.}\quad A_1 = \tfrac{1}{2}(1 - \sqrt{10});\quad B_1 = \tfrac{1}{2}(1 + \sqrt{10});$$

$$\tan\frac{x_1}{x} = 3 + \sqrt{10};\quad x_1^\circ = \frac{3}{\sqrt{[2(50 - 3\sqrt{10})]}};$$

$$y_1^\circ = \frac{9 + 3\sqrt{10}}{\sqrt{[2(170 + 53\sqrt{10})]}};\quad M_1 = 0.$$

CHAPTER X.

TANGENTS AND THEIR PROPERTIES.

317. *Definitions.* A *tangent* to a curve is a straight line which meets or crosses the curve and has the same direction with it at the common point.

The common point at which the curve and its tangent have the same direction is called the *point of contact*, or of *tangency*.

318. *Corollary.* If the position of a straight line which cuts a curve in more than one point be shifted till two or more of the points of intersection coincide, it becomes a tangent. For, at the moment that two points of intersection run together, the straight line cuts the curve in two adjacent points, and, therefore (Geom. § 11), has the same direction with the curve at that part.

Thus, if $P'P^{\text{IV}}$ (Fig. 39) is moved up till the points of intersection which correspond to P'' and P''' coincide, it becomes tangent to the curve at P_1'' or P_1'''. Also, if MM'' (Fig. 69) be turned in the plane of the curve around the point M, till the intersection-point which corresponds to M'' coincides with M, it becomes tangent to the curve of that figure at M.

319. *Corollary.* Only one tangent can be drawn to a curve at any point; for only one straight line can be drawn through a given point in a given direction. Hence the tangent which is obtained by revolving a straight line

about a point which it has in common with a curve, as in § 318, is the only tangent which can be drawn to the curve at that point; so that any tangent may be conceived as a line which cuts a curve in two or more points which coincide.

320. *Corollary.* The number of distinct points which a tangent has in common with a curve is at least one less than the greatest number of points in which a straight line can cut the curve.

Thus, $P_1'P_1^{IV}$ cuts the curve of Fig. 39 in only three points, while $P'P^{IV}$ cuts it in four points.

321. *Problem.* To find the number of points in which a straight line can cut an algebraic curve.

Solution. Suppose the curve to be of the *n*th order. Then its equation, referred to any rectilinear system whatever, is, by § 268, of the *n*th degree, and it is represented by the equation of § 272. For the points in which the curve is cut by the axis of x,

$$y = 0;$$

and the substitution of this value in the equation of the curve reduces it to the form

$$A x^n + B x^{n-1} + C x^{n-2} + \&c. = 0. \qquad (174.)$$

This equation, being of the *n*th degree, has (Alg. art. 269) n roots, each of which gives a point common to the curve and the axis of x. But the axis of x represents any straight line whatever, so that, in general, a straight line cuts an algebraic curve at as many points as there are units in the exponent of the order of the curve.

322. *Corollary.* If two or more of the roots of (174) are *equal*, the corresponding points of intersection coin-

cide in one point, at which the straight line is a *tangent* to the curve.

If any of the roots of (174) are *imaginary*, the corresponding points of intersection disappear. If all the roots are imaginary, the straight line does not cut the curve.

If $A = 0$, in (174), the equation is of the $(n - 1)$ degree and gives only $n - 1$ roots.

Hence, the rule of § 321 only gives *the greatest number* of points at which a straight line can cut a curve.

323. *Corollary.* A straight line cannot cut a curve of the second order in more than two points.

A *tangent* to a curve of the second order will have only one point in common with the curve.

324. *Problem.* To find the direction of a tangent to a curve of the second order.

Solution. Let the tangent be taken for the axis of x in a system of rectangular coördinates, and the point of contact for the origin. Let (123) represent the equation of the curve, referred to this system. Since the origin is a point of the curve, (123) must be satisfied by its coördinates, namely,

$$x = 0, \qquad y = 0;$$

which reduce (123) to

$$M = 0.$$

The substitution of this value in (123) gives

$$A x^2 + B x y + C y^2 + D x + E y = 0. \qquad (175.)$$

For the points at which the axis of x cuts the curve,

$$y = 0;$$

and (175) gives for those points

$$A x^2 + D x = x(A x + D) = 0.$$

If a product equals zero, one of its factors must be zero; so that either

$$x = 0,$$

or

$$A x + D = 0,$$

$$x = -\frac{D}{A}.$$

But, since the axis of x is a tangent, these two roots must, by § 322, be equal to each other; that is,

$$D = 0.$$

If, now, the curve is referred, as in § 299. *a*, to a system in which the axes are parallel to the principal diameters, and the origin is still at the point of tangency, the equation assumes the form (130), and (127) and (128) become

$$D_1 = E \sin \genfrac{}{}{0pt}{}{x_1}{x}; \qquad E_1 = E \cos \genfrac{}{}{0pt}{}{x_1}{x};$$

which give, by division,

$$\tan \genfrac{}{}{0pt}{}{x_1}{x} = \frac{D_1}{E_1}. \tag{176.}$$

But (132) and (133) give

$$D_1 = -2 A_1 x_1^\circ, \qquad E_1 = -2 B_1 y_1^\circ;$$

in which A_1 and B_1 denote the coefficients of x^2 and y^2 in the equation of the curve referred to its principal diameters, and x_1° and y_1° the coördinates of the centre, referred to the system of x_1 and y_1. If the principal diameters are taken for the axes of x and y, and if τ denotes the direction of the tangent, the angle which is called $\genfrac{}{}{0pt}{}{x_1}{x}$ in (176) becomes $\genfrac{}{}{0pt}{}{x}{\tau}$; so that (§ 20. *k*)

$$\tan \genfrac{}{}{0pt}{}{\tau}{x} = -\tan \genfrac{}{}{0pt}{}{x}{\tau} = -\frac{D_1}{E_1} = -\frac{2 A_1 x_1^\circ}{2 B_1 y_1^\circ} = -\frac{A_1 x_1^\circ}{B_1 y_1^\circ}.$$

If x° and y° denote the coördinates of the point of contact, referred to the principal diameters,

$$x^\circ = -x_1^\circ, \qquad y^\circ = -y_1^\circ;$$

and, substituting these values in the above expression for $\tan\frac{\tau}{x}$, we have

$$\tan\frac{\tau}{x} = -\frac{A_1 x^\circ}{B_1 y^\circ}; \qquad (177.)$$

and (177) expresses the condition that a line is tangent to a curve of the second order.

325. *Corollary. For the ellipse* of which the semi-axes are A and B, (172) gives

$$-\frac{A_1}{B_1} = -\frac{B^2}{A^2};$$

and (177) becomes

$$\tan\frac{\tau}{x} = -\frac{B^2 x^\circ}{A^2 y^\circ}. \qquad (178.)$$

326. *Corollary. For the hyperbola* of which the semi-axes are A and B, (173) gives

$$-\frac{A_1}{B_1} = \frac{B^2}{A^2};$$

and (177) becomes

$$\tan\frac{\tau}{x} = \frac{B^2 x^\circ}{A^2 y^\circ}. \qquad (179.)$$

327. *Corollary.* If, *for the parabola,* its axis is taken for the axis of x, and its vertex for the origin, (133) gives, as above,

$$E_1 = -2\,B_1\,y_1^\circ = 2\,B_1\,y^\circ;$$

and (176) gives, by (150),

$$\tan\frac{\tau}{x} = -\frac{D_1}{E_1} = -\frac{D_1}{2\,B_1\,y^\circ} = \frac{2\,p}{y^\circ}. \qquad (180.)$$

328. *Corollary.* The condition of (180) is the same as that of (89); so that, if y° is taken of the same value in both cases, τ and β_1 must denote the same direction; that is, the conjugate to any diameter of a parabola is a tangent to the curve at the vertex of the diameter.

329. *Problem.* To find the equation of a tangent to a curve of the second order.

Solution. If x° and y° denote, as in § 324, the coördinates of the point of contact, (117) gives for the equation of the tangent

$$y - y^\circ = \tan \frac{\tau}{x} (x - x^\circ). \qquad (181.)$$

Substituting (177) in this equation, we have, by reduction,

$$y - y^\circ = -\frac{A_1 x^\circ}{B_1 y^\circ} (x - x^\circ),$$

$$A_1 x^\circ x + B_1 y^\circ y - A_1 x^{\circ 2} - B_1 y^{\circ 2} = 0.$$

But, since the point of contact is a point of the curve, its coördinates satisfy the equation of the curve; so that, by (136),

$$-A_1 x^{\circ 2} - B_1 y^{\circ 2} = M_1;$$

which, substituted in the above equation, gives for the equation of the tangent, referred to the principal diameters of the curve,

$$A_1 x^\circ x + B_1 y^\circ y + M_1 = 0. \qquad (182.)$$

330. *Corollary.* If (182) be divided by $\pm M_1$, it gives for the *ellipse*, by § 299. (1.) *a*, if A and B, instead of A_2 and B_2, denote the semi-axes,

$$\frac{x^\circ x}{A^2} + \frac{y^\circ y}{B^2} = 1; \qquad (183.)$$

for the *hyperbola which has its transverse axis on the axis of x*,

$$\frac{x^\circ x}{A^2} - \frac{y^\circ y}{B^2} = 1; \qquad (184.)$$

for the *hyperbola which has its transverse axis on the axis of y,*

$$\frac{y^\circ y}{B^2} - \frac{x^\circ x}{A^2} = 1. \qquad (185.)$$

331. *Corollary.* For the *parabola,* (180), substituted in (181), gives

$$y - y^\circ = \frac{2p}{y^\circ}(x - x^\circ),$$

$$y^\circ y = 2px + y^{\circ 2} - 2px^\circ.$$

Since x°, y° is a point of the curve,

$$y^{\circ 2} = 4px^\circ;$$

and the equation of the tangent, referred to the system of § 200, is

$$y^\circ y = 2p(x + x^\circ). \qquad (186.)$$

332. *Definitions.* When a tangent is spoken of as having a *definite length,* the part which is included between the point of contact and the axis of x, as MT (Fig. 69), is meant.

The *subtangent* is the projection of the tangent on the axis of x, as PT (Fig. 69).

If x° denotes the abscissa of the point of contact, and x' that of the point at which the tangent cuts the axis of x, the length of the subtangent is, by § 73,

$$x' - x^\circ.$$

333. *Problem.* To find the length of a subtangent of the ellipse or the hyperbola.

Solution. For the point at which the tangent cuts the axis of x, $y = 0$, and the substitution of this value in either (183) or (184) gives for x', the abscissa of the point at which the tangent cuts the axis of x,

$$x' = \frac{A^2}{x^\circ}. \qquad (187.)$$

The substitution of the same value in (185) gives

$$x' = -\frac{A^2}{x^\circ}. \qquad (188.)$$

Hence, the expression for the subtangent of the ellipse or of the x-hyperbola is

$$x' - x^\circ = \frac{A^2}{x^\circ} - x^\circ = \frac{A^2 - x^{\circ 2}}{x^\circ}; \qquad (189.)$$

and that for the subtangent of the y-hyperbola is

$$x' - x^\circ = -\frac{A^2}{x^\circ} - x^\circ = -\frac{A^2 + x^\circ}{x^\circ}. \qquad (190.)$$

334. *Corollary.* Since (189) and (190) involve only A and x°, the length of the subtangent is independent of the length of the axis which is laid off on the axis of y and of the ordinate of the point of contact. Thus, the tangents which are drawn to the ellipses of Fig. 79, which have the common axis $2A$, from points, M', M_1', M_2', in the same straight line parallel to the axis of y, cut the axis of x at the same point P''; and the tangents drawn to the hyperbolas of that figure from M'', M_1'', in the same straight line parallel to the axis of y, likewise cut the axis of x at the common point P'.

335. *Corollary.* In Fig. 79, let x° be the abscissa of M'', M_1'', &c.; let $x' = AP'$ be the abscissa of M', M_1', &c., and also of the points at which the tangents to the hyperbolas drawn from M'', &c. cut the axis of x; and let $x'' = AP''$ be the abscissa of the points at which the axis of x is cut by the tangents to the ellipses drawn from M', &c. (187) gives

$$x' = \frac{A^2}{x^\circ},$$

$$x'' = \frac{A^2}{x'} = A^2 \div \frac{A^2}{x^\circ} = x^\circ;$$

so that AP'' is the abscissa of M'', &c.

Hence, if from the point of contact of an hyperbola with its tangent a perpendicular be dropped on the transverse axis produced, and if from the point at which the tangent cuts the transverse axis a perpendicular to the transverse axis be erected, the latter perpendicular will meet any ellipse described on the transverse axis of the hyperbola at its point of contact with the tangent drawn through the foot of the former perpendicular.

336. *Problem.* To draw a tangent to an ellipse at a given point.

Solution. On either axis, $C'C$ (Fig. 79), of the ellipse, as a diameter, describe a circle. Through the given point, M' or M_2', drop a perpendicular to the diameter, cutting the circumference at M_1'. By Geom. § 150. *a*, draw through M_1' a tangent to the circumference, cutting $C'C$ produced at P''. $P''M'$ or $P''M_2'$ is, by § 334, the required tangent.

337. *Problem.* To draw a tangent to an hyperbola at a given point.

Solution. On the transverse axis, $C'C$ (Fig. 79), of the hyperbola, as a diameter, describe a circle. From the given point, M'', drop a perpendicular on $C'C$ produced. From P'', the foot of the perpendicular, draw a tangent to the circle, by Geom. § 150. *b*, touching it at M_1'. From M_1' drop on $C'C$ a perpendicular, meeting it at P'. $P'M''$ is, by § 335, the required tangent.

338. *Problem.* To find the length of a subtangent of the parabola.

Solution. The substitution of $y = 0$ in (186) gives, if x' denotes the abscissa of the point at which the tangent cuts the axis of x,

$$2\,p\,x' = -\,2\,p\,x^{\circ},$$

$$x' = - x^\circ, \tag{191.}$$

$$x' - x^\circ = - 2\, x^\circ. \tag{192.}$$

339. *Problem.* To draw a tangent to a parabola at a given point.

Solution. From the given point, M (Fig. 71), drop a perpendicular, MP, on the axis. Take

$$CT = PC = - x^\circ,$$

and TM is, by § 338, the required tangent.

340. *Definition.* If two straight lines are drawn through the vertices of the transverse axis of an hyperbola, or of either axis of an ellipse, to a point of the curve, they are called *supplementary chords.*

Thus CM and $C'M$ are supplementary chords of the hyperbola of Fig. 70.

341. *Problem.* To express the condition that two chords of an ellipse or an hyperbola are supplementary.

Solution. Let the diameter from which the chords are drawn be taken for the axis of x and denoted by $2\,A$; let x' and y' be coördinates of that point of the curve at which the chords meet; and let α' and α'' denote the directions of the chords drawn through the left-hand and right-hand vertices respectively. The coördinates of the right-hand vertex are

$$x = A, \qquad y = 0;$$

so that (118) gives

$$\tan {\alpha'' \atop x} = \frac{y'}{x' - A}. \tag{193.}$$

The coördinates of the left-hand vertex are

$$x = - A, \qquad y = 0;$$

so that (118) gives

$$\tan {\alpha' \atop x} = \frac{y'}{x' + A}. \tag{194.}$$

The product of (193) and (194) is

$$\tan{}^{\alpha'}_{x} \tan{}^{\alpha''}_{x} = \frac{y'^2}{x'^2 - A^2}. \qquad (195.)$$

If the curve is an ellipse, x' and y' satisfy (48), which gives

$$y'^2 = -\frac{B^2}{A^2}(x'^2 - A^2);$$

which, substituted in (195), reduces it to

$$\tan{}^{\alpha'}_{x} \tan{}^{\alpha''}_{x} = -\frac{B^2}{A^2}. \qquad (196.)$$

If the curve is an hyperbola, x' and y' satisfy (66), which gives

$$y'^2 = \frac{B^2}{A^2}(x'^2 - A^2),$$

$$\tan{}^{\alpha'}_{x} \tan{}^{\alpha''}_{x} = \frac{B^2}{A^2}. \qquad (197.)$$

(196) and (197) are the required equations of condition.

342. *Corollary.* The conditions expressed in (196) and (197) are identical with those expressed in (52) and (75); so that, if α' in (196) or in (197) denotes the same direction as α_1 or β_1 in (52) or in (75), α'' in (196) or in (197) denotes the same direction as β_1 or α_1 in (52) or in (75).

Hence, if one of two supplementary chords is parallel to a diameter, the other is parallel to the conjugate of that diameter.

343. *Problem.* To draw a diameter of an ellipse or an hyperbola conjugate to a given diameter.

Solution. Let $C_1'C_1$ (Fig. 70) be the given diameter. Draw through either vertex of the transverse axis the chord $C'M$ parallel to $C_1'C_1$, also the supplementary chord CM. $B_1'B_1$, drawn through the centre parallel to CM, is, by § 342, the required diameter.

344. *Scholium.* This method gives only the direction of the diameter; so that, in Fig. 70, the length of $B_1'B_1$ is undetermined.

345. *Problem.* To express the condition that a diameter is drawn to a point of tangency.

Solution. Let α_1 denote the direction of the diameter, and let x° and y° be used as before. Since the coördinates of the centre are

$$x = 0, \qquad y = 0,$$

(118) gives

$$\tan {}^{\alpha_1}_{x} = \frac{y^\circ}{x^\circ}. \tag{198.}$$

346. *Corollary.* *For the ellipse,* (178) and (198) give

$$\tan {}^{\alpha_1}_{x} \tan {}^{\tau}_{x} = -\frac{B^2}{A^2}. \tag{199.}$$

For the hyperbola, (179) and (198) give

$$\tan {}^{\alpha_1}_{x} \tan {}^{\tau}_{x} = \frac{B^2}{A^2}. \tag{200.}$$

347. *Corollary.* The conditions expressed in (199) and (200) are identical with those expressed in (52) and (75) and also in (196) and (197).

Hence, a tangent drawn at the extremity of a diameter is parallel to the conjugate of that diameter; also, if the tangent is parallel to one of two supplementary chords, the diameter is parallel to the other.

348. *Problem.* To draw a tangent to an ellipse or an hyperbola, parallel to a given line.

Solution. Let the given line be KL (Fig. 70). Draw two supplementary chords, CM and $C'M$, one of which, CM, is parallel to the given line. Draw the diameter $C_1'C_1$ parallel to $C'M$. Either $C_1'T'$ or TC_1, drawn parallel to KL, is, by § 347, the required tangent.

349. *Scholium.* Observe that the solutions of §§ 343, 348 apply to the ellipse as well as to the hyperbola.

350. *Corollary.* § 347 gives also a simple method of drawing a tangent to an ellipse or an hyperbola through a given point.

351. *Theorem.* The ellipse makes equal angles at every point with the lines drawn from that point to the foci.

Proof. It is to be proved (Fig. 73) that the angle $C'MF$ and $F'Mm$ are equal.

Let m be a point of the ellipse infinitely near M. Draw Fm and $F'm$. From F' as a centre, with a radius $F'm$, describe an arc, cutting $F'M$ at a; and from F as a centre, with a radius Fm, describe an arc, cutting FM produced at b. Mm, am, and bm, being infinitely small, may be regarded as straight lines; and Mmb and Mma are triangles, right-angled, by Geom. § 120, at b and a.

$$FM + MF' = FM + aF' + Ma;$$
$$Fm + mF' = Fb + aF' = FM + aF' + Mb.$$

By the definition of the ellipse,

$$FM + MF' = Fm + mF';$$
$$Ma = Mb;$$

so that the right-triangles having Mm common are (Geom. § 64) equal, and the angles mMb and $F'Mm$ are equal. But $C'MF$ is vertical to mMb, and therefore also equal to $F'Mm$.

352. *Theorem.* The hyperbola makes equal angles at

every point with the lines drawn from that point to the foci.

Proof. The construction in Fig. 74 is similar to that of the last section.

$$\begin{aligned} FM - MF' &= F\,b + b\,M - M\,a - a\,F' \\ &= F\,m - m\,F' = F\,b - a\,F'; \\ b\,M - M\,a &= 0; \\ b\,M &= M\,a; \end{aligned}$$

so that triangles $M\,m\,a$ and $M\,m\,b$ are equal, and therefore angles

$$F'M\,m = m\,MF = G'M\,m' = m'\,MG.$$

353. *Theorem.* The parabola makes equal angles at every point with the axis and with the line drawn from that point to the focus.

Proof. QM and $q\,m$ (Fig. 75) are parallel to the axis, and $m\,b$ to the directrix. The rest of the construction is similar to that of § 351.

$$\begin{aligned} FM = F\,a + a\,M &= QM = Q\,b + b\,M; \\ F\,m = F\,a &= q\,m = Q\,b; \\ a\,M &= b\,M; \end{aligned}$$

so that triangles $Mm\,a$ and $Mm\,b$ are equal, and angles

$$m\,MF = Q\,M\,m = X_1M\,m'.$$

354. *Corollary.* By § 317, a tangent drawn at M will make, in Figs. 73, 74, equal angles with MF and MF', and, in Fig. 75, equal angles with MF and MX_1, or CX.

355. *Corollary.* § 354 gives a simple method of drawing a tangent to a parabola at a given point, M (Fig. 71). Take $FT = FM$. TM is the required tangent; for, since TFM is isosceles,

$$FTM = TMF.$$

356. *Scholium.* The above theorems are illustrated by the properties of reflectors. The law of reflection is, that *the angle of reflection is equal to the angle of incidence;* the angle of reflection being the angle which the reflected ray makes with a perpendicular to the reflector at the point at which it is reflected, and the angle of incidence being the angle which the same perpendicular makes with the ray before it is reflected.

If the ellipse of Fig. 73 be supposed to revolve about its transverse axis, it will generate a surface which is called the surface of an *ellipsoid.* Suppose that this surface is made a reflector, and that FM is a ray of light which falls on it from the focus F; then $C'MF$ is the complement of the angle of incidence, and, if Ma is the direction of the reflected ray, aMm, the complement of the angle of reflection, must be equal to $C'MF$; so that Ma must point towards the focus F'.

If the hyperbola revolve about its transverse axis, it will generate the surface of an *hyperboloid;* and if this surface be made a reflector, rays FM and $F'M$, which fall on it from the foci, will be reflected respectively in the directions $F'G'$ and FG.

If the parabola of Fig. 75 revolve about its axis, it will generate the surface of a *paraboloid;* and, if this surface be made a reflector, a ray, FM, which falls on it from the focus, will be reflected parallel to the axis.

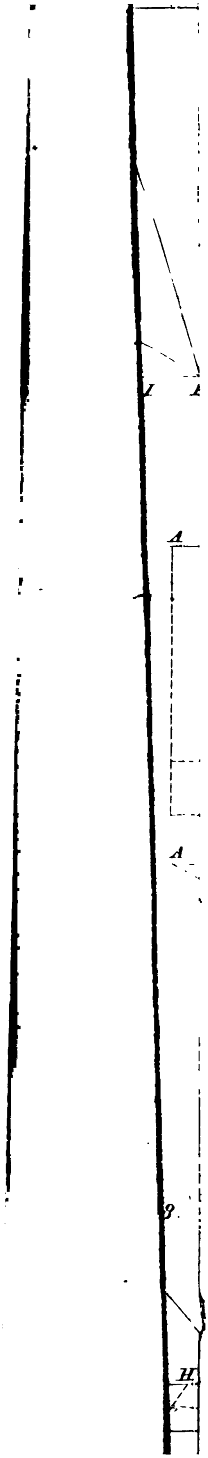
If, therefore, a lamp be placed in either focus of an *elliptic* reflector, the light is reflected to the other focus, as in Fig. 76; if in either focus of an *hyperbolic* reflector, it will be reflected from the other focus, as in Fig. 77; if in the focus of a *parabolic* reflector, it will be reflected parallel to the axis, as in Fig. 78.*

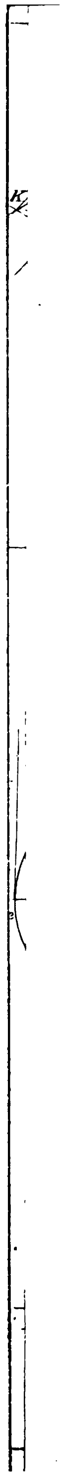
357. *Corollary.* The properties of reflectors illustrate in a new way the relation of the conic sections. In the case of the

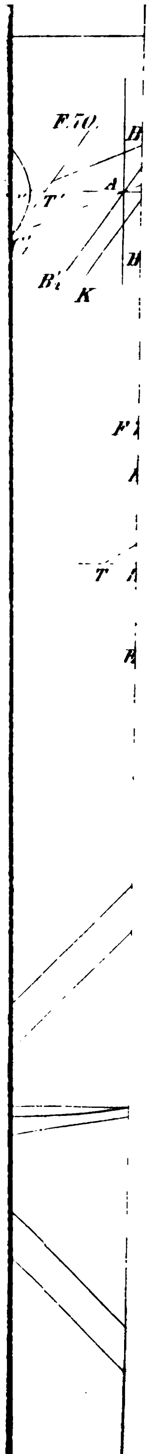
* The law of reflection is a general one, though it has been explained in the language of optics. Thus, F, in Figs. 76, 77, 78, may be a centre of sound or of undulations in water.

elliptic reflector, the rays *converge* after reflection ; in the case of the *hyperbolic* reflector, they *diverge ;* while the *parabolic* reflector gives the intermediate case, in which they neither converge nor diverge, but are *parallel*, and may be conceived to be directed towards an infinitely distant focus on the right, if the parabola is conceived to be an ellipse, or from an infinitely distant focus on the left, if the parabola is conceived to be an hyperbola.

THE END.






F.70.
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