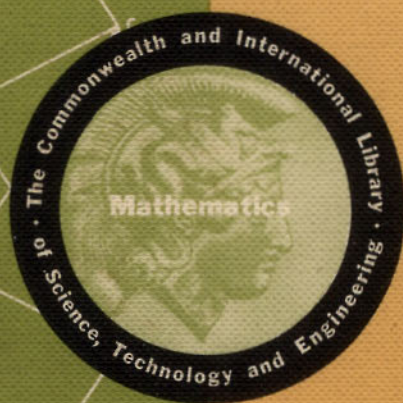


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DEDUCTIVE GEOMETRY

Deductive Geometry

E. A. Maxwell

FELLOW OF QUEENS' COLLEGE
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ROBERT MAXWELL

Publisher at Pergamon Press

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Preface

THE AIM of this book is to give, in concise form, the whole of the geometry of the straight line, circle, plane and sphere, with their associated configurations such as triangle or cylinder, in so far as it is likely to be required for courses in mathematics in the United Kingdom for the G.C.E. at Advanced and Scholarship levels, or for corresponding courses throughout the Commonwealth as required by the appropriate examining boards. The book will be of value also to university undergraduates.

This is a subject which, at the moment of writing, is less popular than it deserves, but I hope that the treatment may help to stimulate interest as well as to satisfy an existing need.

The plan of the book is straightforward; recapitulation of known work, advanced plane geometry, solid geometry with some reference to the geometry of the sphere, a chapter on the nature of space with reference to such properties as congruence, similarity and symmetry, and, finally, a very brief account of the elementary transformations of projection and inversion.

The book is interspersed with a number of examples. Bearing in mind the need for brevity, a number of these are actually standard results which the reader is invited to prove for himself. Such examples are headed *Theorems*.

I would express my thanks to Dr. H. M. Cundy for many valuable suggestions, and also to the staff of Pergamon Press for all their trouble and skill.

E. A. M.

ONE

Introduction and Notation

THE READER who comes to this book is expected to be familiar with the normal concepts of elementary geometry as commonly taught at school: length and angle; similarity and congruence; point, line and circle; area and the theorem of Pythagoras. Such knowledge will, presumably, rest on an empirical basis, leading to an appreciation of the standard theorems and of the general structure of geometrical argument, but without that detailed investigation which was prevalent until the start of this century or even later.

It has recently been realized that the present lack of training in geometrical argument must, for the young student of mathematics, be corrected in some way unless his ability to handle formal mathematical work is to be endangered. A strong candidate for the purpose is formal algebra, which is to be welcomed wholeheartedly. This book seeks to achieve a similar end, but using as alternative subject-matter some topics in geometry which are usually studied in the upper school.

The book will, however, have a somewhat strange look, even to those who are completely familiar with the material, for it is presented in the notation that is in regular use in more modern work in mathematics. *It is emphasized that the number of new symbols is small and that they are introduced not only to serve the*

purposes of this book but also to help pupils to become familiar with their use elsewhere. Experience seems to indicate that, at about the upper school stage, pupils develop a positive enthusiasm for new symbolism, especially when it helps to reduce the burden of writing, and it is hoped that they will readily respond to this approach.

One other procedural innovation should be mentioned. By long tradition, geometrical arguments have been set out under the formal headings, *Given, Required, Construction, Proof*. The discipline has much to commend it, but it is harder to sustain as work progresses; on the other hand, a recognizable structure is helpful both to writer and to reader. In so far as it is possible, therefore, the treatment of each property will begin with a statement: *The Problem*, and this will be followed by a proof under the heading: *The Discussion*.

1. Standard Notation

The following standard notation of elementary geometry will be used regularly:

$\angle ABC$ or $\angle B$	the angle ABC
$\triangle ABC$	the triangle ABC
$\triangle ABC \equiv \triangle PQR$	the triangles are congruent, with the implication $BC = QR$, $CA = RP$, $AB = PQ$
$\triangle ABC \sim \triangle PQR$	the triangles are similar, with the implication $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$
$AB = PQ$	the lengths AB, PQ are equal
$AB \perp PQ$	AB is perpendicular to PQ †

† When the context makes it clear, we assume without explicit statement that " $AB \perp PQ$ " means that B is the foot of the perpendicular from A to PQ .

$AB \parallel PQ$ AB is parallel to PQ $\odot ABC$ the circle ABC

2. Fresh Notation

The notation explained in this paragraph, though now in common usage, will almost certainly be new to pupils in upper schools.

(i) THE SYMBOL OF INCLUSION \in . The symbol \in is used in the sense that " $P \in l$ " means, " P is included among those elements which constitute the set of elements l ".

In a geometrical context, the statement might mean, " P is a point of the line l ".

In practice, a line is often named in terms of two of its points A and B . We then write " $P \in AB$ ".

(ii) THE SYMBOL OF UNION \cup . The symbol \cup is used in the sense that " $AB \cup CD$ " means, "all the points which belong to AB , to CD , or to both". It thus *unites* into the single entity $AB \cup CD$ those points which belong to AB or CD severally.

(This symbol will, in fact, not be used in this book, but it is introduced here because the symbol " \cup " for "union" is natural whereas the next symbol, used more often, is less self-explanatory. Care must be taken not to confuse the two, and the mnemonic " \cup for union" is helpful for this.)

(iii) THE SYMBOL OF INTERSECTION \cap . The symbol \cap is used in the sense that " $AB \cap CD$ " means, "all the points which belong both to AB and to CD ".

For example, it is an immediate consequence of the definitions that

$$AB \cap CD \in AB$$

That is, the intersection of AB and CD belongs to AB .

(The symbols " $AB \cup CD$ " and " $AB \cap CD$ " are sometimes, for obvious reasons, read as " AB cup CD " and " AB cap CD ".)

(iv) THE SYMBOL OF CONSEQUENCE \Rightarrow . The symbol \Rightarrow is used in the sense that a chain of argument like

$$\begin{aligned} & "2x + 5 = 7x + 14 \\ & \Rightarrow 5x = -9 \\ & \Rightarrow x = -\frac{9}{5}" \end{aligned}$$

means,

$$\begin{aligned} & \text{"the equation } 2x + 5 = 7x + 4 \\ & \text{leads to } \quad 5x = -9 \\ & \text{and this leads to } \quad x = -\frac{9}{5}" \end{aligned}$$

The important thing about this symbol is the way the arrow points. The first statement *leads to* the second. Care should be taken not to reverse the argument without ensuring that such reversal is legitimate. For example:

$$\begin{aligned} & ABCD \text{ is a rectangle} \\ & \Rightarrow AB = DC \text{ and } AD = BC; \end{aligned}$$

but it is *not true* that

$$\begin{aligned} & AB = DC \text{ and } AD = BC \\ & \Rightarrow ABCD \text{ is a rectangle.} \end{aligned}$$

On the other hand, both of the statements

$$AB = AC \Rightarrow \angle ACB = \angle ABC$$

and

$$\angle ACB = \angle ABC \Rightarrow AB = AC$$

are true. The notation

$$AB = AC \Leftrightarrow \angle ACB = \angle ABC$$

is often used to denote this fact.

It may be useful to digress for a moment to emphasize one or two points which are probably familiar. Consider, for example, the theorem:

"In $\triangle ABC$, $\triangle PQR$,

$$\begin{aligned} &BC = QR, CA = RP, AB = PQ \\ \Rightarrow &\angle BAC = \angle QPR, \angle CBA = \angle RQP, \angle ACB = \angle PRQ." \end{aligned}$$

The facts *sides equal* and *angles equal* do indeed "go together" in a sense, but the argument

$$\text{sides equal} \Rightarrow \text{angles equal}$$

cannot be reversed: it is NOT true that

$$\text{angles equal} \Rightarrow \text{sides equal.}$$

In other words, it is *not legitimate* to interchange the rôles of data and conclusion in an argument without careful examination.

Definition. A result obtained from a given theorem by interchanging rôles of data and conclusion is called a *converse* of that theorem.

The point we have been making is that *a converse of a true theorem is not necessarily true.*

The symbol \Leftrightarrow to which we have just referred may be used only when theorem and converse are both true. For example:

$$\begin{aligned} &ABCD \text{ is a cyclic quadrilateral} \\ \Leftrightarrow &\text{the sum of opposite angles is } 180^\circ. \end{aligned}$$

Analogous ideas in common use depend on the use of the word "if". The reader is strongly urged to be careful to use this very simple word completely unambiguously. A convenient way to be quite sure is to use the verbal formula "if . . . then . . .". For example:

If $ABCD$ is a cyclic quadrilateral, *then* the sum of opposite angles is 180° .

In this example, the converse is also true:

If the sum of the opposite angles of the quadrilateral $ABCD$ is 180° , *then* the quadrilateral is cyclic.

Note how the formula “if . . . then . . .” follows the sense of the arrow \Rightarrow .

When both theorem and converse are true, the phrase “if and only if” is often used:

The quadrilateral $ABCD$ is cyclic *if and only if* the sum of the opposite angles is 180° .

Thus the formula “if and only if” follows the two senses of the double arrow \Leftrightarrow . The statement so enunciated implies two distinct problems, which often need separate solution.

In many of the arguments which follow, there are steps where the double symbol \Leftrightarrow might be applied legitimately but where only the sense \Rightarrow is relevant. In such cases, the single symbol \Rightarrow is usually adopted.

When two or more conditions must be taken together to lead to a result, they will often be linked by a bracket. Thus:

$$\left. \begin{array}{l} AP = PQ, AC = PR \\ \angle BAC = \angle QPR \end{array} \right\} \Rightarrow \triangle ABC \equiv \triangle PQR \Rightarrow BC = QR.$$

(v) THE SYMBOL OF EXISTENCE \exists . The symbol \exists is used in the sense that

$$“\exists O \text{ such that } OA = OB = OC”$$

means, “there exists a point O —the circumcentre of the triangle ABC —such that $OA = OB = OC$ ”.

Remark. The notation just introduced is, at this stage, merely a notation, with which the reader is expected to become familiar quickly. It use carries no implications of set theory or symbolic logic, though knowledge of it may help when these subjects come forward for study later.

3. Results Assumed Known

The following summary indicates the results on which arguments will be founded and also gives an introduction to some of the notation just explained. For brevity, plentiful use is made of diagrams.

(i) ANGLE PROPERTIES AND PARALLEL LINES.

$$AUB \text{ straight line} \Leftrightarrow \angle AUP + \angle PUB = 180^\circ.$$

$$AUB \text{ straight line} \Leftrightarrow \angle AUP = \angle BUV.$$

$$CD \parallel AB \Leftrightarrow \angle AUP = \angle CVP,$$

$$CD \parallel AB \Leftrightarrow \angle AUV = \angle UVD,$$

$$CD \parallel AB \Leftrightarrow \angle AUV + \angle UVC = 180^\circ.$$

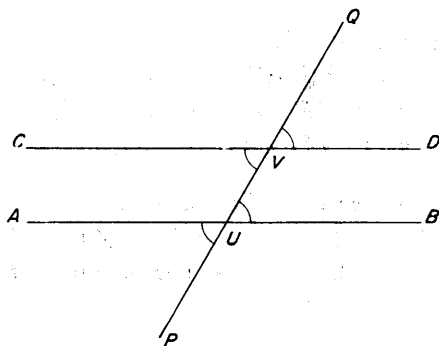


Fig. 1

(ii) ANGLE PROPERTIES
FOR A TRIANGLE.

$$\angle A + \angle B + \angle C = 180^\circ,$$

$$\angle ACD = \angle A + \angle B.$$

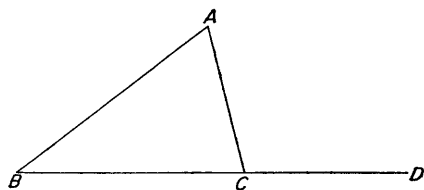


Fig. 2

Extension. The sum of the angles of a polygon of n sides is $2n - 4$ right angles; in particular, the sum of the angles of a quadrilateral is 360° .

(iii) CONGRUENCE OF TRIANGLES.

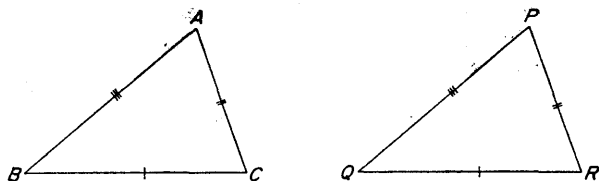


Fig. 3

$$\triangle ABC \equiv \triangle PQR$$

$$\Leftrightarrow AB = PQ, AC = PR, \angle A = \angle P,$$

$$\Leftrightarrow BC = QR, CA = RP, AB = PQ,$$

$$\Leftrightarrow BC = QR, \angle B = \angle Q, \angle C = \angle R.$$

Note the double sense \Leftrightarrow of the arrows.

Note, too, that $\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R \not\Rightarrow \triangle ABC \equiv \triangle PQR$; three equal angles are *not enough* for congruence.

(iv) SIMILARITY OF TRIANGLES.

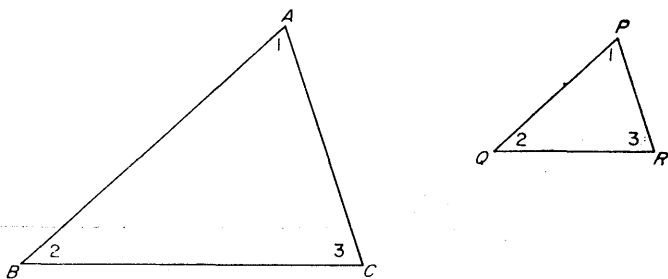


Fig. 4

$$\triangle ABC \sim \triangle PQR$$

$$\Leftrightarrow \angle A = \angle P, \angle B = \angle Q, \angle C = \angle R,$$

$$\Leftrightarrow \frac{BC}{QR} = \frac{CA}{RP} = \frac{AB}{PQ},$$

$$\Leftrightarrow \angle A = \angle P, \quad \frac{BA}{QP} = \frac{CA}{RP}.$$

Note. As a particular case,

$$UV \parallel BC$$

$$\Leftrightarrow \frac{AU}{AB} = \frac{AV}{AC}$$

$$\Leftrightarrow \frac{AU}{UB} = \frac{AV}{VC}$$

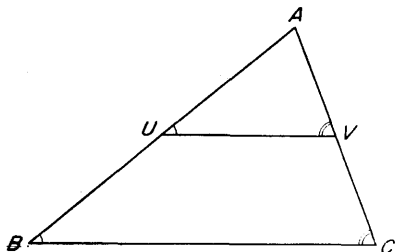


Fig. 5

Still more particularly,

$$\left. \begin{array}{l} U \text{ middle point of } AB \\ V \text{ middle point of } BC \end{array} \right\} \Rightarrow UV \parallel BC \text{ and } UV = \frac{1}{2} BC.$$

(v) AREA.

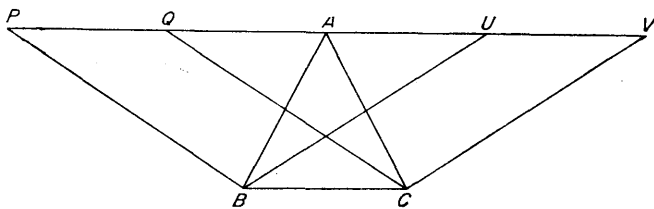


Fig. 6

$$PQAUV \parallel BC$$

$$\Leftrightarrow \text{area } BCQP = \text{area } BCVU \\ = 2 \triangle ABC.$$

(vi) THE THEOREM OF PYTHAGORAS.

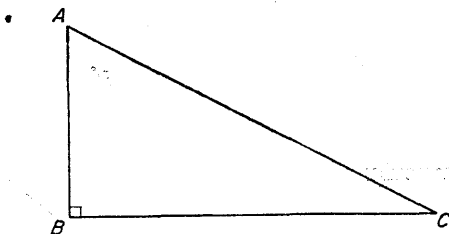


Fig. 7

$$\begin{aligned} \angle ABC &= 90^\circ \\ \Leftrightarrow AC^2 &= AB^2 + BC^2. \end{aligned}$$

(vii) THE TRIANGLE INEQUALITY.

The sum of two sides of a triangle is greater than the third side.
In symbols,

$$AB + AC > BC, \quad BC + BA > CA, \quad CA + CB > AB.$$

Note. $AB + AC = BC \Rightarrow A, B, C$ are collinear.

(viii) SYMMETRY PROPERTY OF A CIRCLE.

$$\begin{aligned} AU &= UB \\ \Rightarrow OU &\perp AB. \end{aligned}$$

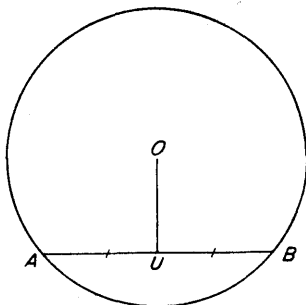


Fig. 8

(ix) ANGLE PROPERTIES OF
A CIRCLE.

$$\angle AOB = 2\angle AUB.$$

$ABVU$ cyclic

$$\Leftrightarrow \angle AUB = \angle AVB.$$

$AWBU$ cyclic

$$\Leftrightarrow \angle AUB + \angle AWB = 180^\circ.$$

AC diameter

$$\Leftrightarrow \angle ABC = 90^\circ.$$

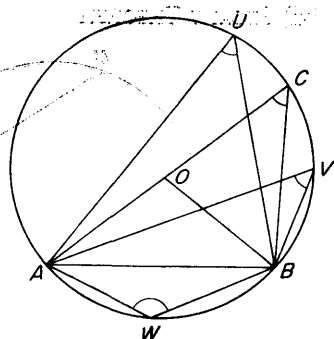


Fig. 9

Note. If $ABCD$ is a quadrilateral in which $\angle B = \angle D = 90^\circ$, then $\angle A + \angle C = 180^\circ$ and the quadrilateral is cyclic with AC as a diameter. This result is often important.

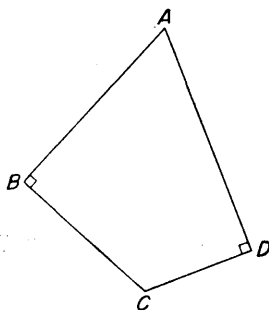


Fig. 10

(x) TANGENCY.

AT is tangent, AO

is radius

$$\Leftrightarrow OA \perp AT$$

$$\Leftrightarrow \angle TAP = \angle AUP.$$

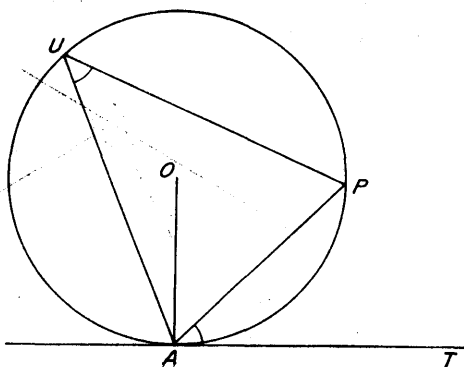


Fig. 11

(xi) SECANT THEOREM.

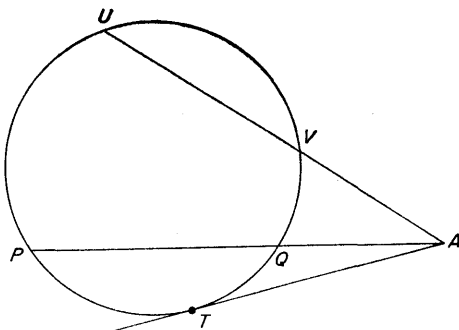


Fig. 12

P, Q, U, V concyclic

$$\Leftrightarrow AP \cdot AQ = AU \cdot AV.$$

AT is tangent at T to $\odot TPQ$

$$\Leftrightarrow AT^2 = AP \cdot AQ.$$

(xii) ANGLE BISECTOR THEOREMS.

$$\frac{BP}{PC} = \frac{BA}{AC} = \frac{BQ}{QC}$$

$\Leftrightarrow AP, AQ$ are the internal and external bisectors of $\angle A$.

Note that $PA \perp AQ$.

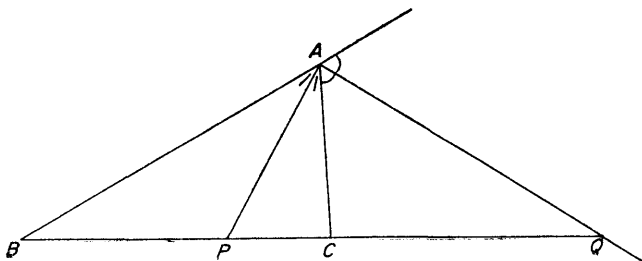


Fig. 13

(xiii) SOME LOCI.

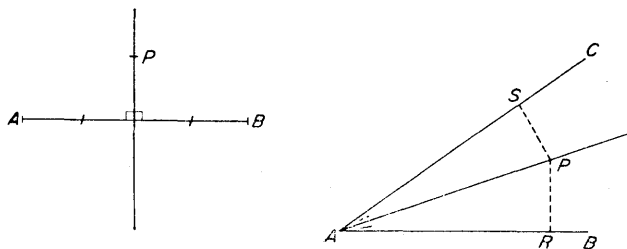


Fig. 14

(a) The locus of a point P equidistant from A , B is the perpendicular bisector of AB .

(b) The locus of a point P equidistant from two lines AB , AC is a bisector of $\angle BAC$.

TWO

The Geometry of the Triangle

1. The Centroid

THE PROBLEM. ABC is a given triangle and A' , B' , C' are the middle points of the sides BC , CA , AB .† It is required to prove that AA' , BB' , CC' meet at a point G where each is trisected.

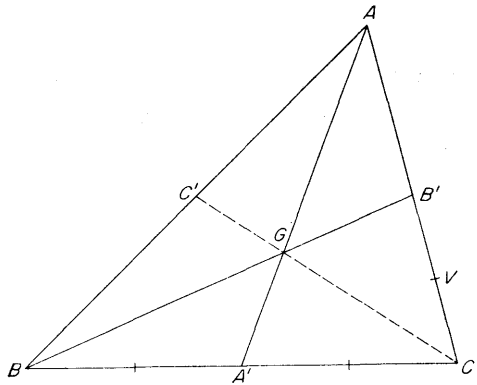


Fig. 15

Definitions. The lines AA' , BB' , CC' are called the *medians* of the triangle ABC .

The point G is called the *centroid* of the triangle ABC .

THE DISCUSSION. Let $G = AA' \cap BB'$, and let V be the middle point of CB' . Then

† In such contexts we omit the word “respectively”, which remains understood.

$$\left. \begin{array}{l} CA' = A'B \\ CV = VB' \end{array} \right\} \Rightarrow A'V \parallel BB' \Rightarrow A'V \parallel GB'.$$

Also

$$\begin{aligned} GB' \parallel A'V &\Rightarrow \frac{AG}{GA'} = \frac{AB'}{B'V} \\ &= \frac{2}{1} \quad (AB' = B'C = 2B'V). \end{aligned}$$

Thus $BB' \cap AA'$ is a point of trisection of AA' .

Similarly $CC' \cap AA'$ is the same point of trisection of AA' .

Hence AA' , BB' , CC' have a common point G which is a point of trisection of AA' and, by similar argument, of BB' and CC' .

Theorems

[Examples with which the reader should attempt to become familiar will be called *Theorems*.]

1. G is the centroid of $\triangle A'B'C'$.

2. If $P \in AB$, $Q \in AC$ such that $PQ \parallel BC$, and if $R = BQ \cap CP$ then $R \in AA'$.

Problems

1. $A'B'AC'$ is a parallelogram.

2. If AA' is produced to U so that $GU = AG$, then $UBGC$ is a parallelogram.

3. $B'C' \cap AG$ is a point of quadrisection of AG .

4. If $ABCD$ is a parallelogram, the centroids of $\triangle BAD$ and of $\triangle BCD$ are the points of trisection of AC .

Also the centroids of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, $\triangle DAB$ are at the vertices of a parallelogram.

5. If G, H are the centroids of two triangles ABC, ABD with a common side AB , then $GH \parallel CD$ and $GH = \frac{1}{3}CD$.

2. The Circumcentre

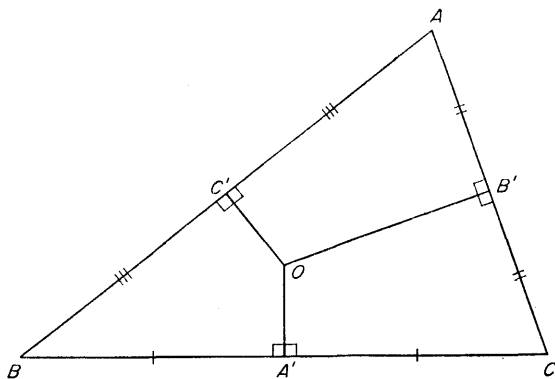


Fig. 16

THE PROBLEM. ABC is a given triangle. It is required to establish the existence of a point O such that $OA = OB = OC$; that is, $\exists O$ such that $OA = OB = OC$.

Definitions. The point O is called the *circumcentre* of $\triangle ABC$. The circle of centre O and radius OA passes through A, B, C and is called the *circumcircle* of $\triangle ABC$.

THE DISCUSSION. Let the perpendicular bisectors of AB, AC meet in O . Then

$O \in$ perpendicular bisector of $\begin{cases} AB \\ AC \end{cases}$ respectively

$\Rightarrow O$ is equidistant from $\begin{cases} A \text{ and } B \\ A \text{ and } C \end{cases}$ respectively

$\Rightarrow O$ is equidistant from A, B and C

$\Rightarrow OA = OB = OC$.

COROLLARIES. (i) $OA' \perp BC$, $OB' \perp CA$, $OC' \perp AB$. (ii) The perpendicular bisectors of the sides of a triangle are concurrent, in the circumcentre.

3. The Orthocentre

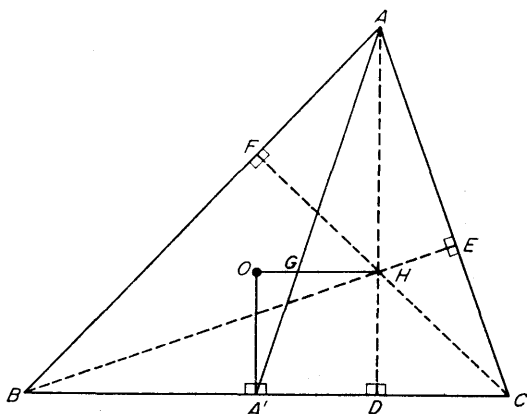


Fig. 17

THE PROBLEM. ABC is a given triangle, and $AD \perp BC$, $BE \perp CA$, $CF \perp AB$. It is required to prove that AD , BE , CF have a common point H .

Definitions. The lines AD , BE , CF are called the *altitudes* of $\triangle ABC$. The point H is called the *orthocentre* of $\triangle ABC$.

THE DISCUSSION. Let A' be the middle point of BC , so that the centroid G is the point on AA' such that $A'G/GA = 1/2$. Also let O be the circumcentre, so that $OA' \perp BC$.

Produce OG to H so that $OG/GH = 1/2$.

Then

$$\begin{aligned} OG/GH &= 1/2 = A'G/GA \\ \Rightarrow OA' &\parallel AH. \end{aligned}$$

But

$$OA' \perp BC,$$

so that

$$AH \perp BC.$$

Thus, the fixed point H on OG produced such that $OG/GH = 1/2$ has the property that $AH \perp BC$; hence AH is an altitude. Identical reasoning shows that BH , CH are altitudes. Hence the altitudes intersect in H .

COROLLARY. The points O , G , H are collinear and such that $OG/GH = 1/2$.

Definition. The line OGH is called the *Euler line* of $\triangle ABC$.

Theorems

1. O is the orthocentre of $\triangle A'B'C'$.
2. The four points A , B , C , H are so related that each is the orthocentre of the triangle whose vertices are the other three.
3. H is outside $\triangle ABC \Leftrightarrow$ one of the angles of the triangle is obtuse.
4. $\odot HBC = \odot HCA = \odot HAB = \odot ABC$ (That is, the radii are equal).
5. $AH \cdot HD = BH \cdot HE = CH \cdot HF$.
6. DA , BC bisect the angles between DE , DF .

Problems

1. The circles on AB , AC as diameters meet on BC .
2. The middle point of OH is equidistant from A' and D .
3. The triangle UVW is drawn so that $A \in VW$, $B \in WU$, $C \in UV$ and $WV \parallel BC$, $UW \parallel CA$, $VU \parallel AB$. Prove that H is the circumcentre of $\triangle UVW$.
4. $ABCD$ is a parallelogram; O is the circumcentre of $\triangle ABC$ and Q is the circumcentre of $\triangle ADC$. Prove that $AOCQ$ is a rhombus.
5. $ABCD$ is a parallelogram and H is the orthocentre of $\triangle ABC$. Prove that $D \in \odot AHC$.
6. The tangent at A to $\odot AEF$ is parallel to BC (Fig. 17).
7. The tangent at A to $\odot ABC$ is parallel to EF (Fig. 17).
8. (Alternative proof of the orthocentre property). In $\triangle ABC$, let BE , CF be altitudes meeting in H , and let AH meet BC in K . Establish the argument:

$$\angle KAE = \angle HFE = \angle KBE \Rightarrow AK \perp BC.$$

4. The Incentre and the Escribed Centres

(i) The incentre

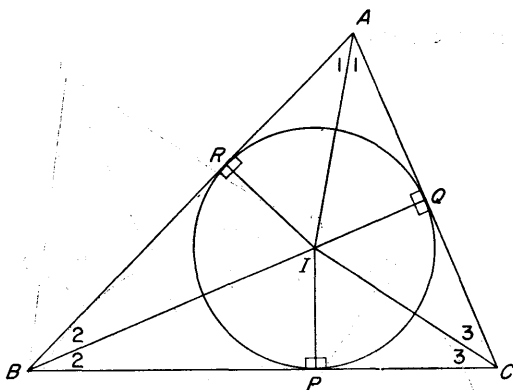


Fig. 18

THE PROBLEM. ABC is a given triangle. It is required to establish the existence of a point I which lies inside the triangle and which is equidistant from BC , CA and AB ; that is, $\exists I$ such that, if $IP \perp BC$, $IQ \perp CA$, $IR \perp AB$, then $IP = IQ = IR$.

Definitions. The point I is called the *incentre* of $\triangle ABC$. The circle of centre I and radius IP is called the *incircle* of $\triangle ABC$; the lines BC , CA , AB are the tangents at P , Q , R .

THE DISCUSSION. Let the internal bisectors of $\angle B$, $\angle C$ meet in I . Then

$I \in$ bisector of $\begin{cases} \angle B \\ \angle C \end{cases}$ respectively

$\Rightarrow I$ is equidistant from the lines $\begin{cases} BA \text{ and } BC \\ CA \text{ and } CB \end{cases}$ respectively

$\Rightarrow I$ is equidistant from BC , CA , AB

C

COROLLARY. The internal bisectors of the angles of a triangle are concurrent, in the incentre.

(ii) The escribed centres

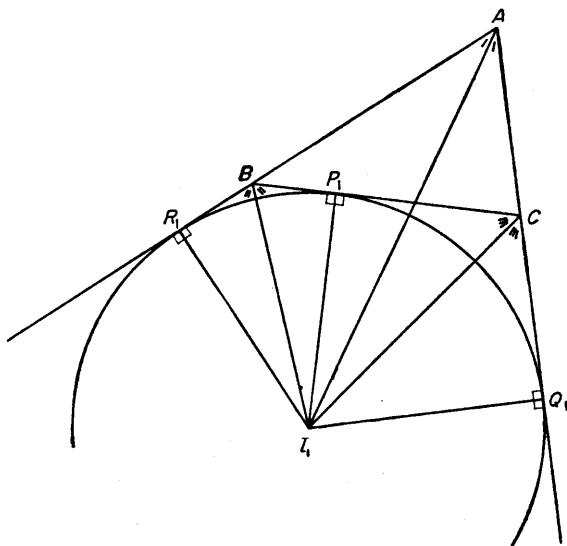


Fig. 19

THE PROBLEM. There are three circles to be described, one opposite each of the vertices A , B , C of the given triangle; the circle opposite A is taken as typical.

It is required to establish the existence of a point I_1 , lying outside the triangle but within the angle formed by AB and AC , such that I_1 is equidistant from BC , CA , AB ; that is, $\exists I_1$ such that, if $I_1P_1 \perp BC$, $I_1Q_1 \perp CA$, $I_1R_1 \perp AB$, then $I_1P_1 = I_1Q_1 = I_1R_1$.

Definitions. The point I_1 is called the escribed centre opposite A of $\triangle ABC$.

The circle of centre I_1 and radius I_1P_1 is called the escribed

circle opposite A of $\triangle ABC$; the lines BC, CA, AB are the tangents at P_1, Q_1, R_1 .

The escribed centres opposite B and C , and the escribed circles opposite B and C , are defined similarly.

THE DISCUSSION. Let the *external* bisectors of $\angle B, \angle C$ meet in I_1 . Then

$I_1 \in$ a bisector of $\begin{cases} \angle B \\ \angle C \end{cases}$ respectively

$\Rightarrow I_1$ is equidistant from the lines $\begin{cases} BA \text{ and } BC \\ CA \text{ and } CB \end{cases}$ respectively

$\Rightarrow I_1$ is equidistant from BC, CA, AB .

(iii) The configuration of these four centres

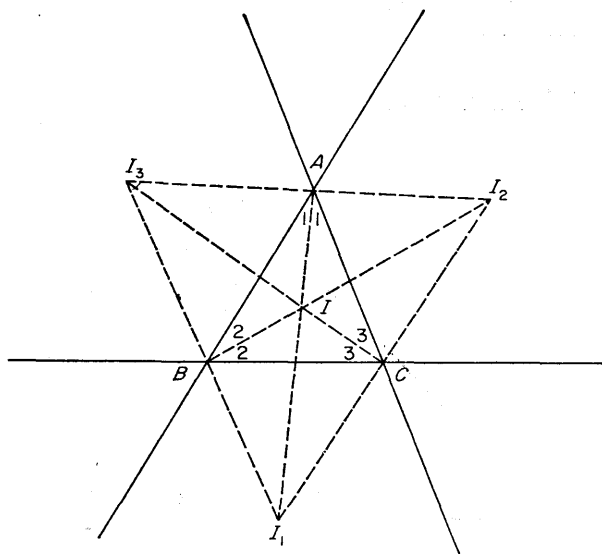


Fig. 20

The diagram shows how the four points I, I_1, I_2, I_3 are related to the vertices A, B, C of the given triangle. The properties which follow are important, but the proofs are left to the reader.

Theorems

1. The points A, I, I_1 are collinear.
2. The points I_2, A, I_3 are collinear.
3. $IA \perp I_2I_3$; $IB \perp I_3I_1$; $IC \perp I_1I_2$.
4. I is the orthocentre of $\triangle I_1I_2I_3$.
5. If $BC = a, CA = b, AB = c, s = \frac{1}{2}(a + b + c)$, then $AQ = AR = s - a, AQ_1 = AR_1 = s, BP_1 = BR_1 = s - c$.
6. If Δ is the area of the triangle ABC , and if the radii of the inscribed and escribed circles are r, r_1, r_2, r_3 , then

$$r = \Delta/s, r_1 = \Delta/(s - a).$$

Problems

1. $I_1 \in \odot BIC$.
2. $AI \cdot II_1 = BI \cdot II_2 = CI \cdot II_3$.
3. $BC = CA = AB \Leftrightarrow I_2I_3 = I_3I_1 = I_1I_2$.

5. The Nine-Points Circle

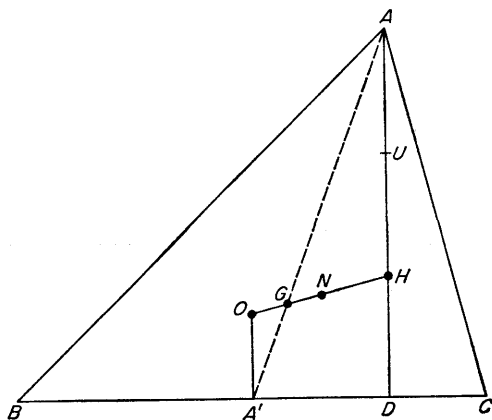


Fig. 21

THE PROBLEM. Let ABC be a given triangle, A', B', C' the middle points of BC, CA, AB ; $AD \perp BC, BE \perp CA, CF \perp AB$; U, V, W the middle points of HA, HB, HC .

(Only A', D, U are shown in the diagram.)

It is required to prove that the nine points $A', B', C', D, E, F, U, V, W$ lie on a circle whose centre is N , the middle point of OH , and whose radius is one-half of that of $\odot ABC$.

Definition. The circle is called the *nine-points circle* and N the *nine-points centre* of $\triangle ABC$.

THE DISCUSSION.

$$\left. \begin{array}{l} A'O \parallel HA \\ A'G = \frac{1}{2} GA \end{array} \right\} \Rightarrow OA' = \frac{1}{2} HA = HU = UA.$$

$$\left. \begin{array}{l} A'O \parallel UA \\ A'O = UA \end{array} \right\} \Rightarrow A'O AU \text{ is a parallelogram}$$

$$\Rightarrow A'U = OA = \text{radius of } \odot ABC.$$

$$\left. \begin{array}{l} A'O \parallel HU \\ A'O = HU \end{array} \right\} \Rightarrow A'OUH \text{ is a parallelogram}$$

$$\Rightarrow A'U \text{ is bisected at } N, \text{ the middle point of } OH$$

$$\Rightarrow NU = NA' = \frac{1}{2} \text{ radius of } \odot ABC.$$

Also

$$\left. \begin{array}{l} OA' \perp BC, HD \perp BC \\ N \text{ is the middle point of } OH \end{array} \right\} \Rightarrow N \text{ is on the perpendicular bisector of } A'D$$

$$\Rightarrow ND = NA' = \frac{1}{2} \text{ radius of } \odot ABC.$$

Hence

$$NA = NU = ND = \frac{1}{2} \text{ radius of } \odot ABC.$$

Similar argument obtains the same values for $NB, NV, NE; NC, NW, NF$.

Hence the nine points lie on a circle of centre N and radius equal to $\frac{1}{2}$ radius of $\odot ABC$.

6. The Nine-Points Circle; Alternative Treatment

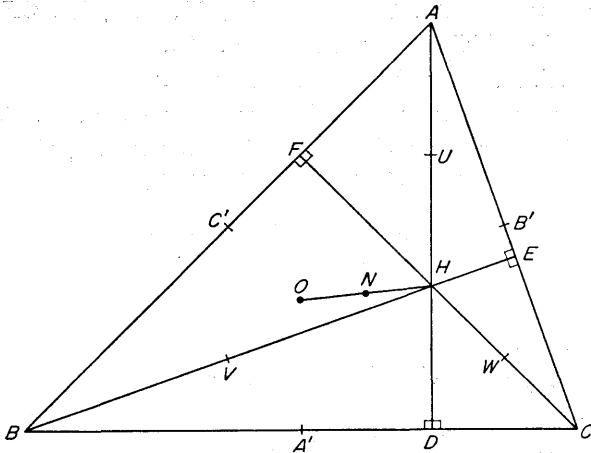


Fig. 22

THE PROBLEM. Let ABC be a given triangle, with circumcentre O and orthocentre H . Let A' , B' , C' be the middle points of BC , CA , AB ; and $AD \perp BC$, $BE \perp CA$, $CF \perp AB$, so that AD , BE , CF meet in the orthocentre H ; finally, let U , V , W be the middle points of HA , HB , HC and N the middle point of OH .

It is required to prove that the points A' , B' , C' , D , E , F , U , V , W all lie on a circle of centre N .

THE DISCUSSION.

$$\left. \begin{array}{l} BA' = A'C' \\ BV = VH \\ CA' = A'B' \\ CW = WH \end{array} \right\} \Rightarrow \left. \begin{array}{l} A'V \parallel CH \\ A'W \parallel BH \end{array} \right\} \Rightarrow \angle VA'W = \angle VHW$$

But

$$\begin{aligned} \angle VHW &= \angle EHF \\ &= 180^\circ - \angle EAF \quad (HF \perp AF \text{ and } HE \perp AE) \\ &= 180^\circ - \angle BAC \end{aligned}$$

Also

$$\left. \begin{array}{l} HV = VB \\ HU = UA \end{array} \right\} \Rightarrow VU \parallel BA$$

$$\left. \begin{array}{l} HW = WC \\ HU = UA \end{array} \right\} \Rightarrow WU \parallel CA$$

$$\Rightarrow \angle VUW = \angle BAC.$$

Thus

$$\angle VA'W + \angle VUW = 180^\circ.$$

Hence $VUWA'$ is cyclic; that is,

$$A' \in \odot UVW.$$

Similarly $B', C' \in \odot UVW$.

Again

$$\left. \begin{array}{l} \angle BDH = 90^\circ \\ V \text{ is middle point of } BH \end{array} \right\} \Rightarrow VH = VD$$

$$\Rightarrow \angle VDH = \angle VHD.$$

Similarly,

$$\angle WDH = \angle WHD.$$

Hence

$$\begin{aligned} \angle VDW &= \angle VDH + \angle WDH \\ &= \angle VHD + \angle WHD \\ &= \angle VHW \\ &= \angle VA'W \quad \text{as before.} \end{aligned}$$

Thus $VA'DW$ is cyclic; that is,

$$D \in \odot VA'W,$$

so that, as above,

$$D \in \odot UVWA'B'C'.$$

Similarly $E, F \in$ that circle. Hence

$$U, V, W, A', B', C', D, E, F$$

lie on a circle.

Finally,

$$\left. \begin{aligned} HN &= \frac{1}{2} HO \\ HU &= \frac{1}{2} HA \end{aligned} \right\} \Rightarrow NU = \frac{1}{2} OA; \text{ that is,} \\ NU = \frac{1}{2} (\text{radius of } \odot ABC).$$

Similar $NV, NW = \frac{1}{2}$ (radius of $\odot ABC$).

Hence $NU = NV = NW$, so that N is the centre of the circle UVW .

Theorems

- O, G, N, H are collinear, and $OG/GN = OH/HN = 2/1$.
- Definition.** The line $OGNH$ is called the *Euler line* of $\triangle ABC$.
- A, B, C, H are the inscribed and escribed centres of $\triangle DEF$.
- The triangles ABC, HBC, HCA, HAB all have the same nine-points circle.

Problems

- $A'U, B'V, C'W$ bisect each other at N .
- $\triangle UVW \equiv \triangle A'B'C'$.
- $B'C'VW$ is a rectangle.
- $UE = UF; A'U \perp EF$.
- OU bisects AA' ; $OU \cap AA' \in B'C'$.
- UE, UF are the tangents from U to $\odot BCEF$.
- If HD is produced to D' and HA' produced to A'' so that $HD = DD', HA' = A'A''$, then D', A'' both lie on $\odot ABC$.
- If $\odot ABC$ meets $\odot AFHE$ again in K , then the line OU is the perpendicular bisector of AK .

If $L = AK \cap BC$, then $L \in \odot EKC$.

By applying the theorem that the common chords of three circles taken in pairs are concurrent, or otherwise, prove that $L = EF \cap BC$.

THREE

Some Circle Theorems

1. The Theorem of Ptolemy

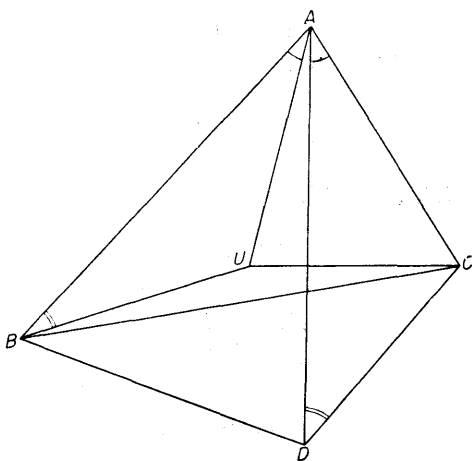


Fig. 23

(i) A quadrilateral inequality

THE PROBLEM. ABC is a given triangle and D an arbitrary point. (The diagram shows a typical position of D ; other cases are possible, but the differences are slight.) A triangle ABU is constructed as in the diagram so that

$$\triangle ABU \sim \triangle ADC.$$

We are to prove that

$$\triangle ACU \sim \triangle ADB$$

and hence that

$$AB \cdot CD + AC \cdot BD > BC \cdot AD.$$

THE DISCUSSION.

$$\begin{aligned} \triangle ABU \sim \triangle ADC &\Rightarrow \angle BAU = \angle DAC \\ &\Rightarrow \angle BAU + \angle UAD = \angle UAD + \angle DAC \\ &\Rightarrow \angle BAD = \angle UAC. \end{aligned}$$

Also

$$\begin{aligned} \triangle ABU \sim \triangle ADC &\Rightarrow \frac{AB}{AD} = \frac{AU}{AC} \\ &\Rightarrow \frac{AB}{AU} = \frac{AD}{AC}. \end{aligned}$$

Hence, in $\triangle ACU$, $\triangle ADB$,

$$\angle UAC = \angle BAD$$

and

$$\frac{AU}{AB} = \frac{AC}{AD},$$

so that

$$\triangle ACU \sim \triangle ADB.$$

Moreover, from the similar triangles,

$$\begin{aligned} \triangle ABU \sim \triangle ADC &\Rightarrow \frac{AB}{AD} = \frac{BU}{DC} \\ &\Rightarrow AB \cdot CD = AD \cdot BU \end{aligned}$$

and

$$\begin{aligned} \triangle ACU \sim \triangle ADB &\Rightarrow \frac{AC}{AD} = \frac{CU}{BD} \\ &\Rightarrow AC \cdot BD = AD \cdot CU \end{aligned}$$

Hence

$$AB \cdot CD + AC \cdot BD = AD (BU + CU) > AD \cdot BC$$

(triangle inequality)

(ii) **The cyclic case; Ptolemy's theorem.**

The final step of the preceding work,

$$BU + CU > BC,$$

presupposed that U was not on BC , a condition that would certainly hold in general. The case of exception must now receive attention.

THE PROBLEM. Suppose that, in the preceding work, $U \in BC$. Then

$$BU + UC = BC$$

so that

$$AB \cdot CD + AC \cdot BD = AD \cdot BC.$$

It is required to prove that

$$\begin{aligned} AB \cdot CD + AC \cdot BD &= AD \cdot BC \\ \Leftrightarrow A, B, C, D &\text{ are concyclic.} \end{aligned}$$

[Note the double arrow \Leftrightarrow .]

Suppose, first, that equality holds, so that $U \in BC$. Then

$$\begin{aligned} \angle ADC &= \angle ABU \\ &= \angle ABC \text{ (} BUC \text{ is a straight line)} \end{aligned}$$

Hence A, B, C, D are concyclic.

Suppose, next, that A, B, C, D are concyclic. Then

$$\begin{aligned} \angle ABU &= \angle ADC \text{ (given, since } \triangle ABU \sim \triangle ADC) \\ &= \angle ABC \text{ (same segment)} \end{aligned}$$

so that BU, BC are the same lines.

Hence $U \in BC$, so that the equality holds.

Note. The four points A, B, C, D can be split in two pairs in three ways,

$$AB, CD; AC, BD; BC, AD.$$

Ptolemy's theorem asserts that, when the four points are concyclic, the sum of products from two of these pairs is equal to the third. The pair which comes "third" is that defined by the diagonals of the cyclic quadrilateral.

[For a more detailed discussion of implications, see E. A. Maxwell, *Fallacies in Mathematics*, Cambridge University Press (1959) p. 28.]

Problems

1. If D is a point on the arc opposite A of the circumcircle of an equilateral triangle ABC , then $AD = BD + CD$.
2. AB, PQ are parallel chords of a circle. Prove that

$$AB \cdot PQ = AQ^2 - AP^2.$$

3. Identify the well-known result which is a special case of the theorem of Ptolemy when $ABCD$ is a rectangle.
4. In $\triangle ABC$, draw $BE \perp AC$, $CF \perp AB$. Prove that $BF = BC \cos B$, $CF = BC \sin B$, and write down similar expressions for CE , BE .

By applying the theorem of Ptolemy to $\odot BCEF$, prove that $EF = BC \cos A$.

5. AB is a diameter of a circle; P, Q are points on the circle, one in each half, so that $\angle PAB = \theta$, $\angle QAB = \phi$. Use the theorem of Ptolemy to prove that $\sin \theta \cos \phi + \sin \phi \cos \theta = \sin(\theta + \phi)$.

2. The Simson Line

(Author's note. It is possible to argue that the Simson line theorem as we know it is slightly unfortunate. Associated with a point P on the circumcircle of a triangle ABC are several parallel lines, of which the most significant is surely the one through

the orthocentre H . For the benefit of pupils taking examinations, the traditional proof is given first. A more general discussion then follows, based on the line to which reference has just been made.)

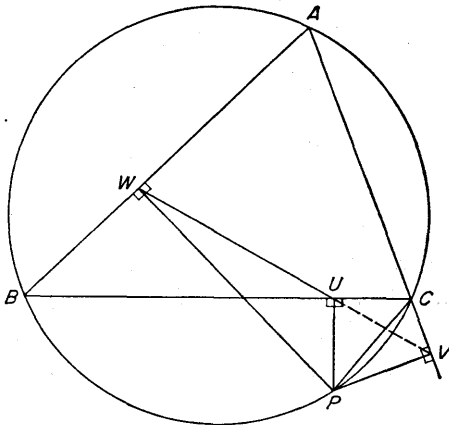


Fig. 24

THE PROBLEM. Take an arbitrary point P on the circumcircle of a triangle ABC . Draw $PU \perp BC$, $PV \perp CA$, $PW \perp AB$. It is required to prove that U, V, W are collinear.

Definition. UVW is known as the *Simson line* of P with respect to the triangle ABC .

THE DISCUSSION.

$$\left. \begin{array}{l} PU \perp BC \\ PV \perp CA \end{array} \right\} \Rightarrow PUCV \text{ cyclic} \Rightarrow \angle CUV = \angle CPV.$$

$$\left. \begin{array}{l} PU \perp BC \\ PW \perp AB \end{array} \right\} \Rightarrow PUWB \text{ cyclic} \Rightarrow \angle BUW = \angle BPW.$$

Also

$$\left. \begin{array}{l} PV \perp CA \\ PW \perp AB \end{array} \right\} \Rightarrow PVAW \text{ cyclic} \Rightarrow \angle VPW = 180^\circ - A$$

and

$$ABPC \text{ cyclic} \Rightarrow \angle BPC = 180^\circ - A.$$

Hence

$$\angle VPW = \angle BPC,$$

so that, subtracting $\angle WPC$ from each,

$$\angle CPV = \angle BPW.$$

Hence

$$\angle CUV = \angle BUW,$$

so that

UVW is a straight line.

Note. The converse theorem is also true:

Given a point P in the plane of a triangle ABC , and $PU \perp BC$, $PV \perp CA$, $PW \perp AB$, then, if U, V, W , are collinear, $P \in \odot ABC$.

The proof is very nearly a reversal of the steps just given.

3. The Simson Property; Alternative Treatment

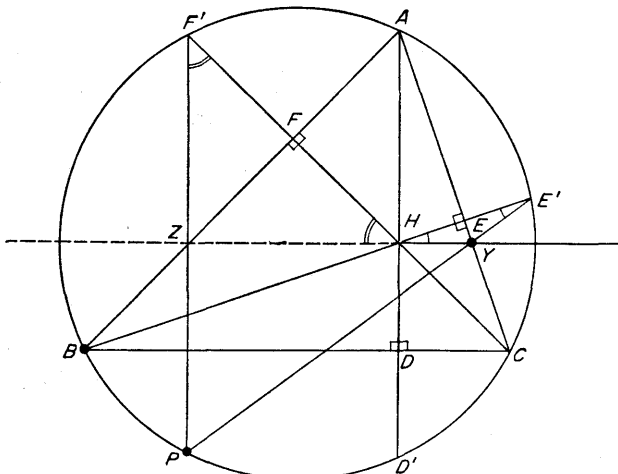


Fig. 25

(i) Lemma

Consider first a theorem which is important in its own right:

THE PROBLEM. The altitudes AHD , BHE , CHF of a triangle ABC meet $\odot ABC$ in D' , E' , F' . It is required to prove that D' , E' , F' are the mirror images of H in BC , CA , AB . Compare Problem 7 on p. 26.

[By saying that D' is the mirror image of H in BC we mean that, if $D = HD' \cap BC$, then $HD' \perp BC$ and $HD = D'D$.]

THE DISCUSSION.

$$\left. \begin{array}{l} HD \perp BC \\ HE \perp CA \end{array} \right\} \Rightarrow HDCE \text{ cyclic} \Rightarrow \angle BHD = \angle ACB.$$

But

$$\angle ACB = \angle AD'B \text{ (same segment),}$$

so that

$$\angle BHD = \angle BD'D \text{ (re-naming the latter angle).}$$

Hence, in $\triangle BHD$, $BD'D$,

$$BD = BD'; \angle BDH = \angle BDD', \angle BHD = \angle BD'D,$$

so that

$$\triangle BHD \equiv \triangle BD'D.$$

In particular,

$$HD = D'D,$$

so that D' is the mirror image of H in BC .

Similarly for E' , F' .

(ii) The main theorem

THE PROBLEM. Let P be an arbitrary point on $\odot ABC$. With the notation of the Lemma, let

$$X = PD' \cap BC, Y = PE' \cap CA, Z = PF' \cap AB.$$

[The point X is not shown in the diagram.]

It is required to prove that X, Y, Z lie on a straight line which passes through H .

THE DISCUSSION. It is an immediate consequence of the preceding work that

$$\begin{aligned}\angle YHE' &= \angle YE'H \\ &= \angle PE'B \quad (\text{re-naming}) \\ &= \angle PAB \quad (\text{same segment})\end{aligned}$$

and that

$$\begin{aligned}\angle ZHF' &= \angle ZF'H \\ &= \angle PF'C \quad (\text{re-naming}) \\ &= \angle PAC \quad (\text{same segment})\end{aligned}$$

Also

$$\angle EHF = 180^\circ - \angle BAC \quad (HE \perp CA, HF \perp AB).$$

Adding corresponding sides of these three equations,

$$\begin{aligned}\angle YHE' + \angle EHF + \angle ZHF' &= 180^\circ + \angle PAB + \angle PAC - \\ &\quad - \angle BAC = 180^\circ.\end{aligned}$$

Hence YHZ is a straight line.

That, is, $Y \in \text{line } ZH$; and similar argument shows that $X \in \text{line } ZH$. Hence X, Y, Z, H are collinear.

Remark: The pith of this proof is the simple observation, "angle on arc BP + angle on arc PC = angle on arc BC ."

(iii) Some consequences

The diagram is now drawn with special reference to the vertex A .

THE PROBLEM. With notation as before, let $PU \perp BC$, and let the line PU meet $\odot ABC$ again in U' . We prove first that the line AU' is parallel to $XYZH$; next that, if Q is the middle point of PH , then UQ is parallel to $XYZH$; and finally, as an immediate consequence, that if V, W are defined similarly to U , then U, V, W all line on the line through Q parallel to $XYZH$.

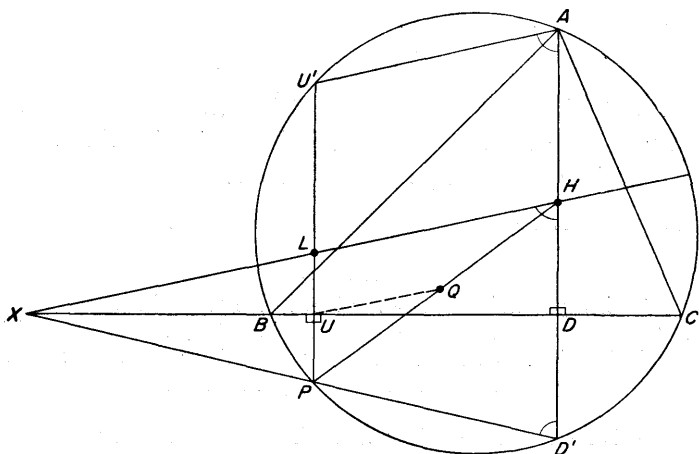


Fig. 26

THE DISCUSSION. By the preceding work,

$$\begin{aligned} \angle XHD &= \angle XD'D \\ &= \angle PD'A \text{ (re-naming)} \\ &= 180^\circ - \angle PU'A \text{ } (\odot AD'PU') \\ &= \angle U'AH \text{ } (PU' \parallel D'A). \end{aligned}$$

But these are corresponding angles, and so

$$HX \parallel AU'.$$

Next, let PU meet $XYZH$ in L . Thus, immediately, since $D'D = DH$, we have

$$PU = UL.$$

Then

$$\left. \begin{aligned} PU &= UL \\ PQ &= QH \end{aligned} \right\} \Rightarrow UQ \parallel LH,$$

so that UQ is parallel to $XYZH$.

THE PROBLEM. Let two given circles have centres A, B and radii a, b , and suppose that their centres are distant d apart. We are to demonstrate precisely, what is, indeed, obvious intuitively, that the circles have two *direct* common tangents PU, QV : that is, lines which are tangents to both circles simultaneously; and (when, as in the diagram, the two circles lie entirely outside each other) two *transverse* common tangents LX, MY . It is also "obvious" that PU, QV meet at a point S on AB and that LX, MY meet at a point T on AB ; further, that AB bisects each of the angles $\angle PSQ, \angle LTM$.

THE DISCUSSION.

(i) *To construct PU, QV .* Draw the circle of centre A and radius $a - b$ (supposing that a is greater than b); construct the tangents BC, BD from B to this circle by the standard method of drawing the circle on AB as diameter to meet it in C, D ; let AC, AD meet the circle of centre A in P, Q ; let lines through B parallel to AP, AQ meet the circle of centre B in U, V . Then PU, QV are the direct common tangents.

To prove this:

$$\left. \begin{array}{l} AC = a - b \\ AP = a \end{array} \right\} \Rightarrow CP = b = BU$$

$$\left. \begin{array}{l} CP = BU \\ CP \parallel BU \end{array} \right\} \Rightarrow CPUB \text{ is a parallelogram}$$

C on circle of diameter $AB \Rightarrow \angle ACB = 90^\circ \Rightarrow \angle PCB = 90^\circ$
 $\Rightarrow CPUB$ is a rectangle
 $\Rightarrow PU \perp AP$ and BU
 $\Rightarrow \begin{cases} PU \text{ is tangent at } P \\ UP \text{ is tangent at } U. \end{cases}$

Similarly QV is tangent at Q and at V .

The construction for LX, MY is very similar, save that the first step is to draw the circle of centre A and radius $a + b$.

(ii) To locate S and T . Let PU meet AB in S . Then

$$\begin{aligned} BU \parallel AP &\Rightarrow \triangle SBU \sim \triangle SAP \\ &\Rightarrow \frac{SB}{SA} = \frac{BU}{AP} = \frac{b}{a} \end{aligned}$$

Thus

$$\frac{SA}{a} = \frac{SB}{b} = \frac{SA - SB}{a - b} = \frac{d}{a - b}$$

so that

$$SA = \frac{ad}{a - b}, \quad SB = \frac{bd}{a - b}$$

Identical argument shows that QV meets AB in precisely the same point.

In the same way, LX , MY meet AB in the point T where

$$\begin{aligned} BX \parallel AL &\Rightarrow \triangle TBX \sim \triangle TAL \\ &\Rightarrow \frac{TB}{TA} = \frac{BX}{AL} = \frac{b}{a} \end{aligned}$$

so that

$$\frac{TA}{a} = \frac{TB}{b} = \frac{TA + TB}{a + b} = \frac{d}{a + b}$$

giving

$$TA = \frac{ad}{a + b}, \quad TB = \frac{bd}{a + b}$$

Note, incidentally, the ratios

$$\frac{SB}{SA} = \frac{TB}{TA} = \frac{b}{a}$$

(iii) The lengths of PU , LX . Using the theorem of Pythagoras, we have

$$\begin{aligned} PU^2 = CB^2 = AB^2 - AC^2 &= d^2 - (a - b)^2 \\ &= (d + a - b)(d - a + b) \end{aligned}$$

and, similarly,

$$LX^2 = d^2 - (a + b)^2 = (d + a + b)(d - a - b).$$

(iv) Similitude

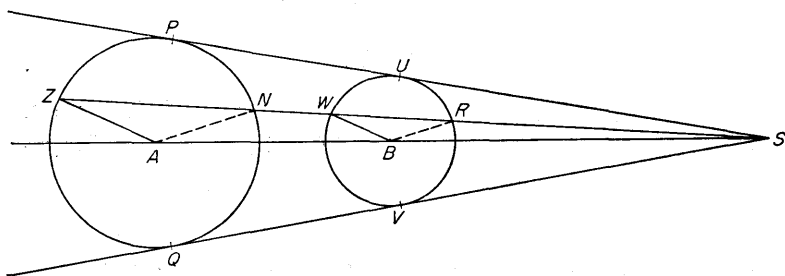


Fig. 28

Definition. The points S, T are called the *centres of similitude* of the two given circles.

The name *similitude* is justified as follows:

Let AZ, BW be parallel radii (in the same sense) of the two circles. We prove that *the line ZW passes through S* . For $AZ \parallel BW \Rightarrow \angle SAZ = \angle SBW$

and

$$\frac{AZ}{BW} = \frac{a}{b} = \frac{SA}{SB} \Rightarrow \frac{SA}{AZ} = \frac{SB}{BW},$$

so that

$$\triangle SAZ \sim \triangle SBW.$$

In particular,

$$\angle ASZ = \angle BSW,$$

so that SWZ is a straight line.

To follow the configuration further, let the line SWZ cut the circles again in N, R . Then we prove that $AN \parallel BR$:

$$AN = AZ \Rightarrow \angle ANZ = \angle AZN,$$

$$AZ \parallel BW \Rightarrow \angle AZN = \angle BWR,$$

$$BW = BR \Rightarrow \angle BWR = \angle BRW,$$

so that

$$\angle ANZ = \angle BRW,$$

giving

$$AN \parallel BR.$$

The two circles are thus, in an obvious sense of the phrase, *similarly placed* with respect to S and (by the same kind of argument) with respect to T .

Theorems

1. *More generally* two circles, however placed, have associated with them two points S, T on their line of centres such that the lines joining ends of parallel radii, one of each circle, pass through one or other of S, T .
The points, S, T are *centres of similitude* for the circles.
2. The orthocentre H and the centroid G are the centres of similitude for the nine-points circle and the circumcircle of a triangle ABC .

Problems

1. The centres of similitude for the incircle and the escribed circle opposite A of $\triangle ABC$ are the vertex A and the point where the internal bisector of $\angle BAC$ meets BC .
2. Two equal circles have only one centre of similitude.

5. Radical Axes

THE PROBLEM. Let two non-intersecting circles be given, of centres A, B and radii a, b . It is required to *prove that the locus of a point P such that the tangents from P to the two circles are equal is a straight line perpendicular to AB .*

Definition. The line is called the *radical axis* of the two circles.

THE DISCUSSION. Let P be such a point, and draw the tangents PT, PU to the circles. Then $PT = PU$.

But

$$\begin{aligned} PT \text{ is a tangent} &\Rightarrow \angle ATP = 90^\circ \\ &\Rightarrow PT^2 = PA^2 - AT^2 = PA^2 - a^2, \end{aligned}$$

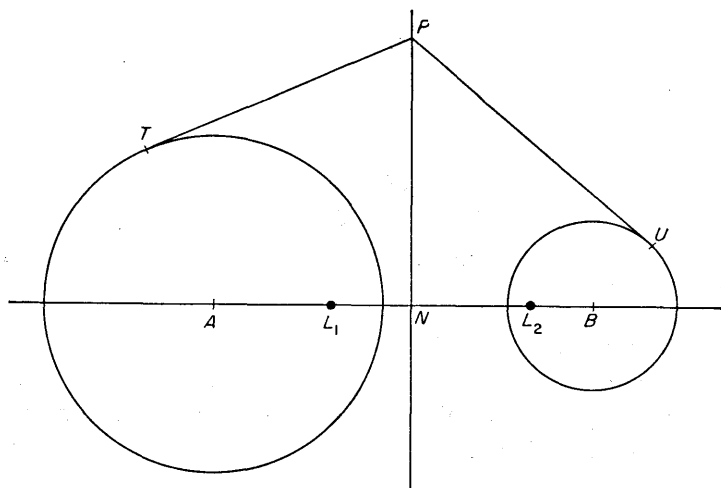


Fig. 29

and, similarly,

$$PU^2 = PB^2 - b^2.$$

Hence

$$PA^2 - a^2 = PB^2 - b^2,$$

or

$$PA^2 - PB^2 = a^2 - b^2.$$

Draw $PN \perp AB$.

Then

$$\begin{aligned} PA^2 - PB^2 &= a^2 - b^2 \\ \Rightarrow (PN^2 + AN^2) - (PN^2 + BN^2) &= a^2 - b^2 \\ \Rightarrow AN^2 - BN^2 &= a^2 - b^2 \\ \Rightarrow (AN + BN)(AN - BN) &= a^2 - b^2. \end{aligned}$$

Thus, if $AB = d$,

$$AN - BN = \frac{a^2 - b^2}{d}.$$

Also

$$AN + BN = d,$$

so that

$$AN = \frac{d^2 + a^2 - b^2}{2d},$$

$$BN = \frac{d^2 - a^2 + b^2}{2d}.$$

Hence N is a fixed point, and so *the point P lies on the fixed line through N perpendicular to AB .*

Further immediate properties.

Let the circle of centre P and radius PT cut the line AB in points L_1, L_2 . We prove that L_1, L_2 are the same for all positions of P on the radical axis.

Definition. L_1, L_2 are called the *limiting points* of the two circles.

By the theorem of Pythagoras,

$$\begin{aligned} NL_1^2 &= PL_1^2 - PN^2 \\ &= PT^2 - (PA^2 - AN^2) \\ &= AN^2 - (PA^2 - PT^2) \\ &= AN^2 - AT^2 \\ &= \left(\frac{d^2 + a^2 - b^2}{2d} \right)^2 - a^2 \\ &= \left(\frac{d^2 + a^2 - b^2}{2d} + a \right) \left(\frac{d^2 + a^2 - b^2}{2d} - a \right). \\ &= \left\{ \frac{(d+a)^2 - b^2}{2d} \right\} \left\{ \frac{(d-a)^2 - b^2}{2d} \right\} \\ &= \frac{(d+a+b)(d+a-b)(d-a+b)(d-a-b)}{4d^2}. \end{aligned}$$

Similar argument obtains the same value for NL_2^2 . Also, the expression on the right-hand side depends only on a, b, d , and so is independent of the position of P on the radical axis.

Hence L_1, L_2 are fixed points, and also $NL_1 = NL_2$.

Remark: The step " $NL_1^2 = AN^2 - AT^2$ " suffices to prove that the position of L_1 is independent of P . The algebraic formulation is added for interest.

Theorems, (i) For intersecting circles

1. If two circles intersect in X, Y , then all points P , such that the tangents from P to the two circles are equal, lie on XY .
2. In Question 1, the circle of centre P and radius $\sqrt{PX \cdot PY}$ does not meet the line of centres of the two given circles.
3. The common chords of three circles taken in pairs are concurrent.

Theorems, (ii) For non-intersecting circles

1. The radical axes of three non-intersecting circles taken in pairs (with non-collinear centres) are concurrent.
2. If a circle cuts the circle of centre A in R, S and the circle of centre B in R', S' , then $RS \cap R'S'$ is on the radical axis of the two given circles.

Theorems (Generalization)

Let P be a point in the plane of a given circle, and let an arbitrary line through P meet the circle in A, B . Then the product $\vec{PA} \cdot \vec{PB}$ (having regard to sign; compare p. 51) is called the *power* of P with respect to the circle. It is independent of the chord selected.

1. The power of P is positive, negative or zero according as P is outside, inside or on the circle.
2. The locus of a point P such that the powers of P with respect to the two circles are equal is a straight line, the *radical axis* of the circles.

Problems

1. In $\triangle ABC$, $AD \perp BC$, $BE \perp CA$, $CF \perp AB$. Verify that the common chords of $\odot ABC$, $\odot BCEF$ is BC , that the common chord of $\odot AEF$, $\odot BCEF$ is EF , and deduce that, if $X = BC \cap EF$ and if AX meets $\odot ABC$ again in Y , then $Y \in \odot AEF$.
2. The radical axis of the inscribed circle and the escribed circle opposite A of $\triangle ABC$ bisects BC .
3. In $\triangle ABC$, $AD \perp BC$, $BE \perp CA$, $CF \perp AB$; $P = BC \cap EF$, $Q = CA \cap FD$, $R = AB \cap DE$. Prove that $PE \cdot PF = PB \cdot PC$, and deduce that P, Q, R lie on a straight line perpendicular to the Euler line $OGNH$ of $\triangle ABC$.

4. Prove that the locus of a point P , which moves so that the length of the tangent from P to a given circle is equal to the distance from P to a given point A outside that circle, is a straight line.
5. In a copy of Fig. 29, everything was obliterated except the three points L_1, L_2, U . Starting from these three points as sole data, show how to construct the circle through U .

6. Orthogonal Circles

Preliminary definition: When two circles pass through a point L , the angle between the circles is defined to be the angle between the two tangents at L .

The two circles have a second common point M . It is clear from the symmetry of the figure, or easy to prove directly (in the diagram, $TL = TM$, $UL = UM \Rightarrow \triangle LTU \cong \triangle MTU$), that the angle between the tangents at L is equal to the angle between the tangents

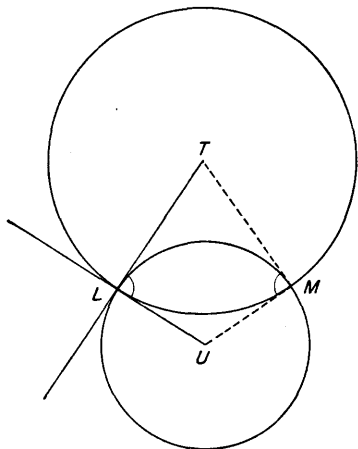


Fig. 30

at M ; that is, *the angle between the circles is the same at each of their common points.*

In particular,

Definition: Two circles are said to be *orthogonal* when they cut at right angles.

THE PROBLEM. Suppose that two orthogonal circles, of centres A, B , cut at L, M . We establish *two tests for orthogonality:*

- (i) *the circles are orthogonal at $L(M)$ if the tangent at $L(M)$ to either passes through the centre of the other;*

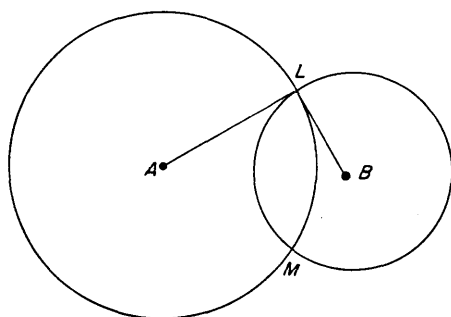


Fig. 31

(ii) *the circles are orthogonal if the square on the distance between their centres is equal to the sum of the squares on their radii.*

THE DISCUSSION (i) Taking the intersection at L as typical, draw the tangent there to the circle of centre A (B). If the circles are orthogonal, the tangent at L to the other circle is perpendicular to it and so passes through the centre of $A(B)$.

(ii) By the theorem of Pythagoras,

$$AL \perp LB \Rightarrow AB^2 = AL^2 + LB^2.$$

Notation: We sometimes write

$$\odot A \perp \odot B$$

(or equivalent) to denote that the circles of centres A and B are orthogonal.

The two tests just established are reversible, for it may be proved directly that

$$\begin{aligned} \odot A \perp \odot B &\Leftrightarrow \text{tangent to either passes through} \\ &\quad \text{centre of other} \\ &\Leftrightarrow AB^2 = a^2 + b^2. \end{aligned}$$

7. Inverse Points

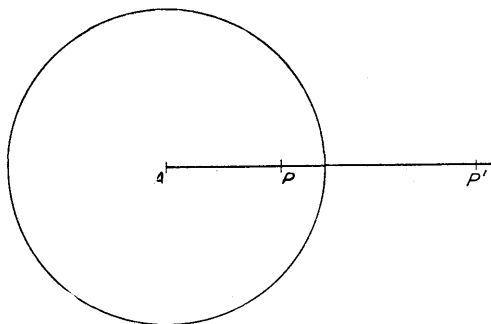


Fig. 32

Definition. Given a circle of centre A and radius a , two points P, P' are said to be *inverse with respect to it* if the points P, P' lie on a line through A , both on the same side of A , and if they are such that

$$AP \cdot AP' = a^2.$$

P' is called the *inverse* of P with respect to the circle; *then P is also the inverse of P' .*

Properties of circles through points inverse with respect to a given circle.

THE PROBLEM. Let P, P' be two points inverse with respect to a given circle of centre A . For convenience of reference, denote this circle by the symbol Ω (Greek *omega*).

We prove that (i) *every circle through P, P' cuts the circle Ω orthogonally*; also that (ii) *any line through A cuts every such circle in two points which are inverse with respect to Ω .*

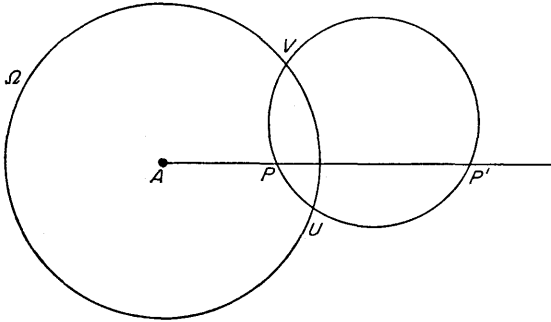


Fig. 33

THE DISCUSSION. (i) Let an arbitrary circle through P, P' cut Ω in points U, V . Then

- P, P' inverse with respect to Ω
- $\Rightarrow AU^2 = AP \cdot AP'$
- $\Rightarrow AU$ is a tangent to the circle UPP'
- \Rightarrow the circles are orthogonal (since AU is a radius of Ω).

(ii) If an arbitrary line through A cuts the circle UPP' (which was drawn arbitrarily through P, P') in points L, M , not shown in the diagram, then

- AU is a tangent to the circle UPP'
- $\Rightarrow AL \cdot AM = AU^2$
- $\Rightarrow L, M$ are inverse with respect to Ω .

8. Further Properties of Radical Axes and Limiting Points

Given two circles of centres A, B and radii a, b , let RN be their radical axis, meeting AB in N , and let L_1, L_2 be the limiting points.

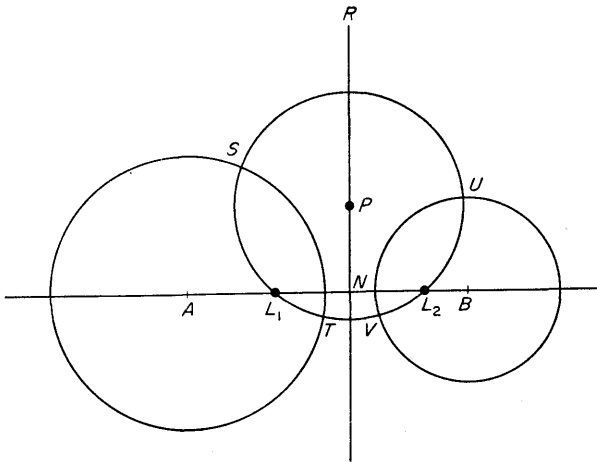


Fig. 34

THE PROBLEM. It is required to prove that L_1, L_2 are inverse points with respect to each of the given circles.

THE DISCUSSION. Take any point P on the radical axis and let the circle of centre P and radius $PL_1 (= PL_2)$ cut the given circles in points S, T and U, V . Then, since (§5) PL_1 is the length of a tangent from P to either circle, PS, PT, PU, PV are tangents; so that

$$\odot P \perp \odot A; \quad \odot P \perp \odot B.$$

But

$$\left. \begin{array}{l} \odot P \perp \odot A \\ AL_1 L_2 \text{ cuts } \odot P \text{ in } L_1, L_2 \end{array} \right\} \Rightarrow L_1, L_2 \text{ are inverse with respect to } \odot A.$$

Similarly L_1, L_2 are inverse with respect to $\odot B$.

COROLLARY. Every circle through the limiting points cuts both the given circles orthogonally.

Theorem

1. In $\triangle ABC$, $AD \perp BC$, $BE \perp CA$, $CF \perp AB$. Then \exists a circle with respect to which B, F ; C, E ; H, D are three inverse pairs. This circle cuts orthogonally each of the circles $BCEF$, $BDHF$, $CDHE$.

Problems

1. In $\triangle ABC$, $BE \perp CA$, $CF \perp AB$. Prove that $\odot AEF \perp \odot BCEF$.
2. P is a point on the circle of diameter AB ; prove that the circle with centre A and radius AP is orthogonal to the circle of centre B and radius BP .

Prove also that the circle of centre A and radius BP is orthogonal to the circle of centre B and radius AP .

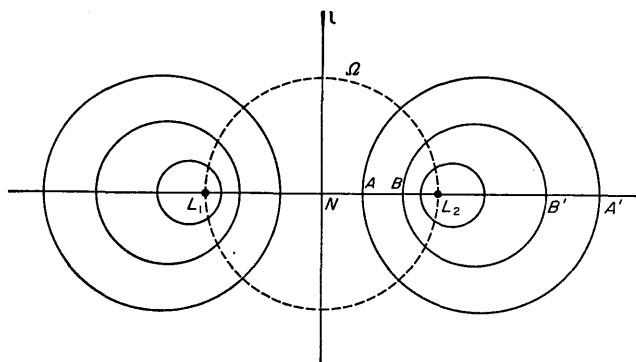
9. Coaxal Circles

Fig. 35

Definition. L_1, L_2 are the extremities of a diameter of a given circle Ω of centre N . A number of pairs of points A, A' ; B, B' ; \dots inverse with respect to Ω are taken on L_1L_2 . The system of circles on AA', BB', \dots as diameters is called a *coaxal system*, and L_1, L_2 are called the *limiting points of the system*.

THE PROBLEM. Let l be the perpendicular bisector of L_1L_2 . The point of the name "coaxal system" is that l is the *radical axis of any two circles of the system*.

THE DISCUSSION. Take, for example, the circles on AA' , BB' as diameters. Then

- A, A' and B, B' are inverse with respect to Ω
- $\Rightarrow NA \cdot NA' = NB \cdot NB'$
- \Rightarrow the tangents from N to the circles are equal
- $\Rightarrow N$ is on the radical axis of the circles
- \Rightarrow the radical axis is the line through N perpendicular to AB
- \Rightarrow the radical axis is l .

Theorems

1. The locus of a point P , which moves so that the ratio PA/PB of its distances from two fixed points A, B has constant value r , is a circle.

Definition. The circle is called the *circle of Apollonius* for the points A, B and the ratio r .

2. If, in Question 1, r is allowed to vary, then the circles for different values of r form a coaxal system with A, B as limiting points.

3. One and only one circle of a given coaxal system can be drawn through an arbitrary point U . Its centre is on the perpendicular bisector of UU' , where U' is the inverse of U with respect to Ω .

4. Every circle through L_1, L_2 cuts orthogonally every circle of the coaxal system.

5. If two circles intersect at A, B , then (p. 43) their radical axis is the line AB . All pairs of circles through A, B have this same radical axis. Conversely, any circle whose radical axis when taken with the first circle is AB and whose radical axis when taken with the second circle is also AB passes through A and B .

6. (Compare Example 4 of this set). The circles through L_1, L_2 of Fig. 35 form a second coaxal system, of *intersecting circles*, and each circle of this system cuts each circle of the given system orthogonally.

FOUR

The Theorems of Ceva and Menelaus

1. The Idea of "Sense" on a Line

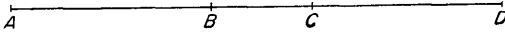


Fig. 36

There are many problems in which it is convenient to superpose on a line a *sense of description*, by which we mean that a symbol such as

$$\vec{AB}$$

denotes not only the length AB but also that it is to be regarded as traced by a point starting at A and moving so as to finish at B . It is then natural to adopt the algebraic symbolism

$$\vec{BA} = -\vec{AB}$$

for the line described in the opposite sense.

With this convention, the equation

$$\vec{AB} = \vec{AC} + \vec{CB}$$

is true for all collinear points A, B, C , whatever the order in which they occur. In particular,

$$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DA} = 0.$$

2. The Theorem of Menelaus

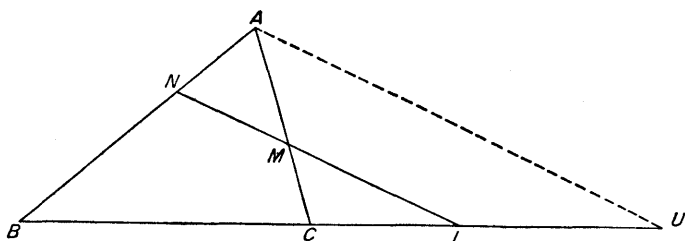


Fig. 37

THE PROBLEM. Let ABC be a given triangle, and suppose that any straight line (a *transversal*) is drawn to cut the sides BC , CA , AB in L , M , N . It is required to *prove that*

$$\frac{\vec{BL}}{\vec{LC}} \cdot \frac{\vec{CM}}{\vec{MA}} \cdot \frac{\vec{AN}}{\vec{NB}} = -1.$$

THE DISCUSSION. Let the line through A parallel to LMN cut BC in U . Then, in whatever order the points occur along the lines,

$$LM \parallel UA \Rightarrow \frac{\vec{CM}}{\vec{MA}} = \frac{\vec{CU}}{\vec{LU}},$$

$$LN \parallel UA \Rightarrow \frac{\vec{AN}}{\vec{NB}} = \frac{\vec{UL}}{\vec{LB}}.$$

Hence

$$\frac{\vec{BL}}{\vec{LC}} \cdot \frac{\vec{CM}}{\vec{MA}} \cdot \frac{\vec{AN}}{\vec{NB}} = \frac{\vec{BL}}{\vec{LC}} \cdot \frac{\vec{CU}}{\vec{LU}} \cdot \frac{\vec{UL}}{\vec{LB}} = -1.$$

3. The Theorem of Ceva

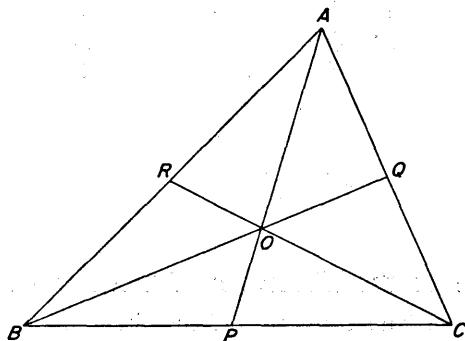


Fig. 38

THE PROBLEM. Let ABC be a given triangle, and O any point in its plane. Suppose that AO, BO, CO meet BC, CA, AB in P, Q, R . It is required to prove that

$$\frac{\vec{BP}}{\vec{PC}} \cdot \frac{\vec{CQ}}{\vec{QA}} \cdot \frac{\vec{AR}}{\vec{RB}} = +1.$$

THE DISCUSSION. Apply the theorem of Menelaus to the triangles APB, APC in turn.

COR is transversal of $\triangle APB$

$$\Rightarrow \frac{\vec{PC}}{\vec{CB}} \cdot \frac{\vec{BR}}{\vec{RA}} \cdot \frac{\vec{AO}}{\vec{OP}} = -1$$

and

BOQ is transversal of $\triangle APC$

$$\Rightarrow \frac{\vec{PB}}{\vec{BC}} \cdot \frac{\vec{CQ}}{\vec{QA}} \cdot \frac{\vec{AO}}{\vec{OP}} = -1.$$

Hence

$$\frac{\vec{PB}}{\vec{BC}} \cdot \frac{\vec{CQ}}{\vec{QA}} = - \frac{\vec{OP}}{\vec{AO}} = \frac{\vec{PC}}{\vec{CB}} \cdot \frac{\vec{BR}}{\vec{RA}},$$

so that

$$\frac{\vec{BP}}{\vec{PC}} \cdot \frac{\vec{CQ}}{\vec{QA}} \cdot \frac{\vec{AR}}{\vec{RB}} = + 1.$$

4. Converse of the Theorems of Ceva and Menelaus

THE PROBLEM. The converses of the theorems of Ceva and Menelaus are also true.

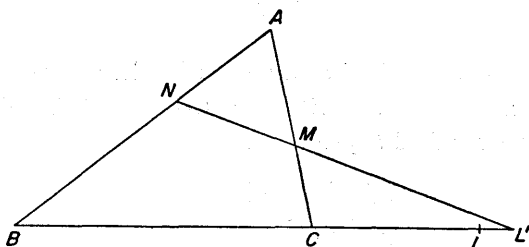


Fig. 39

(i) Given points L, M, N on the sides BC, CA, AB of a triangle ABC such that

$$\frac{\vec{BL}}{\vec{LC}} \cdot \frac{\vec{CM}}{\vec{MA}} \cdot \frac{\vec{AN}}{\vec{NB}} = - 1,$$

to prove that L, M, N are collinear.

Let $L' = MN \cap BC$. Then, by the theorem of Menelaus,

$$\frac{\vec{BL'}}{\vec{L'C}} \cdot \frac{\vec{CM}}{\vec{MA}} \cdot \frac{\vec{AN}}{\vec{NB}} = - 1,$$

so that

$$\frac{\vec{BL}}{\vec{LC}} = \frac{\vec{BL}'}{\vec{L}'C}$$

Hence the points L, L' coincide, and so L lies on the line MN .

(ii) Given points P, Q, R on the sides BC, CA, AB of a triangle ABC such that

$$\frac{\vec{BP}}{\vec{PC}} \cdot \frac{\vec{CQ}}{\vec{QA}} \cdot \frac{\vec{AR}}{\vec{RB}} = +1,$$

to prove that AP, BQ, CR are concurrent.

Let $O = BQ \cap CR$, and $P' = AO \cap BC$. Then, by the theorem of Ceva,

$$\frac{\vec{BP}'}{\vec{P'C}} \cdot \frac{\vec{CQ}}{\vec{QA}} \cdot \frac{\vec{AR}}{\vec{RB}} = +1,$$

so that

$$\frac{\vec{BP}}{\vec{PC}} = \frac{\vec{BP}'}{\vec{P'C}}$$

Hence the points P, P' coincide so that AP also passes through O .

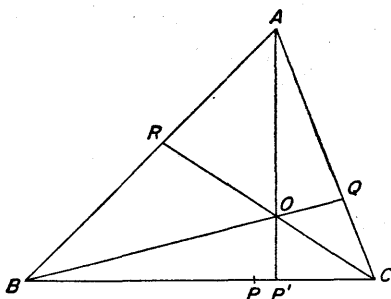


Fig. 40

Problems

1. Use the theorem of Ceva to establish for a given triangle the concurrence of (i) the medians, (ii) the altitudes, (iii) the internal bisectors of the angles. (But the proofs given earlier in this book are more revealing.)

2. Points D, E, F are taken on BC, CA, AB so that AD, BE, CF are concurrent, and $U = EF \cap BC$. Prove that

$$\frac{\vec{BD}}{DC} = -\frac{\vec{BU}}{UC}.$$

3. A point K is taken on the median AA' of $\triangle ABC$; $V = BK \cap AC$, $W = CK \cap AB$. Prove that $VW \parallel BC$.

4. The incircle of $\triangle ABC$ touches BC, CA, AB at P, Q, R . Prove that AP, BQ, CR are concurrent.

5. A point O is taken inside a triangle ABC ; OA, OB, OC are produced to points P, Q, R ; $L = BC \cap QR, M = CA \cap RP, N = AB \cap PQ$. Prove that

$$\frac{\vec{BL}}{LC} \cdot \frac{\vec{CR}}{RO} \cdot \frac{\vec{OQ}}{QB} = -1,$$

and deduce that L, M, N are collinear. (The theorem of *Desargues*.)

FIVE

Harmonic Properties

1. The Definition

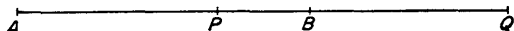


Fig. 41

Let AB be a given line on which there are also two points P, Q . The division of AB at the points P, Q defines two ratios $\frac{\vec{AP}}{\vec{PB}}$ and $\frac{\vec{AQ}}{\vec{QB}}$. There is a very large field of geometrical study in which the *ratio* of these two ratios is of vital importance:

Definition. The expression

$$\frac{\frac{\vec{AP}}{\vec{PB}}}{\frac{\vec{AQ}}{\vec{QB}}}$$

is called the *cross-ratio* of the four collinear points.

In this book, however, we are not concerned with general values of the cross-ratio, but only with the particular case which arises when AB is divided in the same (numerical) ratio by P and by Q , internally and externally respectively, so that

$$\frac{\vec{AP}}{\vec{PB}} = - \frac{\vec{AQ}}{\vec{QB}}.$$

In this case, then, the value of the cross-ratio is -1 .

Because of its great importance, a special name is required for this cross-ratio:

Definitions. A system of points on a straight line is called a *range*; a system of lines passing through a point is called a *pencil*. The line is called the *base* of the range and the point the *vertex* of the pencil.

When four points A, B, P, Q on a line form a range such that

$$\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = -\frac{\overrightarrow{AQ}}{\overrightarrow{QB}},$$

they are said to form a *harmonic range*; P and Q are then *harmonic conjugates* with respect to A and B .

Since

$$\frac{\overrightarrow{PA}}{\overrightarrow{AQ}} = -\frac{\overrightarrow{PB}}{\overrightarrow{BQ}},$$

it follows that A and B are also *harmonic conjugates* with respect to P and Q .

2. Harmonic Pencils

We are to prove a very important theorem, of great generality, proceeding by stages.

(i) THE PROBLEM. Let A, B, P, Q form a harmonic range, so that

$$\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = -\frac{\overrightarrow{AQ}}{\overrightarrow{QB}}.$$

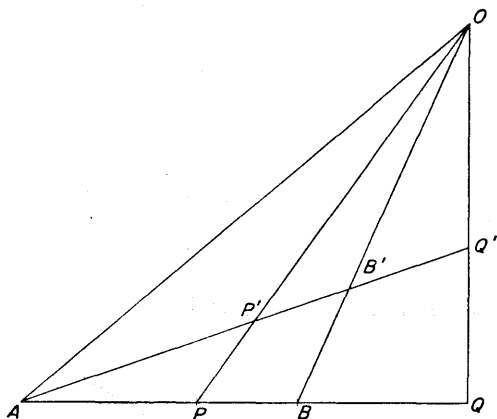


Fig. 42

Take an arbitrary point O and form the pencil of four lines OA, OB, OP, OQ . Let an arbitrary line through A cut the lines of the pencil in A, B', P', Q' . It is required to prove that A, B', P', Q' also form a harmonic pencil.

THE DISCUSSION. By the theorem of Menelaus, $PP'O$ is a transversal of $\triangle ABB'$

$$\Rightarrow \frac{\vec{AP}}{\vec{PB}} \cdot \frac{\vec{BO}}{\vec{OB'}} \cdot \frac{\vec{B'P'}}{\vec{P'A}} = -1,$$

and $QQ'O$ is a transversal of $\triangle ABB'$

$$\Rightarrow \frac{\vec{AQ}}{\vec{QB}} \cdot \frac{\vec{BO}}{\vec{OB'}} \cdot \frac{\vec{B'Q'}}{\vec{Q'A}} = -1.$$

Hence

$$\frac{\vec{AP}}{\vec{PB}} \cdot \frac{\vec{B'P'}}{\vec{P'A}} = - \frac{\vec{OB'}}{\vec{BO}} = \frac{\vec{AQ}}{\vec{QB}} \cdot \frac{\vec{B'Q'}}{\vec{Q'A}}$$

But

$$\frac{\vec{AP}}{\vec{PB}} = -\frac{\vec{AQ}}{\vec{QB}},$$

so that

$$\frac{\vec{AP'}}{\vec{P'B'}} = -\frac{\vec{AQ'}}{\vec{Q'B'}}.$$

(ii) Generalization of the preceding work

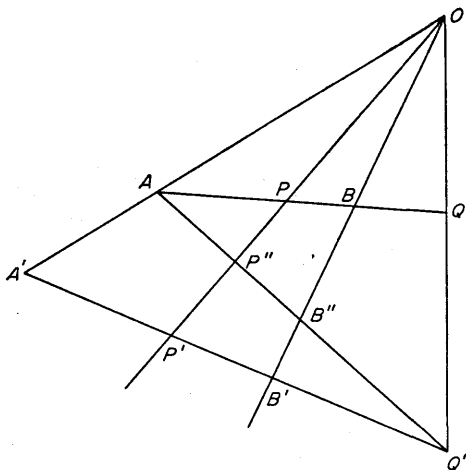


Fig. 43

THE PROBLEM. Once again, let A, B, P, Q , form a harmonic range and take an arbitrary point O . It is required to prove that, if the pencil OA, OB, OP, OQ is met by ANY LINE WHATEVER in points A', B', P', Q' , then the range A', B', P', Q' is also harmonic.

Definition. A pencil of four lines is said to be *harmonic* when it is met by an arbitrary line in the points of a harmonic range.

(By the present theorem, it is sufficient to test the harmonic property for *one* range.)

THE DISCUSSION. Join AQ' , meeting OPP' in P'' and OBB' in B'' . By theorem (i) just proved,

$$A, B, P, Q \text{ harmonic} \Rightarrow A, P'', B'', Q' \text{ harmonic,}$$

and argument similar to that used in (i) proves that

$$A, P'', B'', Q' \text{ harmonic} \Rightarrow A', B', P' Q' \text{ harmonic.}$$

Notation: We write

$$\text{harm. } (AB, PQ)$$

to mean that the range A, B, P, Q is harmonic, with P, Q separating A, B harmonically. Then results such as

$$\text{harm. } (BA, PQ), \text{ harm. } (PQ, AB), \text{ harm. } (QP, BA)$$

follow automatically.

We write similarly

$$\text{harm. } O(AB, PQ)$$

to mean that the pencil OA, OB, OP, OQ is harmonic, with OP, OQ separating OA, OB harmonically.

Problems

1. The internal and external bisectors of $\angle A$ of $\triangle ABC$ meet BC in P, Q . Prove that $\text{harm. } (BC, PQ)$.
2. Points D, E, F are taken on BC, CA, AB so that AD, BE, CF are concurrent; $U = EF \cap BC$. Prove that $\text{harm. } (BC, DU)$.
3. In $\triangle ABC$, with the notation of Chapter 2, §5, $\text{harm. } (GH, ON)$.

3. The Harmonic Tests for Collinearity and Concurrence

(i) THE PROBLEM. Suppose that two harmonic pencils, with vertices U, V have a common "arm", which is necessarily the

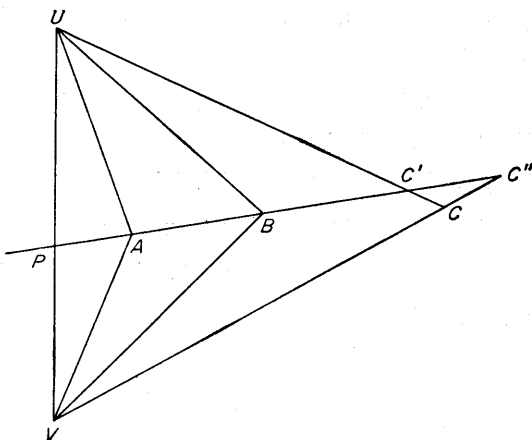


Fig. 44

line UV . It is required to prove that the three other pairs of corresponding arms intersect in collinear points; that is, if the intersections are A, B, C , so that

$$\text{harm. } U(VB, AC), \quad \text{harm. } V(UB, AC),$$

then A, B, C are collinear.

THE DISCUSSION. Let $P = AB \cap UV$, $C' = AB \cap UC$, $C'' = AB \cap VC$. Then

$$\text{harm. } U(PB, AC) \Rightarrow \text{harm. } (PB, AC'),$$

$$\text{harm. } V(PB, AC) \Rightarrow \text{harm. } (PB, AC'').$$

Hence C' and C'' coincide, each being the harmonic conjugate of A with respect to P and B . Since their common point must be on UC and on VC , it must be C , so that $C \in AB$.

(ii) THE PROBLEM. Let A, B, P, Q and A, C, X, Y be harmonic ranges on distinct lines, but having a common point A . It is required to prove that the lines PX, QY, BC are concurrent.

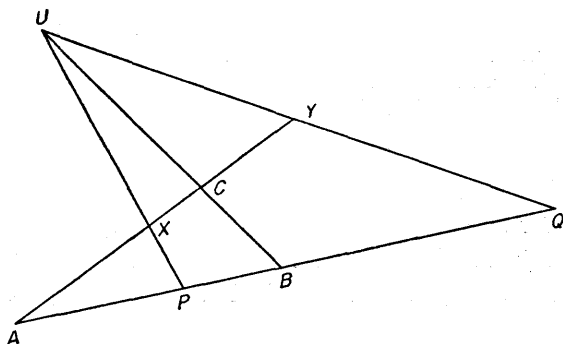


Fig. 45

THE DISCUSSION. Let $U = PX \cap QY$, and suppose that UC meets PQ in a point B' not shown separately in the diagram. Then

$$\begin{aligned} \text{harm. } (AC, XY) &\Rightarrow \text{harm. } U(AC, XY) \\ &\Rightarrow \text{harm. } (AB', PQ). \end{aligned}$$

But we are given that $\text{harm. } (AB, PQ)$, so that B' is at B . Hence $U \in BC$.

Problems

1. Two triangles $ABC, A'BC'$ have a common vertex B , the points B, C, C' not being collinear. The bisectors of $\angle A$ meet BC in P, Q ; the bisectors of $\angle A'$ meet BC' in P', Q' . Prove that

$$PP' \cap QQ' \in CC', PQ' \cap P'Q \in CC'.$$

2. U is a point in the plane of $\triangle ABC$; $E = BU \cap CA, F = CU \cap AB$; $Y \in BU$ such that $\text{harm. } (YE, BU)$ and $Z \in CF$ such that $\text{harm. } (ZF, CU)$. Prove that $EF \cap YZ \in BC$.

3. (AB, CD) and $(A'B', C'D')$ are two harmonic ranges on different lines; O, O' are points on the line AA' . Prove that the points $OB \cap O'B', OC \cap O'C', OD \cap O'D'$ are collinear.

4. $ABCD$ is a rectangle; U is the middle point of AD, V is the middle point of BC ; $X = UV \cap BD, Y = UC \cap BD$. Prove that $\text{harm. } (BY, XD)$.

Prove also that, if L is the harmonic conjugate of C with respect to B and V , then $XL \cap YV \in DC$.

5. In $\triangle ABC$, $P \in AB$, $U \in AB$, $Q \in AC$, $V \in AC$ such that harm. (AU, PB) and harm. (AV, QC) . Prove that, if $L = BV \cap CU$ and $M = BQ \cap CP$, then $A \in LM$.

Prove also that $PQ \cap UV \in BC$.

4. The Quadrangle

The reader who is new to this work will probably not have given much thought to what he means precisely by the word "quadrilateral". There are, in fact, two similar, but distinct, concepts:

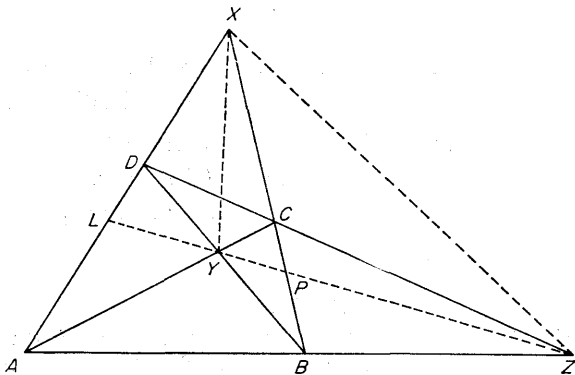


Fig. 46

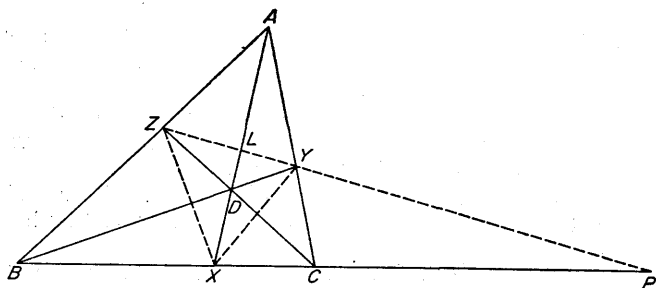


Fig. 47

the quadrangle and the quadrilateral, and it is, indeed, more convenient to begin with the former.

Definitions. The figure defined by four points A, B, C, D and the lines joining them in pairs is called a *quadrangle*, of which A, B, C, D are the *vertices*.

The two diagrams give alternative ways of depicting a typical quadrilateral. They are equally valid for our purposes, but there are sometimes visual advantages in the second, where D is taken inside the triangle ABC .

The six lines

$$BC, AD; CA, BD; AB, CD$$

are called the six *sides* of the quadrangle; we have grouped them in pairs, called *opposite* sides. The points X, Y, Z in which opposite sides meet form a triangle called the *diagonal*, or *harmonic*, triangle of the quadrangle.

The harmonic property of the quadrangle.

The property which follows is basic in the theory of quadrangles and is of very great importance in geometry.

THE PROPERTY. Consider any side, say YZ , of the diagonal triangle of the quadrangle $ABCD$. It meets each of CA, BD at Y and each of AB, CD at Z . Suppose that it meets BC at P and AD at L . It is required to *prove that harm.* (YZ, LP).

THE DISCUSSION. (The second diagram, with D inside the triangle ABC , may be found more convenient for reference.)

PYZ is transversal of $\triangle ABC$

$$\Rightarrow \frac{\vec{BP}}{\vec{PC}} \cdot \frac{\vec{CY}}{\vec{YA}} \cdot \frac{\vec{AZ}}{\vec{ZB}} = -1 \quad (\text{Theorem of Menelaus});$$

AX, BY, CZ meet at D

$$\Rightarrow \frac{\vec{BX}}{\vec{XC}} \cdot \frac{\vec{CY}}{\vec{YA}} \cdot \frac{\vec{AZ}}{\vec{ZB}} = +1 \quad (\text{Theorem of Ceva}).$$

Hence, comparing these results,

$$\frac{\vec{BP}}{\vec{PC}} = - \frac{\vec{BX}}{\vec{XC}},$$

so that

harm. (BC, XP) .

Thus

harm. $A(BC, XP)$

\Rightarrow harm. (ZY, LP)

\Rightarrow harm. (YZ, LP) .

Similar results hold on the sides XY, XZ of the diagonal triangle.

Problems

- XYZ is the diagonal triangle of the quadrangle $ABCD$; a point D' is taken on AD , and $B' = AB \cap YD'$, $C' = AC \cap ZD'$. Prove that $BC \cap B'C' \in YZ$.
- XYZ is the diagonal triangle of the quadrangle $ABCD$; $P = BC \cap YZ$, $L = AD \cap YZ$; $U = BL \cap AP$, $V = BY \cap AP$. Prove that $UX \cap VC \in AB$.
- The altitudes of $\triangle ABC$ are AD, BE, CF , meeting in H . Verify that B, C are two vertices of the diagonal triangle of the quadrangle $AEHF$, and deduce that $EF \cap BC$ is the harmonic conjugate of D with respect to B and C .
- The incentre and the escribed centres of $\triangle ABC$ are I, I_1, I_2, I_3 . Identify the diagonal triangle of the quadrangle I, I_1, I_2, I_3 .

5. Duality

There is a large field of work in which the concept known as *duality* is fundamentally important. This book is not greatly

concerned with such aspects, but a brief account must be given here so that the distinction between a quadrangle and a quadrilateral may be clearly understood.

Much of geometry is concerned with the interrelations of points and lines; for convenience, points may be named by capital letters A, B, C, \dots and lines by lower case letters a, b, c, \dots . Then two points such as A, B define a line, say x ; and two lines such as p, q define a point, say U . The essence of the *principle of duality* is that properties of incidence (such as joins and intersections) for a given figure of points A, B, C, \dots and lines p, q, r, \dots have an exactly analogous counterpart in a figure of lines a, b, c, \dots and points P, Q, R, \dots provided that intersections such as (p, q) are replaced by joins such as PQ while joins such as AB are replaced by intersections such as (a, b) .

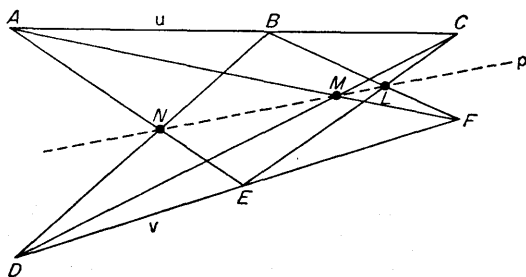


Fig. 48

For example, it can be proved that, *if points A, B, C lie on a line u and points D, E, F on a line v , and if*

$$L = BF \cap CE, M = CD \cap AF, N = AE \cap BD,$$

then L, M, N lie on a line p .

It is an immediate consequence of the principle of duality, or it can be proved directly, that the theorem obtained by interchanging points and lines, joins and intersections, is also true:

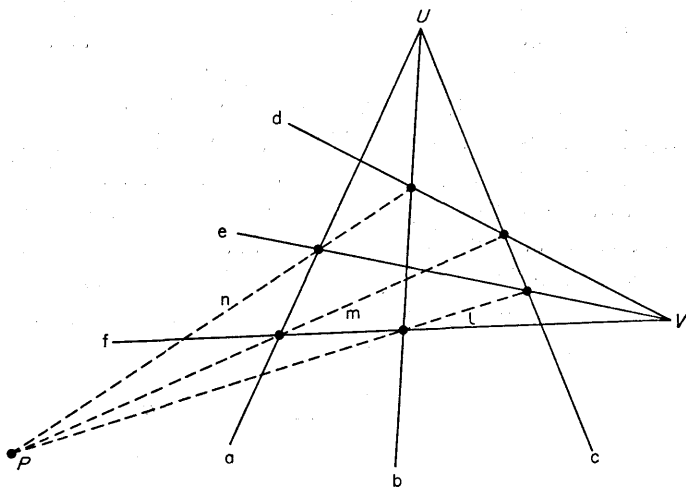


Fig. 49

If lines a, b, c , pass through a point U and lines d, e, f through a point V , and if the line joining the point of intersection (bf) to the point of intersection (ce) is l , with similar notation for m, n , then the lines l, m, n pass through a point P .

6. The Quadrilateral

Definitions. [Compare the dual definitions given in §4.]

The figure formed by four lines a, b, c, d and their six intersections when taken in pairs is called a *quadrilateral*, of which a, b, c, d are the *sides*.

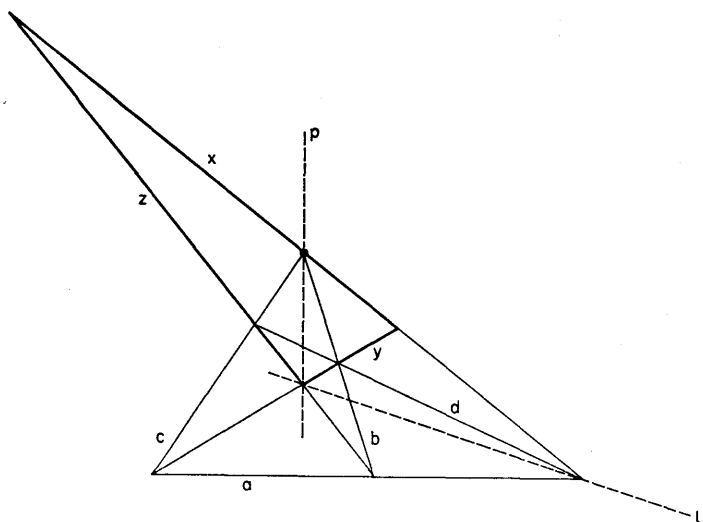


Fig. 50

The six points, which we may conveniently denote by the notation

$$(bc), (ad); (ca), (bd); (ab), (cd)$$

are called the six *vertices* of the quadrilateral; we have grouped them in pairs, called *opposite vertices*. The lines

$$x = \text{join of } (bc), (ad),$$

$$y = \text{join of } (ca), (bd),$$

$$z = \text{join of } (ab), (cd)$$

form a triangle known as the *diagonal* or *harmonic triangle* of the quadrilateral.

Quadrangle and quadrilateral together.

The accompanying diagram can be interpreted to give both a quadrangle and a quadrilateral. As a *quadrangle*, it has vertices

what we have to prove is that $\text{harm. } Y(XZ, UW)$, and this is an immediate consequence of the quadrangle property for the side ZX of the diagonal triangle and its intersections with the lines CA, BD .

COROLLARY. In the "joint" diagram of Fig. 51, the vertices X, Z of the diagonal triangle of the quadrangle are separated harmonically by the vertices U, W of the diagonal triangle of the quadrilateral.

7. Some Illustrations

(i) The polar line of a point with respect to a triangle

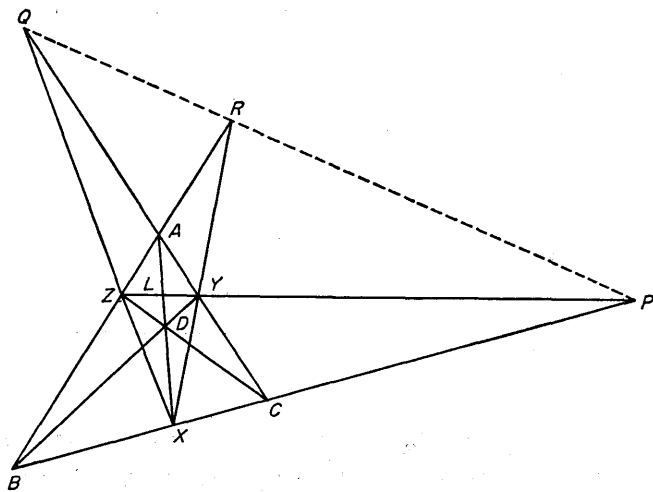


Fig. 52

THE PROPERTY. Let ABC be a given triangle and D a point in its plane. Let $X = BC \cap AD$, $Y = CA \cap BD$, $Z = AB \cap CD$; and then let $P = BC \cap YZ$, $Q = CA \cap ZX$, $R = AB \cap XY$. It is required to prove that P, Q, R are collinear.

Definition. The line PQR is called the *polar line* of D with respect to the triangle ABC .

THE DISCUSSION. If $L = AD \cap YZ$, then XYZ is diagonal triangle of quadrangle $ABCD \Rightarrow$ harm. $(LP, YZ) \Rightarrow$ harm. $X(LP, YZ) \Rightarrow$ harm. (AC, YQ) .
Similarly harm. (AB, ZR) .

But

$$\left. \begin{array}{l} \text{harm. } (AC, YQ) \\ \text{harm. } (AB, ZR) \end{array} \right\} \Rightarrow CB, YZ, QR \text{ collinear}$$

Thus CB, YZ meet on QR , that is,

$$P \in QR.$$

(ii) The angle bisectors of a triangle

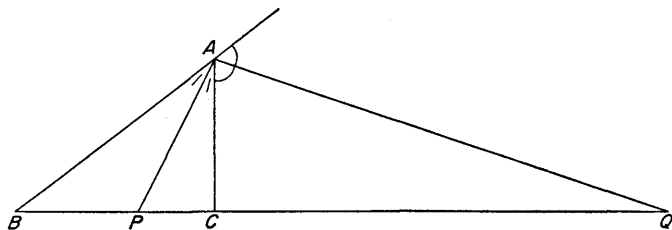


Fig. 53

THE PROBLEM. Given a triangle ABC , let the internal and external bisectors of the angle A meet BC at P, Q . It is required to prove that harm. (BC, PQ) .

THE DISCUSSION:

$$AP \text{ is internal bisector of } \angle A \Rightarrow \frac{\vec{BP}}{\vec{PC}} = + \frac{AB}{AC};$$

$$AQ \text{ is external bisector of } \angle A \Rightarrow \frac{\vec{BQ}}{\vec{QC}} = -\frac{AB}{AC}.$$

Hence

$$\frac{\vec{BP}}{\vec{PC}} = -\frac{\vec{BQ}}{\vec{QC}},$$

so that harm. (BC, PQ) .

8. The Bisection Theorems for a Harmonic Pencil

(i) THE PROBLEM. The pencil $U(AC, BD)$ is given to be harmonic, with B, D separating A, C .

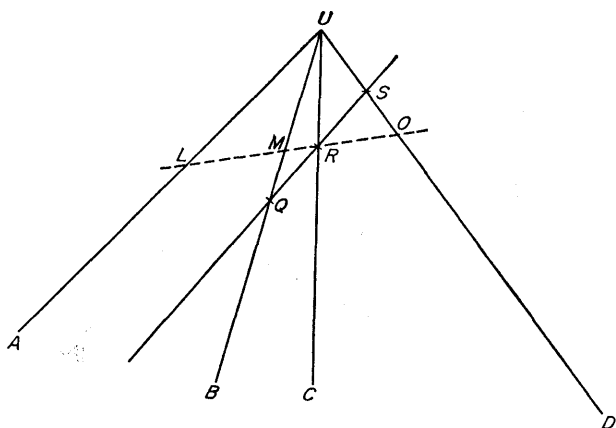


Fig. 54

An arbitrary line is drawn parallel to UA , cutting UB, UC, UD in Q, R, S . It is required to prove that R is the middle point of QS .

THE DISCUSSION. Draw any line through R cutting UA, UB, UD in L, M, O . Then

harm. $U(AC, BD) \Rightarrow$ harm. (LR, MO)

$$\Rightarrow \frac{\vec{LM}}{\vec{MR}} = -\frac{\vec{LO}}{\vec{OR}},$$

so that, numerically,

$$\frac{LM}{MR} = \frac{LO}{OR}.$$

But

$$QR \parallel LU \Rightarrow \frac{LM}{MR} = \frac{LU}{QR}$$

and

$$RS \parallel LU \Rightarrow \frac{LO}{OR} = \frac{LU}{RS}.$$

Hence

$$\frac{LU}{QR} = \frac{LU}{RS},$$

so that

$$QR = RS.$$

(ii) THE PROBLEM. [The converse of Illustration (ii) of §7.]

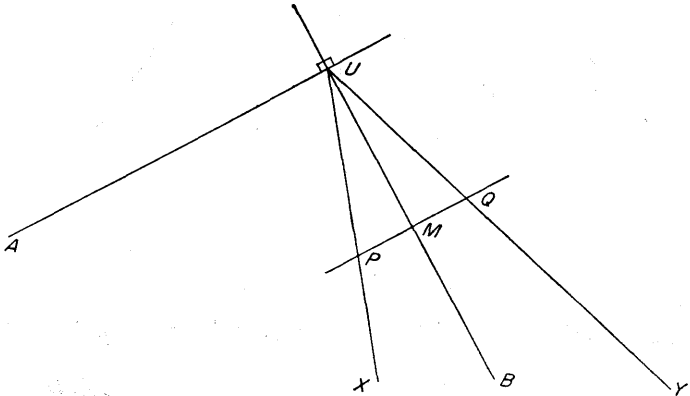


Fig. 55

Let $U(AB, XY)$ be a harmonic pencil, with UX, UY separating UA, UB , and such that

$$AU \perp UB.$$

It is required to prove that UA, UB are the bisectors of angle XUY .

Let a line parallel to UA meet UX, UB, UY in P, M, Q . Then, by the preceding,

$$\left. \begin{array}{l} \text{harm. } U(AB, XY) \\ PMQ \parallel AU \end{array} \right\} \Rightarrow PM = MQ.$$

Also

$$\left. \begin{array}{l} PMQ \parallel AU \\ UM \perp AU \end{array} \right\} \Rightarrow UM \perp PQ.$$

In triangles UMP, UMQ :

$$UM = UM, MP = MQ, \angle UMP = \angle UMQ,$$

so that

$$\triangle UMP \equiv \triangle UMQ,$$

so that

$$\angle PUM = \angle QUM.$$

Hence UM is a bisector of $\angle PUQ$, and so AU , being perpendicular to UM , is the other bisector.

Theorems

1. The converse of §8(i) is also true:

If $U(ABCD)$ is a given pencil, and if a line parallel to UA cuts UB, UC, UD in Q, R, S so that $QR = RS$, then harm. (AC, BD) .

Problems

1. Given a parallelogram $ABCD$ and a line AU such that $AU \parallel BD$, prove that harm. $A(BD, UC)$.

2. $ABCD$ is a parallelogram; P, Q are the middle points of AB, AD . Prove that harm. $B(AD, QC)$ and harm. $D(AB, PC)$, and deduce that $BQ \cap DP \in AC$.

3. XYZ is the diagonal triangle of the quadrangle $ABCD$. The line

through Y parallel to XZ cuts BC , AB , AD , CD in L , N , P , R . Prove that $LN = PR$.

4. Points D , E , F are taken on the sides BC , CA , AB of $\triangle ABC$ so that AD , BE , CF meet in a point O ; $U = EF \cap BC$. The line through D parallel to AU cuts AB , AC in M , N ; the line through D parallel to OU cuts OB , OC in QR . Prove that $MQNR$ is a parallelogram.

5. The diagonals AC , BD of a parallelogram meet in O ; through O are drawn lines l , m so that $l \parallel AB$, $m \parallel AD$, and any point R is taken on m . The line RC meets l in P and DB in U ; the line RD meets l in Q and AC in V . Prove that harm. (RP, CU) and harm. (RQ, DV) , and deduce that $UV \parallel AB$.

9. A Test for a Harmonic Range

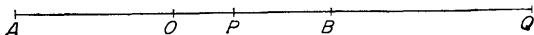


Fig. 56

THE PROBLEM. Let A , P , B , Q be four collinear points, and let O be the middle point of AB . It is required to prove that

$$\text{harm. } (AB, PQ) \Leftrightarrow OA^2 = OP \cdot OQ,$$

the arrow of consequence going in both senses.

THE DISCUSSION. Since O is the middle point of AB ,

$$\vec{AO} = \vec{OB}.$$

Now

$$\begin{aligned} \text{harm. } (AB, PQ) &\Leftrightarrow \frac{\vec{AP}}{\vec{PB}} = -\frac{\vec{AQ}}{\vec{QB}} \\ &\Leftrightarrow \vec{AP} \cdot \vec{QB} + \vec{AQ} \cdot \vec{PB} = 0 \\ &\Leftrightarrow (\vec{AO} + \vec{OP})(\vec{QO} + \vec{OB}) \\ &\quad + (\vec{AO} + \vec{OQ})(\vec{PO} + \vec{OB}) = 0 \end{aligned}$$

$$\begin{aligned}
 &\Leftrightarrow (\vec{OB} + \vec{OP})(\vec{OB} - \vec{OQ}) \\
 &\quad + (\vec{OB} + \vec{OQ})(\vec{OB} - \vec{OP}) = 0 \\
 &\Leftrightarrow \vec{OB}^2 + \vec{OB}(\vec{OP} - \vec{OQ}) - \vec{OP} \cdot \vec{OQ} \\
 &\quad + \vec{OB}^2 - \vec{OB}(\vec{OP} - \vec{OQ}) - \vec{OP} \cdot \vec{OQ} = 0 \\
 &\Leftrightarrow 2\vec{OB}^2 - 2\vec{OP} \cdot \vec{OQ} = 0 \\
 &\Leftrightarrow \vec{OA}^2 = \vec{OB}^2 = \vec{OP} \cdot \vec{OQ}.
 \end{aligned}$$

Problems

1. A, B are two given points; a circle is drawn through A, B and a point P is taken on it; the tangent at P meets the line AB in U . Prove that the circle of centre U and radius UP cuts AB in points X, Y such that harm. (AB, XY) .

2. In $\triangle ABC$, $AD \perp BC$, $BE \perp CA$, $CF \perp AB$, $P = EF \cap BC$ and A' is the middle point of BC . Prove that $A'E$ is the tangent at E to $\odot EDP$.

3. In $\triangle ABC$, $\angle A = 90^\circ$ and $AD \perp BC$. The circle of centre B and radius BA cuts BC in U, V . Prove that harm. (UV, BC) .

4. In $\triangle ABC$, $\angle A = 90^\circ$, $AD \perp BC$ and the tangent at A to $\odot ABC$ meets BC in X . Prove that harm. (BC, DX) .

5. A, B, P, Q are four collinear points such that harm. (AB, PQ) . An arbitrary circle through P, Q cuts the circle on AB as diameter in U, V . Prove that OU, OV are the tangents from O to $\odot PUQV$.

6. l is a given line and A, B two points not on it, both on the same side of l . Prove that \exists two points $U, V \in l$ such that l is a tangent to $\odot ABU$ and to $\odot ABV$.

An arbitrary circle through A, B cuts l in P, Q . Prove that harm. (PQ, UV) .

7. Two circles cut orthogonally. A diameter AB of one cuts the other in points P, Q . Prove that harm. (AB, PQ) .

SIX

Pole and Polar

1. The Polar of a Point with Respect to a Circle

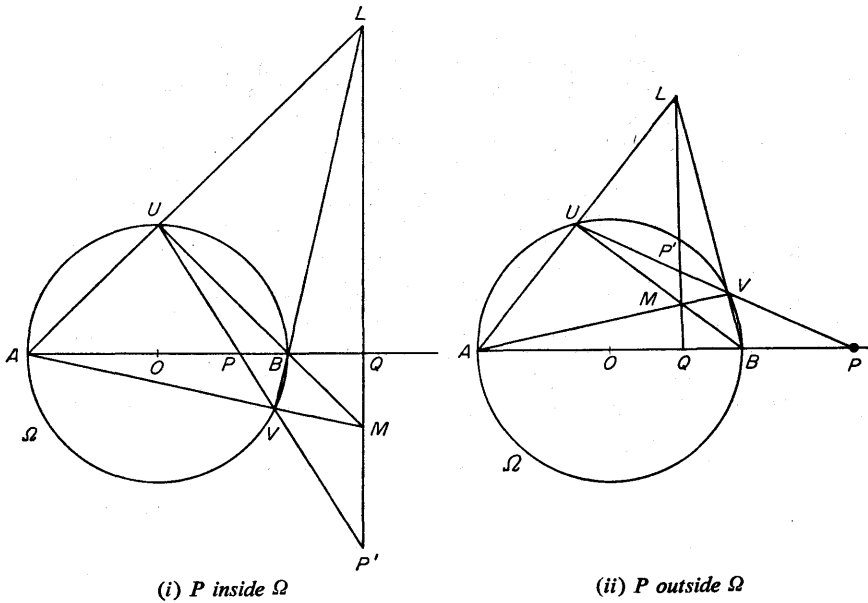


Fig. 57

Definitions. Two points P, P' in the plane of a circle Ω of centre O are said to be *conjugate* with respect to Ω if the line PP' meets the circle in points U, V such that harm. (UV, PP') .

[We shall have to extend this definition later to cover the cases where the line PP' does not meet the circle.]

When P is fixed, the locus of points P' such that P, P' are conjugate with respect to Ω is called the *polar* of P with respect to Ω . The point P itself is called the *pole* of its polar.

THE PROBLEM. It is required to prove that the polar of P is a straight line perpendicular to the diameter through P .

THE DISCUSSION. Draw the diameter through P , meeting Ω in the points A, B . Let an arbitrary chord through P meet the circle in U, V and let P' be the harmonic conjugate of P with respect to U, V ; then P' lies on the polar of P . Finally, let Q be the harmonic conjugate of P with respect to A, B ; in particular, Q lies on the polar of P .

Now

$$\left. \begin{array}{l} \text{harm. } (UV, P'P) \\ \text{harm. } (AB, QP) \end{array} \right\} \Rightarrow UA, VB, P'Q \text{ concurrent, say in } L.$$

Also

$$\left. \begin{array}{l} \text{harm. } (UV, P'P) \\ \text{harm. } (BA, QP) \end{array} \right\} \Rightarrow UB, VA, P'Q \text{ concurrent, say in } M.$$

Further,

AB diameter of Ω

$$\Rightarrow BU \perp AL, AV \perp BL$$

$$\Rightarrow AV, BU \text{ meet in the orthocentre of } \triangle ABL$$

$$\Rightarrow M \text{ is the orthocentre of } \triangle ABL$$

$$\Rightarrow LM \perp AB$$

$$\Rightarrow P'Q \perp AB.$$

Thus P' lies on the line through the *fixed* point Q perpendicular to the *fixed* line AB , so that the locus of P' is † the straight line through Q perpendicular to AB .

† But see §2.

Note:

$$\left. \begin{array}{l} \text{harm. } (AB, PQ) \\ O \text{ middle point of } AB \end{array} \right\} \Rightarrow OA^2 = OP \cdot OQ$$

$$\Rightarrow P, Q \text{ inverse points with respect to } \Omega.$$

Hence the polar of P is the line through the inverse of P with respect to Ω and perpendicular to the diameter through P .

COROLLARY. It is an immediate consequence of the definition that if the polar of a point P passes through a point P' , then the polar of P' passes through P . For if the line PP' cuts† the circle Ω in points U, V , then $\text{harm. } (UV, PP') \Leftrightarrow \text{harm. } (UV, P'P)$.

2. The Case when P is Outside the Circle Ω

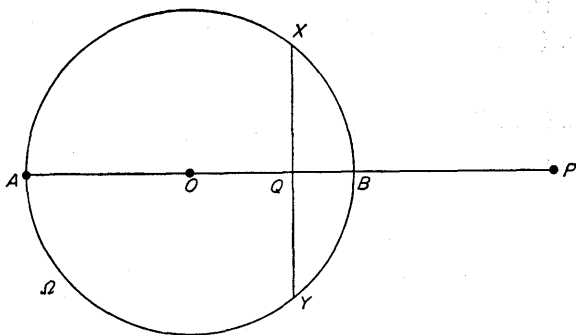


Fig. 58

In §1 we ignored a complication, with which we now deal, occurring when P lies outside the circle Ω . But first we prove a theorem of some importance in itself.

† But see §2.

THE PROBLEM. Let the polar of a point P outside the circle Ω meet the circle in points X, Y . It is required to prove that PX, PY are the tangents from P to Ω .

THE DISCUSSION. As in §1, P, Q are inverse points with respect to Ω . Hence

$$\begin{aligned} OP \cdot OQ &= OX^2 \\ \Rightarrow \frac{OX}{OP} &= \frac{OQ}{OX} \end{aligned}$$

so that, since also $\angle XOP = \angle QOX$,

$$\begin{aligned} \triangle XOP &\sim \triangle QOX \\ \Rightarrow \angle OXP &= \angle OQX \\ &= 90^\circ \\ \Rightarrow XP &\perp OX \\ \Rightarrow XP &\text{ is the tangent at } X. \end{aligned}$$

Similarly YP is the tangent at Y .

The difficulty in the stated definition for the polar of P is that an arbitrary line through P may not meet the circle at all: indeed, it will not do so unless it lies "within" the angle XPY . The locus of P' , as defined, is only *that part of the straight line which lies between X and Y* . We have nevertheless adopted the given definition since it is hoped that the reader will later come to study complex projective geometry where that definition stands and where the difficulty does not arise (because of so-called "imaginary" points of intersection of line and circle).

To resolve the dilemma, such as it is, we can replace the given definition by the equivalent property that P' is on the line through the inverse of P and perpendicular to the diameter through P . All positions of P' are then accounted for, even when the line PP' does not meet Ω .

We must check that, in the excluded case, it is still true that,

Theorem

1. A, B, C, D are four points on a circle, and XYZ is the polar triangle of the quadrangle $ABCD$. Then $\triangle XYZ$ is such that each side is the polar of the opposite vertex. (Compare §3 below.)

Problems

1. A, B, P are three collinear points. Prove that the polars of P with respect to all circles through A, B have a common point.

2. Prove that the angle between the polars of A and B is equal to the angle subtended at the centre by AB .

3. A, B, C, D are four collinear points such that harm. (AB, CD) ; their polars with respect to a circle Ω are a, b, c, d . Prove that the lines a, b, c, d form a pencil such that harm. (ab, cd) .

4. The sides of $\triangle ABC$ are the polars of the vertices of $\triangle PQR$ with respect to a circle Ω . Prove that the sides of $\triangle PQR$ are the polars of the vertices of $\triangle ABC$.

5. Given two points A, B and a circle Ω of centre O ; draw $BX \perp$ polar of A and $AY \perp$ polar of B . Prove that $AY/AO = BX/BO$.

[A possible method is to draw $OM \perp AY$ and $OL \perp BX$; then $\triangle AMO \sim \triangle BLO$.]

Corollary. If $AY = 0$, so that A is on the polar of B , then $BX = 0$, so that B is on the polar of A .

3. Self-Polar (Self-Conjugate) Triangle with Respect to Ω

THE PROBLEM. Let P be any point (inside or outside the circle) in the plane of a circle Ω of centre O , and let Q be any point (inside or outside the circle) on the polar of P . Let the polars of P, Q meet in R . It is required to prove that PQ is the polar of R .

THE DISCUSSION. The proof is direct:

R is on the polar of $P \Rightarrow$ the polar of R is through P ;

R is on the polar of $Q \Rightarrow$ the polar of R is through Q .

Hence the polar of R is PQ .

Definition. A triangle such as PQR , in which each side is the polar of the opposite vertex, is said to be *self-polar* or *self-*

conjugate with respect to Ω . Each two vertices are then conjugate with respect to Ω . The following result is immediate:

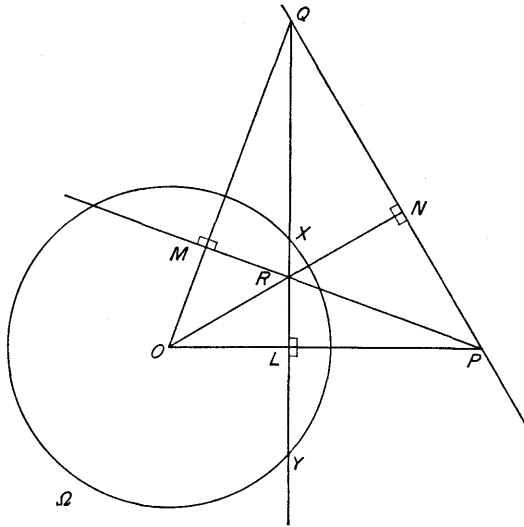


Fig. 60

THE PROBLEM. *To prove that the four points O, P, Q, R are so related that each is the orthocentre of the triangle formed by the other three.*

THE DISCUSSION. QR is the polar of $P \Rightarrow OP$ is perpendicular to QR . Similarly OQ, OR are perpendicular to RP, PQ . By definition of orthocentre, this is the required result.

We now prove a basic property of self-polar triangles:

THE PROBLEM. *It is required to prove that a triangle self-polar with respect to a circle is necessarily obtuse-angled.*

THE DISCUSSION. Observe first that a triangle self-polar with respect to Ω must have at least two vertices outside Ω ; for if there is one vertex at all, say R , inside Ω , then the polar of R lies entirely outside Ω , so that both P and Q must be outside.

If, say, P is one of the outside vertices, then its polar passes through the points of contact X, Y of the tangents from P to Ω and, since Q, R lie on XY and are harmonically conjugate with respect to X, Y , one of them, say Q , is outside and the other, R , is therefore inside. Let OP, OQ, OR meet QR, RP, PQ in L, M, N ; then $L, P; M, Q; N, R$ are inverse pairs with respect to Ω , so that L, P are on the same side of O ; M, Q are on the same side of O ; N, R are on the same side of O . When, however, $\triangle PQR$ is acute, O lies between L, P , between M, Q , and between N, R , so this case is not possible. Hence $\triangle PQR$ is obtuse-angled.

4. Harmonic Pencils on a Circle

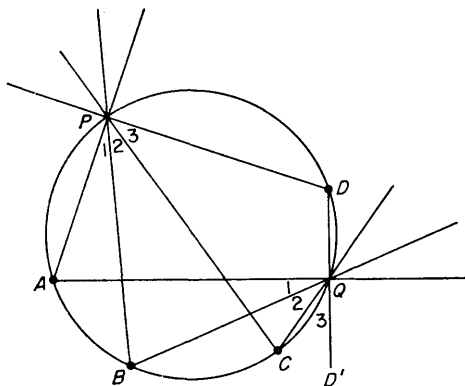


Fig. 61

(i) We start with a lemma:

THE PROBLEM. Let A, B, C, D be four fixed points and P a variable point on a given circle. It is required to prove that the

pencil PA, PB, PC, PD remains constant, in the sense that the angles between those four lines (produced in both directions) have the same values for all positions of P .

THE DISCUSSION. Suppose, for example, that P is on the arc AD as shown. The angles APB, BPC, CPD remain constant for all such positions of P .

If, however, P moves into, say, the arc CD , to the position marked Q , then, if DQ is produced beyond Q to D' ,

$$\angle AQB = \angle APB \text{ (same segment)}$$

$$\angle BQC = \angle BPC \text{ (same segment)}$$

$$\angle CQD' = \angle CPD \text{ (external angle theorem)}$$

so that the pencils can be superposed, in the way implied by the diagram. Corresponding angles are therefore equal.

Note: The result is also true when P is at A, B, C or D , provided that (for, say, P at A) the line PA is interpreted as the tangent at A .

The proof is the same as in the general case save that, if AT is the tangent at A ,

$$\angle TAB = \angle APB$$

because of the theorem on the angle between tangent and chord.

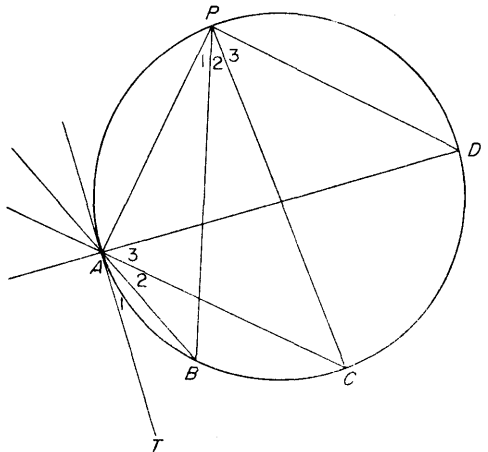


Fig. 62

(ii) Harmonic separation on a circle

We have seen what is meant by harmonic separation on a straight line, and this concept must now be extended from straight line to circle.

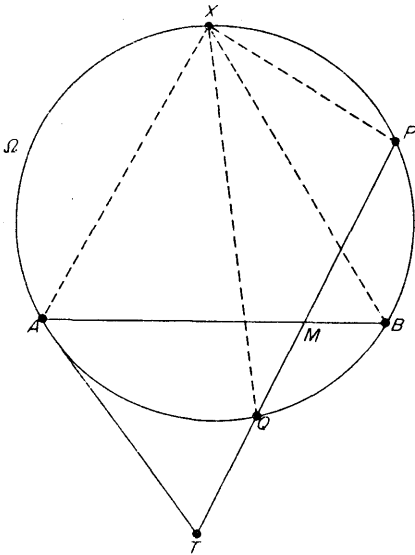


Fig. 63

THE PROBLEM. Let A, B, P, Q be four given points on a circle Ω . If X is a varying point of the circle, then we have just proved that the pencil $X(AB, PQ)$ remains constant in shape for all positions of X . What we are to prove is that *the pencil $X(AB, PQ)$ is harmonic when the four points are so related that each of the chords AB, PQ passes through the pole of the other.*

THE DISCUSSION. Having in mind the note at the end of (i), let the tangent at A meet PQ in T . Let $M = PQ \cap AB$. Then

pencil harmonic \Rightarrow harm. $A(TB, PQ)$
 \Rightarrow harm. (TM, PQ) .

Moreover, if the tangent at B meets PQ in T' , then, in the same way,

harm. $(T'M, PQ)$,

so that T, T' are the same point.

Thus PQ passes through T , the pole of AB ; and, consequently, AB also passes through the pole of PQ .

Note: The converse theorem is also true:

If AB, PQ are chords such that each passes through the pole of the other, then $X(AB, PQ)$ is a harmonic pencil for any position of X on Ω .

It is, indeed, sufficient to take X at A , so that we may refer again to Fig. 63, where T is now known to be the pole of AB . Then

$$\begin{aligned} T \text{ is pole of } AB &\Rightarrow \text{harm. } (TM, PQ) \\ &\Rightarrow \text{harm. } A(TM, PQ) \\ &\Rightarrow \text{harm. } A(AB, PQ) \\ &\Rightarrow \text{harm. } X(AB, PQ) \end{aligned}$$

Definitions. Two chords such that each passes through the pole of the other are said to be *conjugate* with respect to Ω .

Four points such as A, B, P, Q which subtend a harmonic pencil at every point of the circle (so that, as just proved, the chords AB, PQ are conjugate) are said to be *harmonic* on the circle; we also say that A, B separate P, Q harmonically.

Theorem

1. XYZ is the diagonal triangle of a cyclic quadrilateral $ABCD$ (notation of p. 64). The tangents at B, C meet in U , the tangents A, D meet in V , the tangents at A, B meet in L , the tangents at C, D meet in M . Then Y, Z, U, V are collinear and X, Y, L, M are collinear.

Problems

1. AB is a diameter of a circle and CD a chord perpendicular to it. Prove that harm. (AB, CD) on the circle.

Deduce that, if $U = BC \cap AD$, $V = AC \cap BD$, then the tangents at C and D meet on UV .

2. The tangents at points A, B on a circle meet in T and a line through T meets the circle in UV . Prove that, if $L = AU \cap TB$ and $M = AV \cap TB$, then harm. (TB, LM) .

Prove also that, if $X = BU \cap TA$, and $Y = BV \cap TA$, then $LX \cap MY \in AB$ and $LY \cap MX \in AB$.

3. The chords UV, PQ of a circle meet in T and A, B are the points of contact of the tangents from T to the circle. Prove that $PU \cap QV \in AB$ and $PV \cap QU \in AB$.

4. A point U is taken on a circle of which AB is a diameter. The tangents at B, U meet in T , and AU meets BT in S . By first proving that the lines AU, AB divide harmonically the line AT and the tangent at A , prove that T is the middle point of BS .

Prove this result also by elementary geometry.

5. AB is a diameter of a circle of centre O . A line through B cuts this circle in U and also cuts the circle Ω on OB as diameter in V . The line through O parallel to UB cuts Ω again in L ; the line UO cuts the circle Ω again in M . Prove that harm. (LV, MB) on Ω .

SEVEN

Line and Plane

1. Preliminary Ideas

A detailed study of the properties of lines and planes in space is lengthy and, at this stage, somewhat tedious. We propose to pass lightly over some of the things that intuition, not necessarily correctly, regards as obvious, reserving closer study for those matters that are more likely to be found troublesome.

The basic concepts are the *point* and the *straight line*, which we assume to be familiar. From them we derive the *plane*, which is defined to be a surface such that, if A, B are any two points whatever upon it, then the line AB lies wholly in it. Whether such a surface can exist effectively is not as clear as might be thought, but that is a consideration over which we do not linger.

We enunciate without proof a number of *propositions of incidence*, whose truth is to be regarded as evident. Certain complications arising from ideas of parallelism are ignored for the present. (But see below, p. 91.)

- (i) Two lines lying in a plane have a point in common.
- (ii) Two lines meeting in a point lie in a plane; all the points of that plane can be constructed by taking two variable points, one on each given line, and drawing the line joining them (see (iii) below).
- (iii) A unique line joins any two points in space.
- (iv) Two planes in space intersect in the points of a straight line.
- (v) A straight line and a plane meet, in general, in a single point.

(vi) Given a point and a line, there is, in general, a unique plane passing through each of them.

(vii) Three planes meet, in general, in a single point.

(viii) Three non-collinear points determine a unique plane.

Notation: We shall often use capital letters A, B, C, \dots to denote points, small italic letters a, b, c, \dots to denote lines, and Greek letters $\alpha, \beta, \gamma, \delta, \dots$ (alpha, beta, gamma, delta, \dots) to denote planes.

Thus two planes α, β might meet in a line u which, in its turn, might meet a plane γ in a point P .

2. Parallel Lines and Skew Lines

Let l, m be two given straight lines. According to the properties listed in §1, they lie in a plane if they meet and they cannot lie in a plane if they do not meet.

We did, however, indicate in §1 that those properties, while perfectly true in general, were subject to certain exceptions. These we must now study.

Suppose that l, m do indeed lie in a plane. They necessarily intersect, with the one case of exception: they may be *parallel*. We are therefore led to consider two distinct types of non-intersecting lines:

Definitions. Two straight lines which lie in a plane but do not meet are called *parallel*.

Two straight lines which do not lie in a plane (cannot meet and) are called *skew*.

For example, in the "box" shown in the diagram,

$$\begin{aligned} AD &\parallel BC \parallel A'D' \parallel B'C', \\ AB &\parallel DC \parallel A'B' \parallel D'C', \\ AA' &\parallel BB' \parallel CC' \parallel DD'. \end{aligned}$$

Examples of pairs of skew lines are:

$$AB, CC'; A'B', DD'; CD, BB'.$$

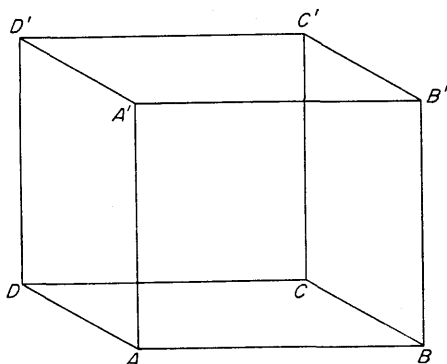


Fig. 64

Theorems

1. If u, v are two skew lines and p, q are two lines each meeting each of them, then p, q are skew.
2. Given two skew lines u, v and a point O not on either, a unique line (a *transversal* of u, v) can be drawn through O to meet both u and v .
3. Given three mutually skew lines u, v, w , a line can be drawn through an arbitrary point of u to cut v and w (a *transversal* of u, v, w). Any two such lines are skew.
4. Two lines each parallel to a third are parallel to one another.
5. If two triangles $ABC, A'B'C'$ in different planes are so related that AA', BB', CC' have a common point O (the triangles then being said to be *in perspective*), the points $L = BC \cap B'C', M = CA \cap C'A', N = AB \cap A'B'$ exist and L, M, N are collinear. (Theorem of *Desargues*.)

Problems

1. A, B, C, D, O are five points in general position in space. A transversal from O meets DB in M and CA in Q ; another meets DC in N and AB in R . Prove that $MN \cap QR \in BC, NQ \cap MR \in AD$.
2. Three mutually skew lines u, v, w are each met by each of three other mutually skew lines p, q, r . Notation such as $p \cap u$ denotes the point common to p, u ; notation such as $p \wedge u$ denotes the plane containing p, u . By considering the lines of intersection of the planes $p \wedge u$,

$q \wedge v, r \wedge w$ taken two at a time, prove that the three lines $(q \cap w, r \cap v), (r \cap u, p \cap w), (p \cap v, q \cap u)$ are concurrent.

3. If u, v are two skew lines and a line p parallel to u meets v while a line q parallel to v meets u , prove that p, q are skew.

4. A line l meets a plane π in a single point O . Prove that l is skew to every line in π not passing through O .

3. The Angle Between Two Lines

Let AB, CD be two skew lines. The angle between them is defined as follows:

Definition. Let O be an arbitrary point, and draw the lines through O parallel to AB, CD . The angle between AB and CD is defined to be the angle between these two (coplanar) lines through O .

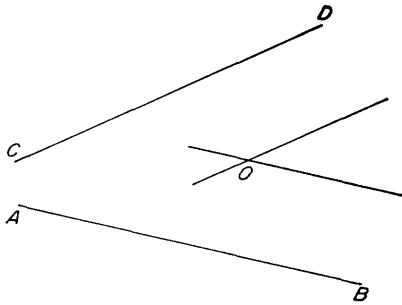


Fig. 65

THE PROBLEM. It is required to prove that the angle so defined is independent of the position of O .

THE DISCUSSION. If O' is an alternative position, choose a point P and a point Q on the two lines through O ; on the corresponding parallel lines through O' , choose P', Q' so that $OP' = OP, OQ' = OQ$. Then $OO'P'P, OO'Q'Q$ are parallelograms, so that PP', QQ' (being equal and parallel to OO') are equal and parallel; hence $PQQ'P'$ is a parallelogram, so that $PQ = P'Q'$.

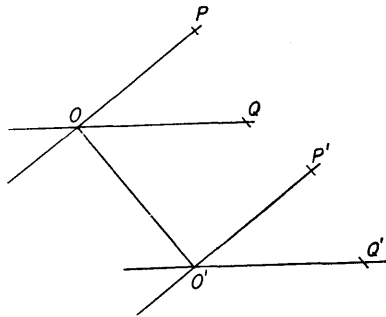


Fig. 66

Hence

$$\begin{aligned}\triangle OPQ &\equiv \triangle O'P'Q' \text{ (3 sides)} \\ &\Rightarrow \angle POQ = \angle P'OQ'.\end{aligned}$$

Note: There are two angles between the lines through O , being supplementary angles. In practice, this seldom causes confusion.

4. Perpendicular Lines and Normals to Planes

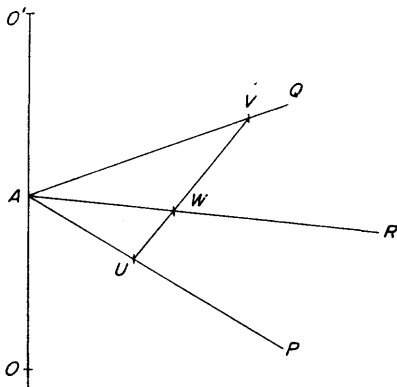


Fig. 67

(i) THE PROBLEM. Let OA be a given line, and AP, AQ two lines each perpendicular to it. The lines AP, AQ define a plane and it is required to prove that OA is perpendicular to every line through A in the plane.

THE DISCUSSION. Let AR be any line through A in the plane APQ . We have to prove that $OA \perp AR$.

Produce OA its own length to O' and let an arbitrary line be drawn to cut AP in U and AQ in V . Then:

$$\begin{aligned}UV \text{ meets } AP, AQ &\Rightarrow UV \text{ is in the plane } APQ \\ &\Rightarrow UV \text{ meets } AR, \text{ say in } W.\end{aligned}$$

Then

$$\left. \begin{array}{l} \triangle OAU \equiv \triangle O'AU \Rightarrow OU = O'U \\ \triangle OAV \equiv \triangle O'AV \Rightarrow OV = O'V \end{array} \right\} \Rightarrow \triangle OUV \equiv \triangle O'UV.$$

But

$$\triangle OUV \equiv \triangle O'UV \Rightarrow \angle OUW = \angle O'UW,$$

and

$$\left. \begin{array}{l} \angle OUW = \angle O'UW \\ OU = O'U \\ UW = UW \end{array} \right\} \Rightarrow \triangle OUW \equiv \triangle O'UW$$

$$\Rightarrow OW = O'W$$

$$\Rightarrow OA \perp AW \text{ (since } OA = O'A)$$

Note that the *converse result* is also true:—

(ii) **THE PROBLEM.** A line OA is given in space, and AP , AQ , AR are three distinct lines perpendicular to it. It is required to prove that AP , AQ , AR are coplanar.

THE DISCUSSION. The lines AP , AQ define one plane and the lines OA , AR define another (different from the first since OA cannot have two distinct lines perpendicular to it and lying with it in one plane). Let the two planes meet in a line AS . Then

$$\left. \begin{array}{l} OA \perp AP, OA \perp AQ \\ AS \text{ in plane } PAQ \end{array} \right\} \Rightarrow OA \perp AS.$$

But

$$\left. \begin{array}{l} OA, AR, AS \text{ coplanar} \\ OA \perp AR, OA \perp AS \end{array} \right\} \Rightarrow AR, AS \text{ coincide.}$$

Hence AR is in the plane APQ .

Definition. Given a point A in a plane, a line through A is said to be *perpendicular* or *normal* to the plane if it is perpendicular to every line through A in the plane.

Test: The line will, in fact, be perpendicular to the plane if it is perpendicular to two lines through A in the plane.

This follows from the preceding work.

(iii) Existence of a normal

THE PROBLEM. Given a point A in a plane, it is necessary to establish the existence of a line which is normal at A to the plane.

THE DISCUSSION. Let AP , AQ be two lines through A in the plane. Draw the plane through A having AP as normal (by drawing two lines through A perpendicular to AP) and the plane through A having AQ as normal. Let these planes meet in a line AO .

Then

$$AO \text{ in plane with } AP \text{ as normal} \Rightarrow AO \perp AP$$

$$AO \text{ in plane with } AQ \text{ as normal} \Rightarrow AO \perp AQ$$

and

$$\left. \begin{array}{l} OA \perp AP \\ OA \perp AQ \end{array} \right\} \Rightarrow OA \perp \text{plane } APQ,$$

so that OA is the required normal.

Theorem

1. There cannot be two distinct lines through a point A normal to a given plane, whether or not A is in the plane.

Problems

1. In the figure of the cube (p. 92) find the angles between the following pairs of lines:

(i) AB , CC' ; (ii) AC , $A'B'$; (iii) BD , $A'C'$; (iv) $A'D$, $B'C'$.

2. Prove that an arbitrary point P on the normal to a plane π at a point A in the plane is equidistant from every point of any given circle of centre A lying in π .

3. Three points A , B , C in a plane π are equidistant from a point O not in π . The centre of $\odot ABC$ is U . Prove that $OU \perp \pi$.

5. Parallel Planes

Definitions. Two planes which do not meet, however far they are produced in any direction, are said to be *parallel*.

Remark: Strictly speaking, we do not assert that such planes can exist; we have merely provided a name for them if they do.

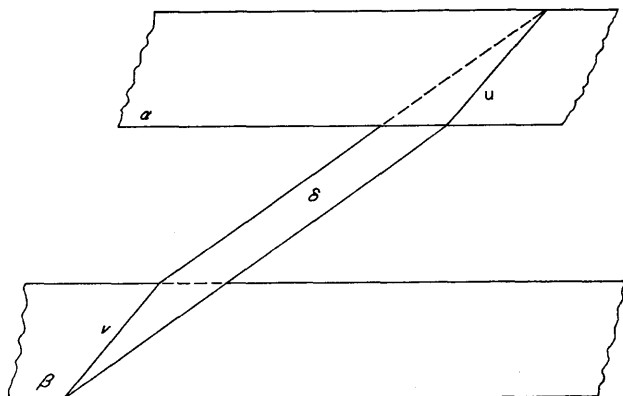


Fig. 68

(i) **THE PROBLEM.** Let α, β be two given parallel planes and let δ be an arbitrary plane meeting them in lines u, v . It is required to prove that $u \parallel v$.

THE DISCUSSION. The lines u, v cannot meet, otherwise each plane α, β would pass through their common point so that the planes could not be parallel. Also, u, v are coplanar, lying in δ . Hence $u \parallel v$.

(ii) **THE PROBLEM.** It is required to prove that two planes which have a common point O have a whole line in common.

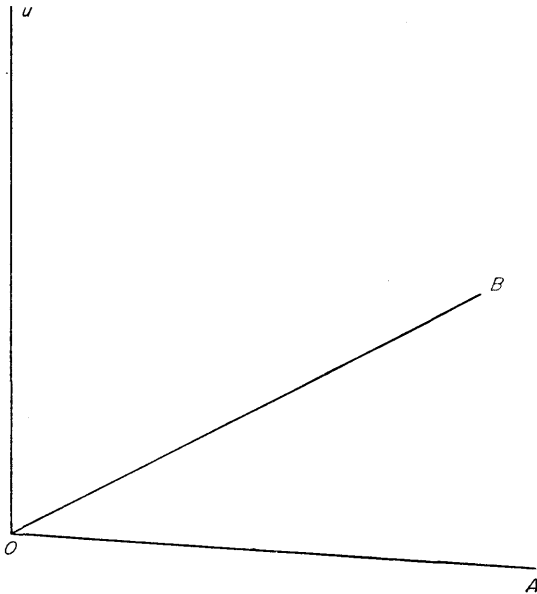


Fig. 69

THE DISCUSSION. Let the given planes α , β (not shown in the diagram) have a common point O , and let OA , OB be the normals at O to α , β . Let u be the line perpendicular to OA and OB .

Then

$$u \perp OA \Rightarrow u \in \alpha,$$

$$u \perp OB \Rightarrow u \in \beta.$$

Hence α , β meet in the line u .

(iii) THE PROBLEM. The actual existence of parallel planes can be confirmed by giving a precise construction:

Let α be a given plane and A a point not in it. It is required to define a plane through A parallel to α .

Take two non-parallel lines $u, v \in \alpha$ and write $P = u \cap v$. In the plane containing A, u draw the line m through A parallel

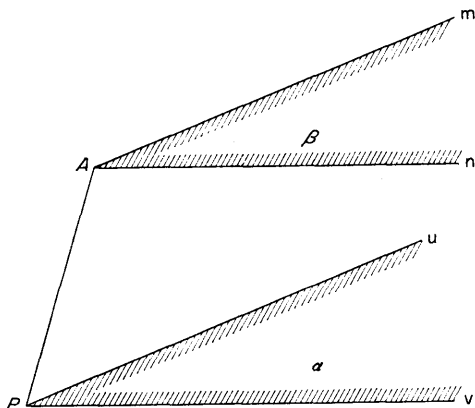


Fig. 70

to u ; in the plane containing A, v draw the line n through A parallel to v . Then *the plane β through m, n is parallel to α .*

THE DISCUSSION. If the planes α, β are not parallel, they meet, by (ii), in a line x . Now the plane through m, u meets x in a point M which, being by definition in α , must lie on u and which, being by definition in β , must lie on m . But $u \parallel m$, so that M cannot exist. Thus x, u lie in α but do not meet; hence $x \parallel u$. Similarly $x \parallel v$. But u, v are not parallel, and so the existence of x is contradicted.

Hence

$$\beta \parallel \alpha.$$

Definition. A line u is said to be *parallel* to a plane α if there is no point common to u and α .

Theorem

1. If $u \parallel \alpha$, then an arbitrary plane through u meets α in a line parallel to u .

(iv) The following result enables us to draw a straight line u through a given point A parallel to a given plane δ . (There are many solutions, all such lines u lying in the plane through A parallel to δ . See the Theorem below.)

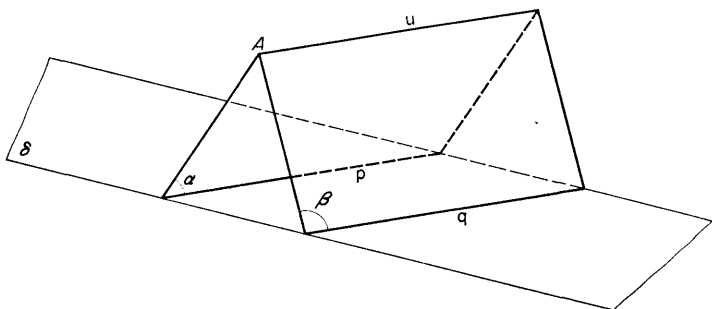


Fig. 71

THE PROBLEM. Let p, q be two parallel lines lying in the plane δ . Draw the planes through A, p and A, q , meeting in a line u , necessarily through A . It is required to prove that the line $u \parallel$ the plane δ .

THE DISCUSSION. Suppose that u does meet δ , say in a point O . Then O , being on u , is also in each of the planes α, β . But

$$\left. \begin{array}{l} O \text{ in } \alpha \\ O \text{ in } \delta \end{array} \right\} \Rightarrow O \text{ on } p,$$

$$\left. \begin{array}{l} O \text{ in } \beta \\ O \text{ in } \delta \end{array} \right\} \Rightarrow O \text{ on } q.$$

But O cannot be on each of the parallel lines p, q , so that the assumption that O exists is false. Thus u does not meet δ , so that $u \parallel \delta$.

Theorem

1. The lines through a given point A parallel to a plane α line in a plane β parallel to α .

6. Properties of Normals

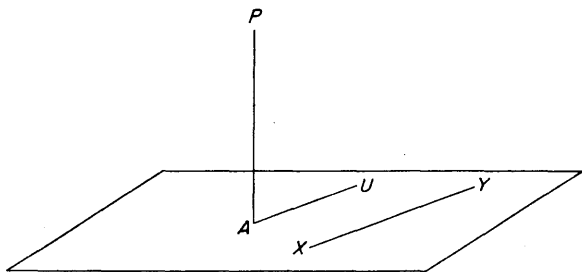


Fig. 72

(i) **THE PROBLEM.** Let AP be the normal to a given plane at a point A . We have proved that AP is perpendicular to every line through A in the plane. We now observe that, more generally, AP is perpendicular to EVERY line in the plane.

THE DISCUSSION. Let XY be a line in the plane, not through A , and draw AU , in the plane, parallel to XY .

Then, by definition (§3), the angle between AP and XY is equal to the angle between AP and AU . Since AP is normal to the plane, this is a right angle.

(ii) **THE PROBLEM.** In a similar way we prove that a line AP is normal to a given plane if it is perpendicular to ANY TWO (non-parallel) lines in the plane.

THE DISCUSSION. Draw through the point A , where the given line meets the plane, the two lines parallel to the lines in the plane. Then, as in (i), AP is perpendicular to each of these lines and therefore normal to the plane.

(iii) THE PROBLEM. Two *perpendicular* skew lines AB , PQ are given. It is required to prove that there exists a unique plane through PQ having AB as normal.

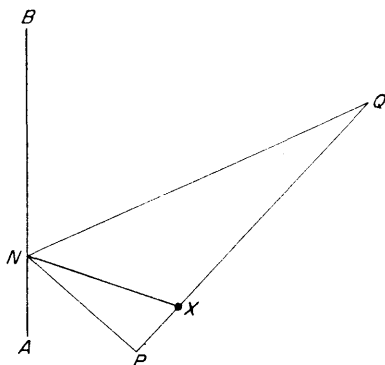


Fig. 73

THE DISCUSSION. Let X be an arbitrary point of PQ , and draw $XN \perp AB$. Then

$$\left. \begin{array}{l} AB \perp PQ \\ AB \perp NX \end{array} \right\} \Rightarrow AB \perp \text{plane } NPQ.$$

The required plane is thus determined.

Also it is unique; for if there were a second plane, cutting AB in N' , then $XN' \perp AB$, which is impossible unless $N' = N$.

(iv) The theorem of the three perpendiculars

THE PROBLEM. Let AB be normal at A to a given plane and PQ an arbitrary line in the plane. It is required to prove that, if $BM \perp PQ$, then $AM \perp PQ$ or, alternatively, that, if $AM \perp PQ$, then $BM \perp PQ$.

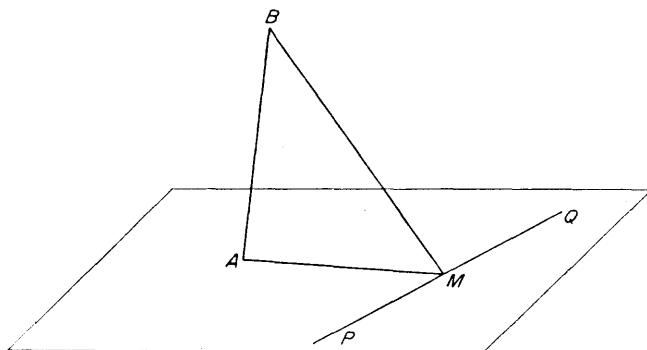


Fig. 74

THE DISCUSSION.

AB normal to the plane $\Rightarrow AB \perp PQ \Rightarrow PQ \perp AB$.

Then

$$\left. \begin{array}{l} PQ \perp AB \\ PQ \perp BM \end{array} \right\} \Rightarrow PQ \perp \text{plane } ABM \\ \Rightarrow PQ \perp AM.$$

Conversely,

$$\left. \begin{array}{l} PQ \perp AB \\ PQ \perp AM \end{array} \right\} \Rightarrow PQ \perp \text{plane } ABM \\ \Rightarrow PQ \perp BM.$$

(v) THE PROBLEM. It is required to prove that all normals to a given plane are parallel.

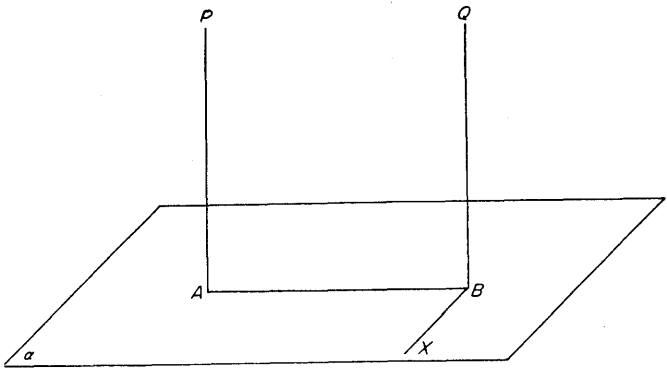


Fig. 75

THE DISCUSSION. Let AP, BQ be normal at A, B to a given plane α . Draw BX in the plane α so that $XB \perp AB$.

Then, by the theorem of the three perpendiculars,

$$\left. \begin{array}{l} PA \perp \alpha \\ AB \perp XB \end{array} \right\} \Rightarrow PB \perp XB.$$

Thus XB is perpendicular to BA, BP, BQ , so that those three lines are coplanar. In particular, AP, BQ are coplanar and, each being perpendicular to AB , are therefore parallel.

6. The Common Perpendicular of Two Skew Lines

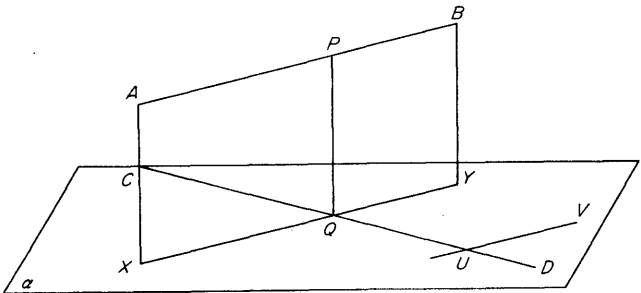


Fig. 76

THE PROBLEM. Two skew lines AB, CD are given. It is required to construct a line PQ meeting them (with $P \in AB$ and $Q \in CD$) so that PQ is perpendicular to both AB and CD .

THE DISCUSSION. Draw through CD the plane α parallel to AB ; this is done by drawing through any point U of CD the line UV parallel to AB and taking the plane $CDUV$. Draw $AX, BY \perp \alpha$, and let XY meet CD in Q . Then the line through Q perpendicular to AB is the required common perpendicular, meeting AB in P .

The proof is immediate:

$$\left. \begin{array}{l} AB \parallel \alpha \Rightarrow AB \parallel XY \\ AB \parallel XY \\ AX \perp XY, BY \perp XY \end{array} \right\} \Rightarrow ABYX \text{ is a rectangle} \\ \Rightarrow XA \perp AB.$$

Also $ABYX$ is a rectangle and PQ is in its plane, so that

$$\left. \begin{array}{l} XA \perp AB \\ QP \perp AB \end{array} \right\} \Rightarrow PQ \parallel AX \Rightarrow PQ \perp \alpha \\ \Rightarrow PQ \perp CD.$$

Theorems

1. If A, B are two given points, the locus of a point P such that $PA = PB$ is a plane to which AB is perpendicular.
2. If A, B, C are three given points, the locus of a point P such that $PA = PB = PC$ is, in general, a line perpendicular to the plane ABC .
3. There is, in general, one single point equidistant from four given points A, B, C, D .
4. (*A converse of the theorem of the three perpendiculars*). Given a point B and a plane α not through it, if PQ is a line in α and $BM \perp PQ$, and if, further, MX is drawn in α so that $XM \perp PQ$, then the line BA such that $BA \perp XM$ is the perpendicular from B to α .
5. If a line u is normal to two distinct planes α, β , then $\alpha \parallel \beta$.
6. The common perpendicular of two skew lines is the shortest distance between them.
7. Two straight lines with two distinct common perpendiculars are parallel.

Problems

1. The common perpendicular of two skew lines u, v meets u in A and v in B . Prove that the locus in space of points equidistant from A, B is a plane α such that $u \parallel \alpha, v \parallel \alpha$.

Prove that $P \in u, Q \in v \Rightarrow$ the middle point of $PQ \in \alpha$.

2. Given three mutually skew lines u, v, w , prove that a line can be found meeting them in U, V, W such that U is the middle point of VW .

3. Given two skew lines u, v and variable points $P \in u, Q \in v$, prove that the locus of the middle point of PQ is a plane to which the common perpendicular of u, v is normal.

4. The common perpendicular of two perpendicular skew lines u, v meets u in P and v in Q . Points A, B are taken on u so that P is the middle point of AB . Prove that every point of v is equidistant from A and B .

5. Given four skew lines, u, v, u', v' such that $u \parallel u', v \parallel v'$, prove that the common perpendicular of u, v is parallel to the common perpendicular of u', v' .

6. Three parallel planes α, β, γ are given; u is a line in α, v a line in β, w a line in γ . Prove that, if β lies between α and γ , then the sum of the common perpendiculars of u, v and of v, w is equal to the common perpendicular of u, w .

EIGHT

Some Standard Solid Bodies

1. The Parallelepiped

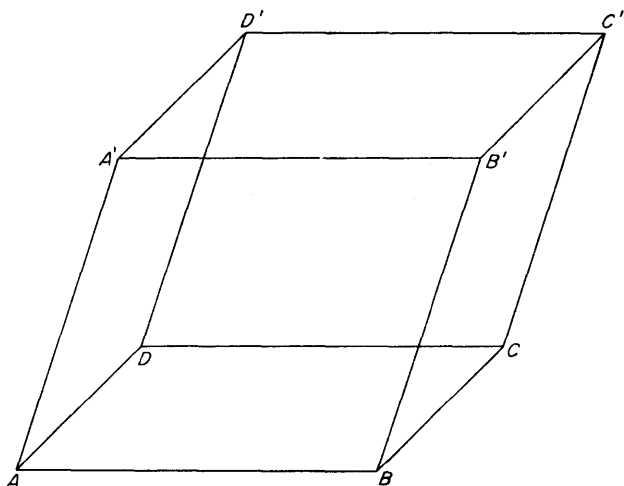


Fig. 77

(The purpose of this account is to make the reader familiar with the basic properties of the figures. The definitions are presented in a correct logical order, but proofs are sketched rather than given in detail since that seems more appropriate to the present level of work.)

(i) Three pairs of parallel planes serve to define a “skew box” known as a parallelepiped. In the diagram, the pairs of parallel planes are

$$\begin{aligned} ABCD &\parallel A'B'C'D', \\ ADD'A' &\parallel BCC'B', \\ ABB'A' &\parallel DCC'D'. \end{aligned}$$

These six planes determine the *faces* of the parallelepiped, in each of which is a parallelogram cut out by the four faces not parallel to it.

There are eight *vertices* $A, B, C, D, A', B', C', D'$ and twelve *edges* grouped in three parallel sets:

$$\begin{aligned} AB &\parallel DC \parallel A'B' \parallel D'C', \\ AD &\parallel BC \parallel A'D' \parallel B'C', \\ AA' &\parallel BB' \parallel CC' \parallel DD'. \end{aligned}$$

There are four *diagonals* AC', BD', CA', DB' , and they *bisect each other* at a point O known as the *centre* of the parallelepiped.

(ii) THE RECTANGULAR PARALLELEPIPED (or “box”). If the three angles at A are all right angles, then all the angles of all the faces are right angles and the parallelepiped is called *rectangular*.

A special feature of this case is that *the four diagonals are all equal*. A sphere can be drawn with centre O to pass through the eight vertices.

(iii) THE CUBE. If the three edges through A of a rectangular parallelepiped are all equal, then all the twelve edges are equal, and the figure is called a *cube*.

2. The Tetrahedron

Four planes in general position serve to define a figure known

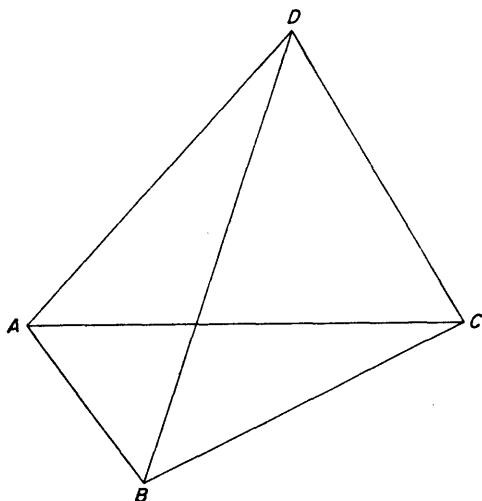


Fig. 78

as a *tetrahedron*. It has four *faces*, each cut in a triangle by the other three, so that the faces are

$$DBC, DCA, DAB, ABC.$$

There are six *edges*, grouped in three *opposite* pairs:

$$AD, BC; BD, CA; CD, AB.$$

The four points A, B, C, D are called *vertices*.

Problems

1. A parallelepiped is named as in Fig. 79 so that A, A' ; B, B' ; C, C' ; D, D' are pairs of opposite vertices. Verify that the parallelepiped can be split up into the five tetrahedra

$$A'BCD + AB'CD + ABC'D + ABCD' + ABCD.$$

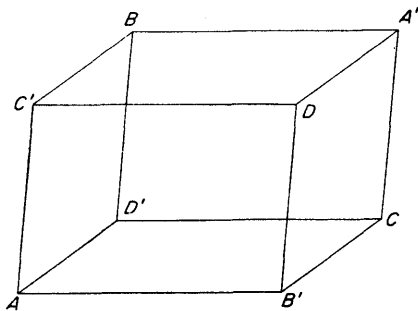


Fig. 79

2. Prove, in the diagram, that $AD'A'D$ is a parallelogram. Hence prove that AA' , BB' , CC' , DD' have a common point O .
3. Prove that, if the parallelepiped is rectangular, then

$$AA'^2 = AB'^2 + AC'^2 + AD'^2.$$

4. Prove that the middle points of AB' , $B'C$, CD' , $D'A$, $A'B$, BC' , $C'D$, DA' are at the vertices of a parallelepiped.

3. Centroid Properties of the Tetrahedron

Consideration of the middle points of the edges of a tetrahedron leads to an exciting chain of properties. Denote by P , Q , R the middle points of BC , CA , AB and by L , M , N the middle points of AD , BD , CD .

(i) THE PROPERTIES. We are to prove that

(a) $QRMN$, $RPNL$, $PQLM$ are parallelograms, whose sides are parallel to the appropriate edges of the tetrahedron.

(b) LP , MQ , NR bisect each other at a point G .

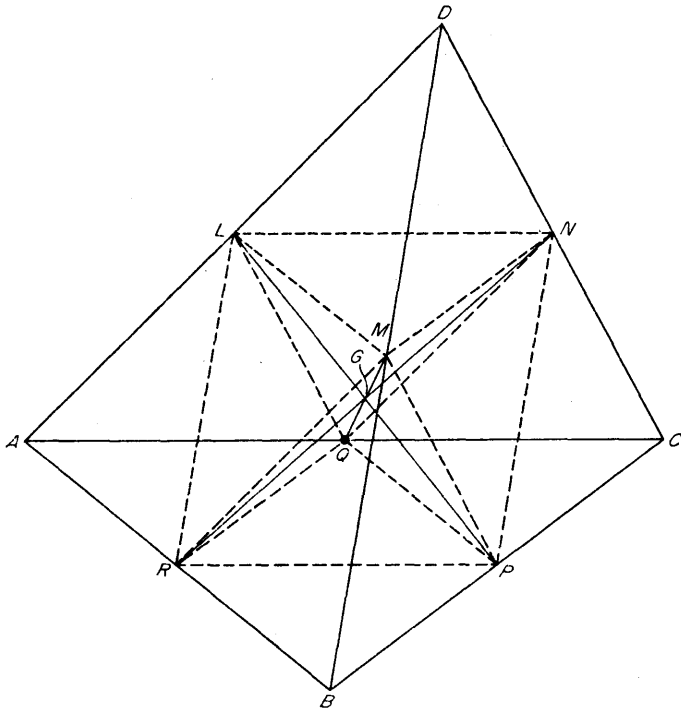


Fig. 80

THE DISCUSSION.

$$\left. \begin{array}{l} AR = RB \\ AQ = QC \end{array} \right\} \Rightarrow QR \parallel BC \text{ and } QR = \frac{1}{2} BC$$

$$\left. \begin{array}{l} DM = MB \\ DN = NC \end{array} \right\} \Rightarrow MN \parallel BC \text{ and } MN = \frac{1}{2} BC.$$

Hence

$$\begin{aligned} & QR \parallel MN, QR = MN \\ \Rightarrow & QRMN \text{ is a parallelogram.} \end{aligned}$$

Similarly

$RPNL, PQLM$ are parallelograms

Further

$QRMN$ is a parallelogram

$\Rightarrow QM, RN$ have a common middle point G .

Also

$RPNL$ is a parallelogram

$\Rightarrow LP, RN$ have a common middle point, which is also G .

Thus LP, MQ, NR bisect each other at G .

(ii) An alternative treatment links these properties from a somewhat different point of view

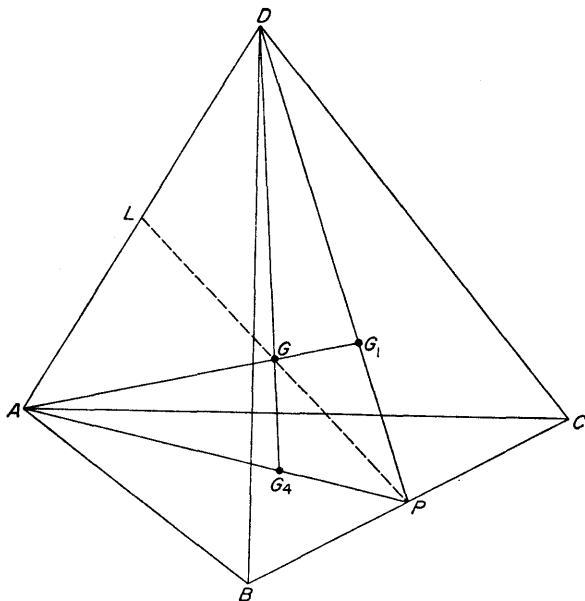


Fig. 81

THE PROBLEM. Let P be the middle point of BC . Then the centroids of $\triangle DBC$, $\triangle ABC$ are points G_1 , G_4 on DP , AP such that $PG_4 = \frac{1}{3}PA$, $PG_1 = \frac{1}{3}PD$. It is required to prove that, if points G_2 , G_3 are defined similarly for $\triangle CAD$, $\triangle ABD$, then AG_1 , BG_2 , CG_3 , DG_4 have a common point G .

THE DISCUSSION. Let AG_1 meet DG_4 in a point which we shall call G .

Then

$$\left. \begin{aligned} PG_4 &= \frac{1}{3}PA \\ PG_1 &= \frac{1}{3}PD \end{aligned} \right\} \Rightarrow G_4G_1 \parallel AD \text{ and } G_4G_1 = \frac{1}{3}AD.$$

$$\Rightarrow G_4G = \frac{1}{3}GD \text{ and } G_1G = \frac{1}{3}GA.$$

Hence AG_1 meets DG_4 in the points of quadrisection furthest from A and D .

Similarly BG_2 , CG_3 , defined in the same way, pass likewise through G .

Thus AG_1 , BG_2 , CG_3 , DG_4 have a common point G which is, for each, the point of quadrisection farthest from the corresponding vertex of the tetrahedron.

(iii) THE PROBLEM. We have to identify the point G as defined in (ii) with the point G as defined in (i). In case (ii), let $L = PG \cap AD$. Then it can be proved by ratios, or by the theorem of Ceva, that L is the middle point of AD ; for example,

$$\frac{\vec{AG}_4}{\vec{G}_4P} \cdot \frac{\vec{PG}_1}{\vec{G}_1D} \cdot \frac{\vec{DL}}{\vec{LA}} = 1$$

$$\Rightarrow \frac{3}{1} \cdot \frac{1}{3} \cdot \frac{\vec{DL}}{\vec{LA}} = 1$$

$$\Rightarrow DL = LA.$$

Thus G , as defined in (ii), lies on AP and, similarly, on BQ , CR . It is thus the same as G as defined in (i).

Definition. The point G is called the *centroid* of the tetrahedron.

4. Orthocentral Properties of the Tetrahedron

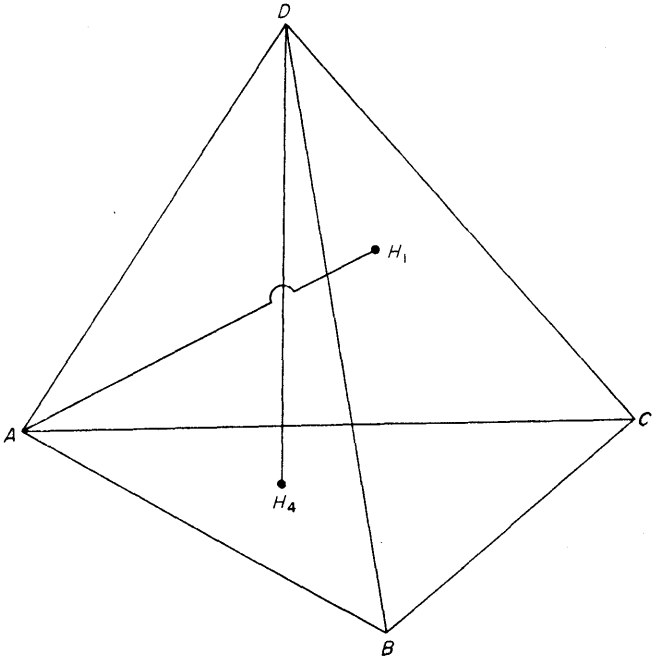


Fig. 82

Definition. The lines AH_1 , BH_2 , CH_3 , DH_4 drawn from the vertices of a tetrahedron perpendicular to the opposite faces are called the *altitudes* of the tetrahedron.

The properties of the altitudes with which we shall be concerned presuppose a specialisation of the tetrahedron which must now be considered. The tetrahedron will be of the type known as *orthogonal*.

THE PROBLEM. Suppose that the tetrahedron has the property that two pairs of opposite edges are perpendicular: say $BD \perp CA$, $CD \perp AB$. It is required to prove that the remaining edges are also perpendicular, so that $AD \perp BC$.

Definition. A tetrahedron whose opposite edges are perpendicular is called *orthogonal*.

THE DISCUSSION. Referring to Fig. 80,

$$\begin{aligned} BD \perp CA &\Rightarrow LR \perp LN \\ &\Rightarrow LNPR \text{ is a rectangle} \\ &\Rightarrow LP = NR. \\ CD \perp AD &\Rightarrow MP \perp ML \\ &\Rightarrow MLQP \text{ is a rectangle} \\ &\Rightarrow LP = MQ. \end{aligned}$$

Hence

$$\begin{aligned} MQ &= NR \\ &\Rightarrow NMRQ \text{ is a rectangle} \\ &\Rightarrow NQ \perp NM \\ &\Rightarrow AD \perp BC. \end{aligned}$$

For the rest of this paragraph, we assume that the tetrahedron $ABCD$ is orthogonal.

(i) **THE PROBLEM.** Let $ABCD$ be an orthogonal tetrahedron, so that

$$AD \perp BC, BD \perp CA, CD \perp AB.$$

It is required to prove that the four altitudes AH_1 , BH_2 , CH_3 , DH_4 meet in a point H .

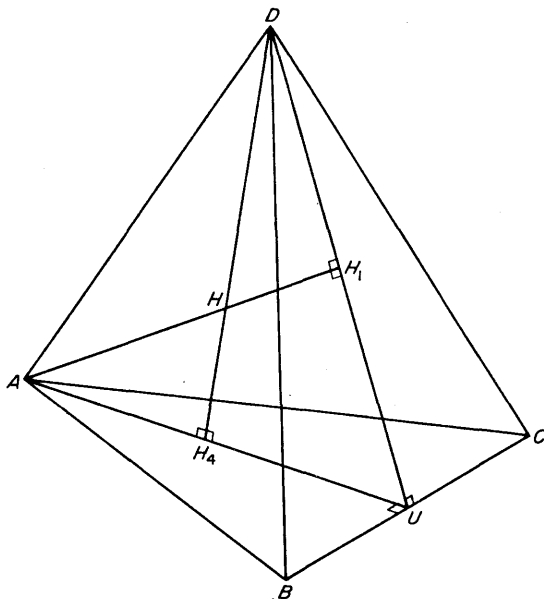


Fig. 83

Definition. The point H is called the **orthocentre** of the tetrahedron.

THE DISCUSSION. Draw $AU \perp BC$. Then

$$\left. \begin{array}{l} BC \perp AU \\ BC \perp AD \end{array} \right\} \Rightarrow BC \perp \text{plane } ADU \Rightarrow BC \perp DU$$

Thus $DU \perp BC$.

Now draw $AH_1 \perp BC$, $DH_4 \perp AU$, and let AH_1 , DH_4 , in the plane ADU , meet in H . Then

$$BC \perp \text{plane } ADU \Rightarrow BC \perp DH_4,$$

and

$$\left. \begin{array}{l} DH_4 \perp BC \\ DH_4 \perp AU \end{array} \right\} \Rightarrow DH_4 \perp ABC.$$

Thus DH_4 is the altitude from D ; similarly AH_1 is the altitude from A .

In other words, AH_1 meets DH_4 .

Similarly BH_2 meets DH_4 . If, then, BH_2 does *not* pass through H , it must lie in the plane of AH_1 and DH_4 : that is, in the plane ADU , which is impossible. Hence BH_2 passes through H . Similarly CH_3 passes through H , so that AH_1, BH_2, CH_3, DH_4 all pass through H .

COROLLARY. H_1, H_2, H_3, H_4 are the orthocentres of $\triangle DBC, \triangle DCA, \triangle DAB, \triangle ABC$. For H_4 is on the altitude from A , and similar argument would have obtained it on the altitudes from B, C ; and similarly for the other triangles.

(ii) There is a test for an orthogonal tetrahedron in terms of the lengths of the sides

THE PROBLEM. It is required to prove that

$$ABCD \text{ is orthogonal} \Leftrightarrow DA^2 + BC^2 = DB^2 + CA^2 = DC^2 + AB^2.$$

THE DISCUSSION. By (i),

$$ABCD \text{ orthogonal} \Rightarrow AU \perp BC, DU \perp BC$$

$$\Rightarrow AB^2 - AC^2 = BU^2 - UC^2 = DB^2 - DC^2$$

$$\Rightarrow DB^2 + CA^2 = CD^2 + AB^2,$$

and the result follows.

If, conversely, $DB^2 + CA^2 = CD^2 + AB^2$,
then

$$DB^2 - DC^2 = AB^2 - AC^2,$$

so that, if

$$DU_1 \perp BC, AU_2 \perp BC,$$

it follows that

$$U_1 \equiv U_2 \equiv U, \text{ say.}$$

Thus

$$\begin{aligned} & BC \perp DU, BC \perp AU \\ \Rightarrow & BC \perp \text{plane } ADU \\ \Rightarrow & BC \perp AD. \end{aligned}$$

Similarly

$$CA \perp BD, AB \perp CD.$$

Theorems

1. If O_1, O_2, O_3, O_4 are the circumcentres of $\triangle DBC, \triangle DCA, \triangle DAB, \triangle ABC$, then the lines through O_1, O_2, O_3, O_4 perpendicular to the planes containing those points have a common point O which is equidistant from A, B, C, D .

Definition. The point O is called the *circumcentre* of the tetrahedron $ABCD$.

2. The diagram shows the section of an orthogonal tetrahedron $ABCD$ by the plane through D and the Euler line $O_4G_4H_4$ of $\triangle ABC$. The circumcentre and centroid of the tetrahedron are O, G and OG meets DH_4 in a point temporarily called H^* . Prove that $GH^* = OG$ and deduce (by symmetry of argument) that H^* is the orthocentre H of the tetrahedron.

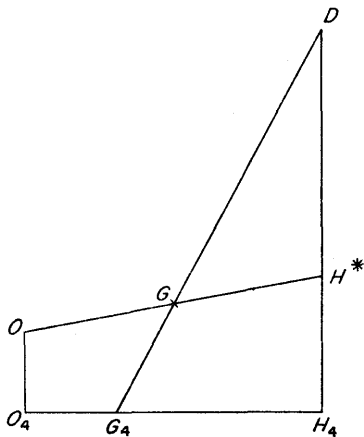


Fig. 84

3. The middle point of the edges of an orthogonal tetrahedron lie on a sphere.

Problems

1. (Notation of §3). Prove that, if $AD = BC, BD = CA, CD = AB$, then LP, MQ, NR are mutually orthogonal.

By proving first that $\triangle LBC$ and $\triangle PAD$ are both isosceles, or otherwise, prove that LP is the common perpendicular of AD, BC .

2. Given a tetrahedron $ABCD$ and a point O , the transversal from O to BC , AD cuts BC in P and AD in L ; the transversal from O to CA , BD cuts CA in Q and BD in M ; the transversal from O to AB , CD cuts AB in R and CD in N . Prove that AP , BQ , CR meet on DO , and that BN , CM , DP meet on AO .

3. Given a tetrahedron $ABCD$, points M , N are taken on CA , AB and points Q , R are taken on DB , DC so that $MN \parallel BC$, $QR \parallel BC$. Prove that $NQ \cap MR \in AD$.

4. A plane meets the edges of a tetrahedron in six points. Prove that they are the vertices of a quadrilateral in the plane.

The tetrahedron is $ABCD$ and the plane meets AB in P , BC in Q , CD in R , DA in S . Prove that (having regard to sign)

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = 1.$$

State and prove the converse of this property.

5. $ABCD$ is a tetrahedron in which $AB = AC$, $DB = DC$. Prove that $AD \perp BC$.

Prove also that any point of AD is equidistant from B and C .

6. $ABCD$ is a tetrahedron in which $DA = DB = DC$. Prove that the foot of the perpendicular from O to the plane ABC is the centre of $\odot ABC$.

NINE

Angles between Lines and Planes.

WE HAVE already (p. 93) considered the angle between two skew lines. We pass now to some further definitions.

1. The Angle Between a Line and a Plane

Let u be a given line and α a given plane meeting it in a point A .

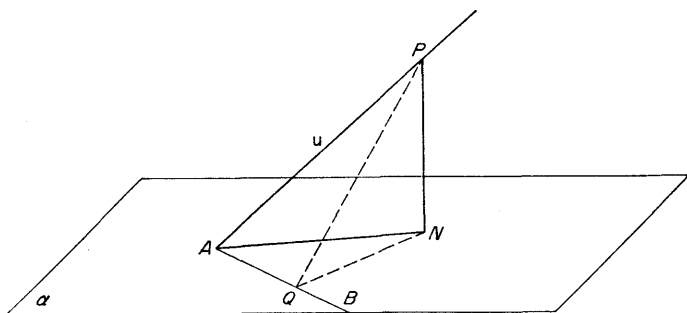


Fig. 85

Definition. The angle between the line u and the plane α is defined to be the angle NAP , where N is the foot of the perpendicular to the plane from an arbitrary point P of the line.

It is an elementary exercise in similar triangles to verify that this angle is independent of the position of P on u .

THE PROBLEM. Let AB be an arbitrary line through A in the plane α . It is required to prove that

$$\angle PAN < \angle PAB$$

and that

$$\angle BAN < \angle PAB.$$

THE DISCUSSION. Draw $PQ \perp AB$. By the theorem of the three perpendiculars, $NQ \perp AB$.

Thus

$$\begin{aligned} PN < PQ \quad (\text{right-angled triangle } NPQ) \\ \Rightarrow \angle PAN < \angle PAQ \end{aligned}$$

and

$$\begin{aligned} QN < PQ \quad (\text{right-angled triangle } NPQ) \\ \Rightarrow \angle QAN < \angle PAQ. \end{aligned}$$

2. The Angle Between Two Planes

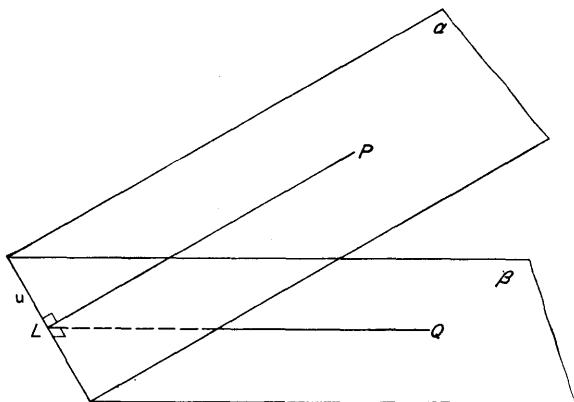


Fig. 86

Let α, β be two given planes meeting in a straight line u .

Definition. The angle between the two planes is defined to be the angle PLQ , where L is an arbitrary point of u and LP, LQ are the lines in α, β perpendicular to u .

Then $PRSQ$ is a rectangle, so that $RS = PQ$, and also, since $\angle LPR = 90^\circ$, $LR > LP$.

Thus, in the language of trigonometry,

$$\begin{aligned}\frac{PQ}{LP} &> \frac{RS}{LR} \\ \Rightarrow \sin \angle PLQ &> \sin \angle RLS \\ \Rightarrow \angle PLQ &> \angle RLS.\end{aligned}$$

Definition. The lines in the plane α which are perpendicular to u are called the *lines of greatest slope* of the plane.

Remark: The language of trigonometry is not essential to the argument, but it helps to make statements more concise.

Problems

1. $ABCD$ is a regular tetrahedron. Prove that, if θ is the angle between DA and the plane ABC , then $\cos \theta = 1/\sqrt{3}$ and that, if ϕ is the angle between the planes DBC , ABC , then $\cos \phi = \frac{1}{3}$.

2. A cube has two parallel square faces $ABCD$, $A'B'C'D'$, so that the edges perpendicular to those faces are AA' , BB' , CC' , DD' . The face $ABCD$ is horizontal. Prove that

(i) the plane $DAB'C'$ makes an angle of 45° with the horizontal,

(ii) if AC' makes an angle θ with the horizontal, then $\sin \theta = 1/\sqrt{3}$,

(iii) if the plane $D'AC$ makes an angle ϕ with the horizontal, then $\cos \phi = 1/\sqrt{3}$.

3. $ABCD$ is a tetrahedron in which $BC = CA = AB = 4$, $DB = DC = 5$, $DA = 3$. Prove that, if θ is the angle between the planes DBC , ABC , then $\cos \theta = 2/\sqrt{7}$, and that, if ϕ is the angle between DB and the plane ABC , then $\cos \phi = \frac{4}{5}$.

4. $ABCD$ is a tetrahedron in which $\angle BDC = \angle CDA = \angle DAB = 90^\circ$, and $DA = a$, $DB = b$, $DC = c$. Prove that the angle between AD and the plane ABC is θ , where $\tan \theta = bc/a\sqrt{(b^2 + c^2)}$, and prove that the angle between the planes DBC , ABC is $90^\circ - \theta$.

Verify that the tetrahedron is orthogonal.

5. $ABCD$ is a tetrahedron in which $BC = CA = AB$ and in which also $DA = DB = DC$. Prove that, if the line AD makes an angle of 60° with the plane ABC , then the angle between the planes DBC , ABC is ϕ , where $\tan \phi = 2\sqrt{3}$.

4. Angles at a Point

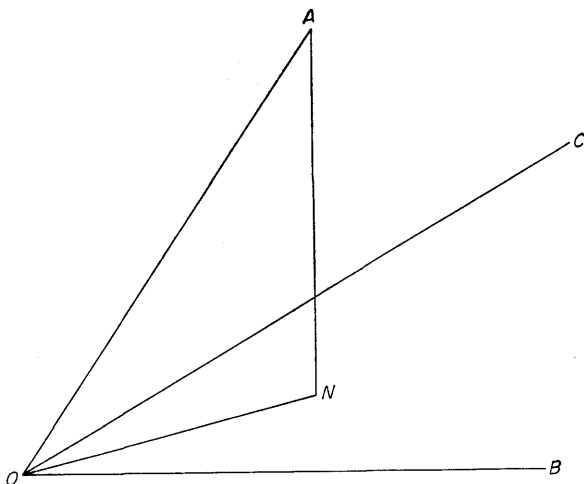


Fig. 88

Let three planes meet at a point O , their lines of intersection in pairs being OA , OB , OC . The three plane angles BOC , COA , AOB form a unit known as a *trihedral angle*.

Two properties are important:

(i) **THE PROBLEM.** It is required to prove that the sum of any two of $\angle BOC$, $\angle COA$, $\angle AOB$ is greater than the third: say

$$\angle AOB + \angle AOC > \angle BOC.$$

THE DISCUSSION. Draw $AN \perp$ plane BOC . Then (pp. 120-1)

$$\angle AOB > \angle NOB,$$

$$\angle AOC > \angle NOC,$$

so that

$$\angle AOB + \angle AOC > \angle NOB + \angle NOC.$$

If, as in the diagram (Fig. 88), ON lies in the angle BOC , it follows at once that

$$\angle NOB + \angle NOC = \angle BOC,$$

so that

$$\angle AOB + \angle AOC > \angle BOC.$$

If ON lies outside the angle BOC , then one or other of $\angle NOB$, $\angle NOC$ is greater than $\angle BOC$, so that

$$\angle NOB + \angle NOC > \angle BOC,$$

and, again,

$$\angle AOB + \angle AOC > \angle BOC.$$

Note: The case of equality occurs only when OA, OB, OC are coplanar with OA "inside" the angle BOC . Then

$$\angle AOB + \angle AOC = \angle BOC.$$

(ii) THE PROBLEM. It is required to prove that

$$\angle BOC + \angle COA + \angle AOB < 360^\circ.$$

By what we have just proved,

$$\angle OAB + \angle OAC > \angle BAC,$$

$$\angle OBC + \angle OBA > \angle CBA,$$

$$\angle OCA + \angle OCB > \angle ACB.$$

Add and rearrange:

$$\begin{aligned} &(\angle OBC + \angle OCB) + (\angle OCA + \angle OAC) + \\ &+ (\angle OAB + \angle OBA) > \angle BAC + \angle CBA + \angle ACB, \end{aligned}$$

so that

$$\begin{aligned} (180^\circ - \angle BOC) + (180^\circ - \angle COA) + (180^\circ - \angle AOB) \\ > 180^\circ, \end{aligned}$$

or

$$\angle BOC + \angle COA + \angle AOB < 360^\circ.$$

5. The Five Regular (Platonic) Solids

THE PROBLEM. The regular tetrahedron has all its faces equilateral triangles and the cube has all its faces squares. It is a matter of interest *to prove that there are only five convex solids whose faces are regular polygons.*

THE DISCUSSION. The possible regular polygons, with the sizes of corresponding angles, are, in the first instance,

triangle	60°
square	90°
pentagon	108°
hexagon	120° , etc.

the angles increasing with the number of sides.

Now there must be at least 3 faces at a vertex, and there may be more. On the other hand, the theorem of §4 can be extended to prove that the sum of the angles at a vertex is in all cases less than 360° . Hence, if there are n faces meeting at a typical vertex, we have

$$n \geq 3$$

and

triangles,	$60n < 360$,
squares,	$90n < 360$,
pentagons,	$108n < 360$,
hexagons,	$120n < 360$.

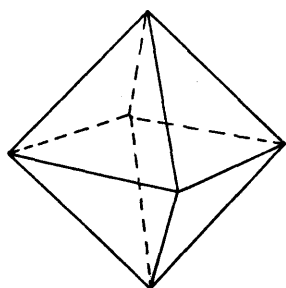
The possibilities are thus:

triangles	3, 4, 5 at a vertex
squares	3 at a vertex
pentagons	3 at a vertex.

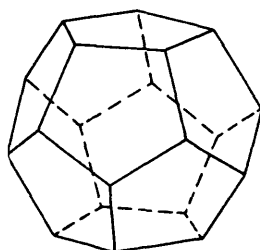
The figures defined in this way are called the *regular* or *Platonic* solids. Several books may be consulted for further details. In particular, the reader will find an excellent account of how to

construct them in *Mathematical Models* by H. Martyn Cundy and A. P. Rollett, Clarendon Press, 1952.

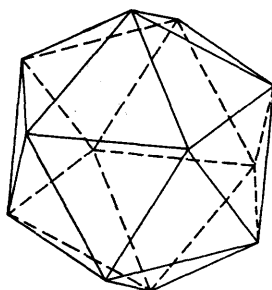
The regular tetrahedron (triangles, $n = 3$) and the cube (squares, $n = 3$) are familiar. The following diagrams of the other solids are based, with permission which we gratefully acknowledge, on *Mathematical Models*.



Octahedron
Triangles, $n=4$



Dodecahedron
Pentagon, $n = 3$



Icosahedron
Triangles, $n=6$

Fig. 89

Problems

1. Copy the cube shown in the diagram and mark the middle points of the edges.

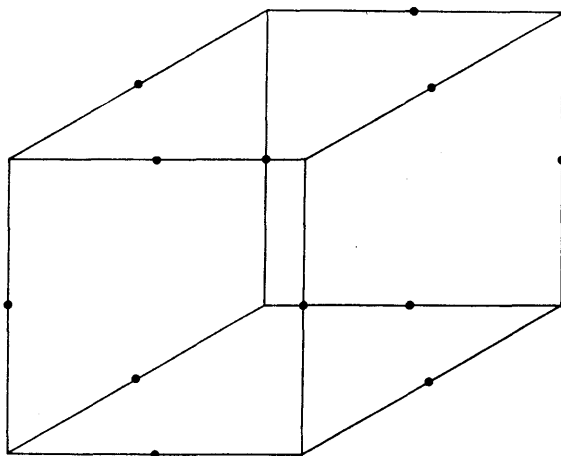


Fig. 90

A new solid is formed by removing the 8 tetrahedra each of which has as its four vertices one vertex of the cube and the three middle points nearest to it. Verify that the resulting solid has 12 vertices, 24 edges, 14 faces of which 6 are square and 8 triangular. Prove also that the sum of the angles at any vertex is 300° .

2. Copy the regular tetrahedron shown in the diagram and mark the points of trisection of the edges.

A new solid is formed by removing the 4 tetrahedra each of which has as its

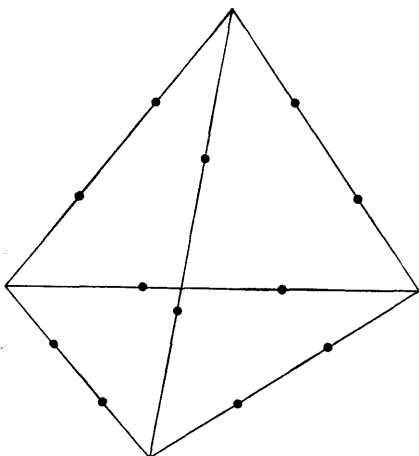


Fig. 91

four vertices one vertex of the given tetrahedron and the three points of trisection nearest to it. Verify that the resulting solid has 12 vertices, 18 edges, 8 faces of which 4 are hexagonal and 4 triangular. Prove that the sum of the angles at any vertex is 300° .

Remark: A famous theorem due to Euler states that, for any such convex body, the number of vertices + the number of faces exceeds by 2 the number of edges.

TEN

The Sphere

1. Definition and First Properties

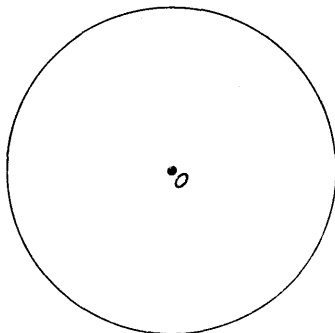


Fig. 92

Definitions. The *sphere* is a surface traced in space by a point whose distance from a fixed point O (the *centre*) has a constant value (the *radius*). Any chord through the centre is called a *diameter* and any plane through the centre is called a *diametral plane*.

(i) **THE PROBLEM** A general plane may or may not meet the sphere. It does meet it when its distance from the centre is less

than the radius. It is required to prove that a plane meeting the sphere does so in the points of a circle.

THE DISCUSSION. Let α be a given plane in the presence of a sphere of centre O and radius a . Draw $OA \perp \alpha$, and let $OA = p$. Then the plane cuts the sphere if $p < a$.

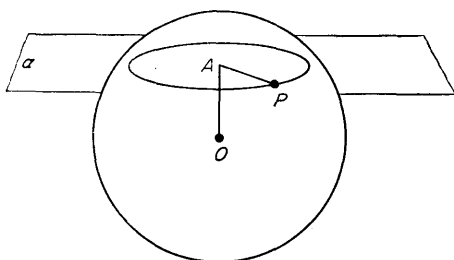


Fig. 93

Now let P be any point common to α and the sphere.

Then $OA \perp \alpha \Rightarrow OA \perp AP$

$$\Rightarrow AP^2 = OP^2 - OA^2 = a^2 - p^2 = \text{constant.}$$

Thus P , lying in α , is at constant distance $\sqrt{(a^2 - p^2)}$ from the fixed point A , so that the locus of P is a circle.

(ii) The tangent plane at a point Q

As a particular case of (i), suppose that $p = a$. Then the foot of the perpendicular from O to the plane is on the sphere, at a point which we now call Q . The plane α is the *tangent plane at Q* , meeting the circle at the point Q , and only at Q . The radius OQ is perpendicular to every line in the tangent plane at Q .

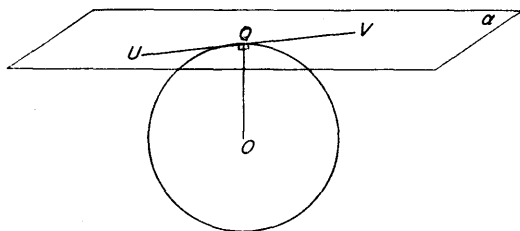


Fig. 94

Theorems

1. Two circles in different planes but having two points in common define a sphere on which both lie.
2. If two spheres pass through a point A and have the same tangent plane there, the distance between their centres is either the sum or the difference of their radii.

Definition. The two spheres are said to *touch at A* .

3. Two spheres which intersect do so in the points of a circle; the distance between their centres is less than the sum of their radii.

Problems

1. The line joining the centres of two circles cut on a sphere by parallel planes is perpendicular to each plane.
2. The centres of the circles of given radius on a sphere lie on a concentric sphere.
3. The centres of the circles on a sphere whose planes pass through a fixed point A lie on the sphere having OA as a diameter, where O is the centre of the given sphere.
4. The larger the radius of a circle on a given sphere, the less is its distance from the centre.

2. Circles on the Sphere

We have seen that a plane cutting the sphere does so in a circle.

Definitions. A circle whose plane passes through the centre of the sphere is called a *great circle*. Its radius is equal to that of the sphere.

A circle whose plane does not pass through the centre of the sphere is called a *small circle*. Its radius is less than that of the sphere.

Given a *great circle* lying in a plane α , the diameter perpen-

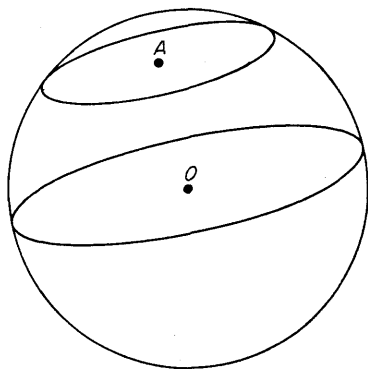


Fig. 95

dicular to a cuts the sphere in two points N, S called the *poles* of the great circle. The plane a is then called the *polar plane* of N and of S . Any plane through the line NS cuts the sphere in a great circle whose plane is perpendicular to a .

It is sometimes convenient, for obvious reasons, to call the great circle in a the *equator* and N, S the *north* and *south* poles. The small circles in planes parallel to a are *circles of latitude* and the great circles through NS are *circles of longitude*.

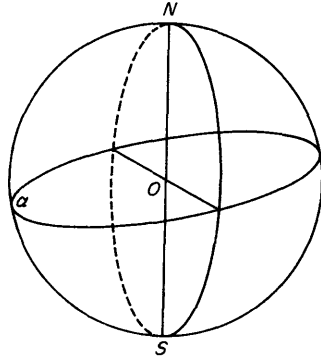


Fig. 96

Problems

1. If A lies in the polar plane of B , then B lies in the polar plane of A .
2. Every point of the sphere has a unique polar plane.
3. Two given points, not at the ends of a diameter, define a unique great circle and so a point on whose polar plane both lie.
4. Given three points A, B, U on a sphere (in general position on it) such that UA, UB both subtend a right angle at the centre O , then U is the pole of the great circle through A, B .

3. Spherical Triangles

It is not our aim to study in great detail the geometry of circles on a sphere, but one or two basic ideas may be helpful.

It is assumed throughout this paragraph that all circles mentioned are great circles.

Let Ω be a given sphere, and draw three diametral planes meeting it in great circles α, β, γ . There are two points common to each pair of circles and they lie at the ends of the diameter

common to their planes. Suppose that β, γ meet in A, A' ; γ, a meet in B, B' ; a, β meet in C, C' . (In the diagram, a is taken as the plane of the paper.)

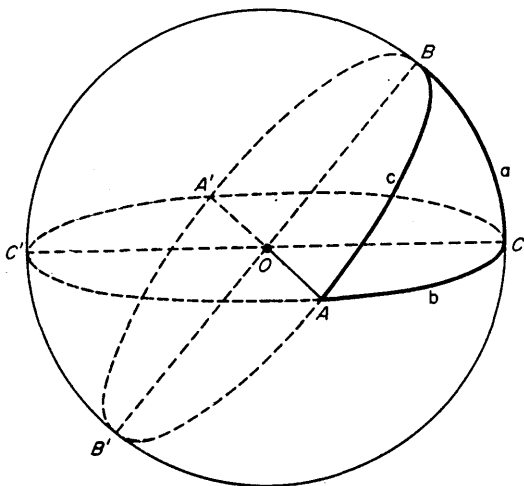


Fig. 97

Definition. The figure on a sphere bounded by arcs of three great circles is called a *spherical triangle*. (But see the last sentence of this paragraph.)

In the diagram, there are 8 triangles, $ABC, A'BC, AB'C, ABC', AB'C', A'BC', A'B'C, A'B'C'$, grouped in diametrically opposite pairs. We focus attention on one of them, say ABC .

For the purposes of calculations, mainly in spherical trigonometry with which we do not deal, the ground sphere Ω is taken to be of unit radius. Thus a *spherical triangle* is defined by three great circles on a sphere of unit radius.

The lengths of the sides of the triangle are the lengths of the

arcs of the defining circles, say $BC = a$, $CA = b$, $AB = c$. Note that the length of the arc BC is, by standard formula, equal to the radius of the circle $BCB'C'$ multiplied by the radian measure of the angle subtended by BC at the centre O . It is in this sense that the side BC is often spoken of as an *angle* (in radian measure), namely the angle subtended by it at the centre when the sphere has unit radius. With this convention, a formula involving $\sin a$ or $\cos a$ has a clear meaning.

The *angles of the triangle* are defined to be the angles between the tangents at the vertices to the defining sides. For example, the angle A is defined to be the angle between the tangents at A to the circles AB , AC . Since these tangents are both perpendicular to OA , this angle is, by definition, the angle between the *planes* containing the circles AB , AC .

Finally, we remark that, given the vertices of a spherical triangle ABC , there still remains ambiguity, since there are *two* arcs of great circles joining, say, B , C . It is agreed by convention that the triangle is so selected that its sides and angles are all less than π .

4. The Polar Triangle of a Spherical Triangle

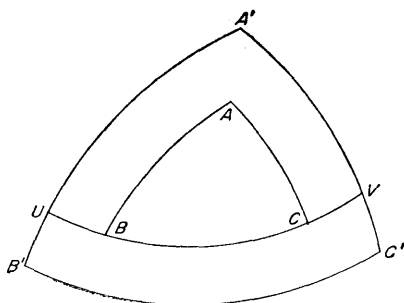


Fig. 98

Given a spherical triangle ABC whose sides a, b, c lie in planes α, β, γ , let A', B', C' be the poles of α, β, γ , selected in each case to lie on the same side of the plane as the corresponding vertex.

Definition. The spherical triangle with vertices A', B', C' is called the *polar triangle* of ABC .

(i) **THE PROBLEM.** It is required to prove that ABC is the polar triangle of $A'B'C'$.

THE DISCUSSION.

B' is the pole of $AC \Rightarrow B'A = \frac{1}{2}\pi$ (on unit sphere)

C' is the pole of $AB \Rightarrow C'A = \frac{1}{2}\pi$.

Thus

$$AB', AC' = \frac{1}{2}\pi.$$

$\Rightarrow A$ is the pole of the great circle $B'C'$.

Moreover,

A' on the same side of BC as $A \Rightarrow AA' < \frac{1}{2}\pi$

$\Rightarrow A$ on the same side of $B'C'$ as A' .

Hence, since similar results hold for B and C , the triangle ABC is the polar triangle of $A'B'C'$.

(ii) **THE PROBLEM.** It is required to prove that, if the sides of the triangle ABC (in radian measure, for a unit sphere) are a, b, c and its angles A, B, C , and if a', b', c', A', B', C' , are the corresponding magnitudes for the polar triangle $A'B'C'$, then

$$a + A' = b + B' = c + C' = \pi,$$

$$a' + A = b' + B = c' + C = \pi.$$

THE DISCUSSION. Let BC meet $A'B'$ in U and $A'C'$ in V . Then

A' is the pole of BC

$$\Rightarrow A'U = A'V = \frac{1}{2}\pi$$

$\Rightarrow UOV$ (where O is the centre of the sphere) is the angle between the planes $A'B'$, $A'C'$

$$\Rightarrow UV = A' \text{ (in radian measure on the unit sphere).}$$

But

B is the pole of $A'C'$

$$\Rightarrow BV = \frac{1}{2}\pi;$$

C is the pole of $A'B'$

$$\Rightarrow CU = \frac{1}{2}\pi.$$

Then

$$BV + CU = \pi$$

$$\Rightarrow (BC + CV) + CU = \pi$$

$$\Rightarrow BC + (CV + CU) = \pi$$

$$\Rightarrow BC + UV = \pi$$

$$\Rightarrow a + A' = \pi.$$

The results $b + B' = \pi$, $c + C' = \pi$ follow similarly. The other three formulae follow at once since ABC is the polar triangle of $A'B'C'$, so that rôles can be reversed.

Theorems

- Two spherical triangles are congruent which have
 - three sides equal,
 - two sides and the included angles equal,
 - three angles equal.
- If a spherical triangle has two sides equal, the corresponding angles are also equal.

Problems

- The angles of a spherical triangle ABC , on a unit sphere, are all $\frac{1}{2}\pi$. Prove that the sides are all $\frac{1}{2}\pi$.

U is the point of BC produced such that $CU = \frac{1}{4}\pi$; V is the point of CB produced such that $CV = \frac{1}{4}\pi$. Prove that UV is a diameter of the sphere, and that a great circle can be drawn through U , V bisecting AB at P and AC at Q .

Prove that the triangles BVP , APQ , CQU are congruent, and deduce that $PQ = \frac{1}{3}\pi$.

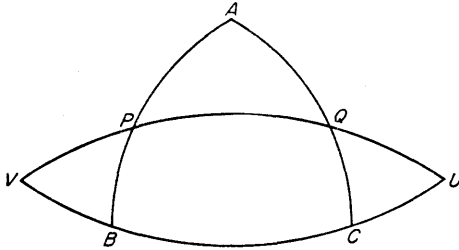


Fig. 99

2. In a spherical triangle ABC it is given that $AB = AC$, and U is the middle point of BC . Prove that $AU \perp BC$.
3. A spherical triangle has all its sides equal. Prove that its polar triangle also has all its sides equal.

5. Area on a Sphere

(i) The area of a lune

Definition. A *lune* on a sphere is that portion of the surface which is cut off between two diametral planes.

(See the diagram, Fig. 100.)

If the angle between the two planes is θ , then we may call θ the *angle of the lune*.

As a matter of simple proportion, the area of the lune bears to the whole sphere the same ratio as θ (in *radian* measure) bears to 2π . Further, it is known that the area of a sphere of radius a is $4\pi a^2$, and so the *area of the lune* is $(\theta/2\pi)4\pi a^2$, or

$$2\theta a^2.$$

In particular, the area of a lune of angle θ on a unit sphere is 2θ .

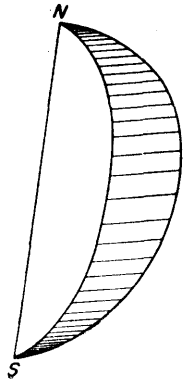


Fig. 100

(ii) The area of a spherical triangle

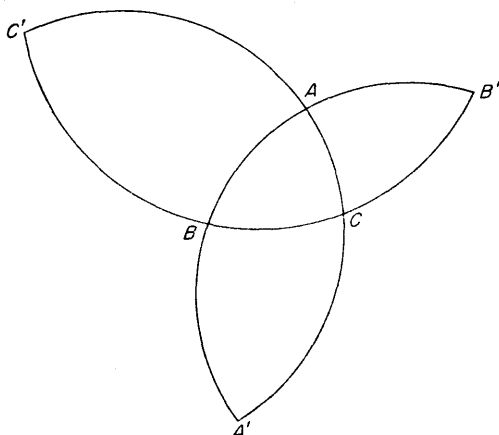


Fig. 101

THE PROBLEM. It is required to prove that the area of a spherical triangle (on a unit sphere) whose angles in radian measure are \hat{A} , \hat{B} , \hat{C} is given by the formula:

$$\text{area} = A + B + C - \pi.$$

THE DISCUSSION. Let the other ends of the diameters through the vertices A , B , C of the given triangle be A' , B' , C' . Then (Fig. 101) $ABA'C$, $BCB'A$, $CAC'B$ are lunes of angles A , B , C so that the sum of their areas is $2(\hat{A} + \hat{B} + \hat{C})$.

Refer now to Fig. 97 (p. 134). The sum of the three lunes is the sum of the areas of the spherical triangles

$$\begin{aligned} & (ABC + A'BC) + (ABC + AB'C) + (ABC + ABC') \\ & = (ABC + A'BC + AB'C + ABC') + 2ABC. \end{aligned}$$

But $\triangle AB'C \equiv \triangle A'BC'$, and so the sum in brackets is (re-arranging)

$$ACB + CA'B + A'C'B + C'AB,$$

which, by inspection, is the sum of the four triangles in the "upper" hemisphere of the diagram. Hence

$$\begin{aligned} 2(\hat{A} + \hat{B} + \hat{C}) &= \text{hemisphere} + 2\triangle ABC \\ &= 2\pi + 2\triangle ABC, \end{aligned}$$

so that

$$\triangle ABC = \hat{A} + \hat{B} + \hat{C} - \pi.$$

COROLLARY. Since the area is necessarily positive, *the sum of the angles of a spherical triangle is greater than π .*

Definitions. The quantity $\hat{A} + \hat{B} + \hat{C} - \pi$ is called the *spherical excess* of the spherical triangle ABC .

Problems

1. If OA, OB, OC are three mutually perpendicular radii of a sphere of unit radius, the area of the spherical triangle ABC is $\frac{1}{2}\pi$.
2. Prove that, in a spherical triangle ABC ,

$$\hat{A} + \hat{B} + \hat{C} > \pi.$$

By considering the corresponding result for the polar triangles, deduce that

$$a + b + c < 2\pi.$$

6. The Right Circular Cone

Definitions. Given a point O and a curve not lying in a plane through it, the surface traced out (*generated*) by lines passing through O and a variable point of the curve is called a *cone* of vertex O , the variable lines being called *generators*.

When the curve is a circle whose centre D is the foot of the perpendicular from O to the plane of the circle, the cone is said to be *right circular*. The line OD is called the *axis* of the cone.

By congruent triangles, the angle between the axis and a generator has a constant value, known as the *semi vertical angle* of the

cone. If $OD = h$ and the radius of the circle is a , then the semi-vertical angle θ satisfies the relation

$$a = h \tan \theta.$$

In some contexts, the cone is regarded as truncated to lie

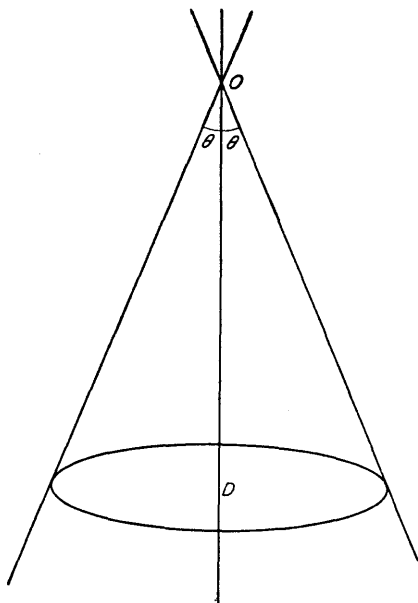


Fig. 102

between O and the plane of the defining circle. Then OD is called the *height*, the circle is called the *base* and the length of the segment of a generator intercepted between O and the base is called the *slant height*.

Theorems

1. A plane through O cuts the cone *either* at O only *or* in two generators. Exceptionally, the plane may cut the cone in a single generator, when

its plane cuts the base in a straight line touching the circle there. Such a plane is called a *tangent plane* to the cone.

2. Every plane parallel to the base cuts the cone in a circle. Each tangent line to the cone cuts such a plane in a line which is a tangent to the circle of section.

3. The tangent lines to a sphere which pass through a point O outside the sphere are the generators of a right circular cone of vertex O . The points common to the cone and the sphere lie on a circle.

7. The Right Circular Cylinder

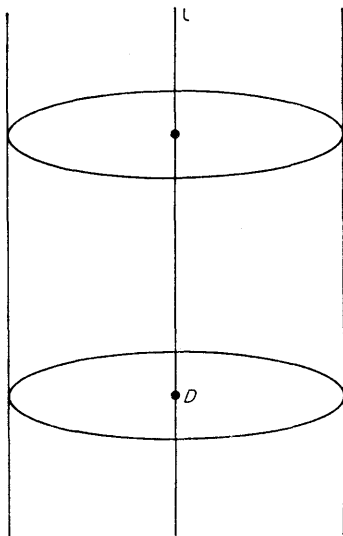


Fig. 103

Definitions. Given a line l and a curve not lying in a plane through it, the surface *generated* by lines parallel to l and passing through a variable point of the curve is called a *cylinder* of which the variable lines are *generators*.

When the curve is a circle whose centre D lies on l and whose

plane is perpendicular to l , the cylinder is said to be *right circular*. The line l is the *axis* of the cylinder.

Theorems

1. The planes perpendicular to the axis cut the cylinder in circles of constant radius.
2. The tangent lines to a sphere which are parallel to a given direction are the generators of a right circular cylinder. The points common to the cylinder and the sphere lie on a circle.

Problems

1. A right circular cone of base radius a and height h is cut by a right circular cylinder of radius $\frac{1}{2}a$ whose axis coincides with that of the cone. Prove that the points common to the cylinder and the cone (extended "beyond its vertex") lie on one or other of two equal circles.
2. A right circular cone has vertex A , axis AB and vertical angle 60° ; another right circular cone has vertex B , axis BA and vertical angle 30° . Prove that the points common to the cones lie on one or other of two circles whose radii are in the ratio 1:2.
3. Prove that the radius of the largest sphere that can be inscribed in a right circular cone of height h and base radius a is

$$a\{\sqrt{(a^2 + b^2)} - a\}/h.$$

4. Prove that the points common to a right circular cone and a sphere whose centre is on the axis lie on one or other of two circles.

Prove the corresponding result when the cone is replaced by a right circular cylinder.

5. Prove that a right circular cone and a right circular cylinder having the same axis meet in the points of two circles (one on either side of the vertex of the cone).

ELEVEN

The Nature of Space

THE THEOREMS outlined in Chapter 1 have a dual purpose: they form a basis for the study of geometrical relationships founded on logical argument, and, even more fundamentally, they seek to describe the structure of space itself in so far as it is concerned with such matters as size and relative position. The ideas of point, line, length, angle, are undoubtedly abstract, but they are designed to agree as closely as possible with the physical world of sight and touch.

There are certain relationships of particular importance which may usefully be emphasised here. It is probable that they will present little that is new, but the attempt to classify them helps towards clearer thinking.

For simplicity of statement we confine ourselves mainly to geometry in a plane.

1. Translation

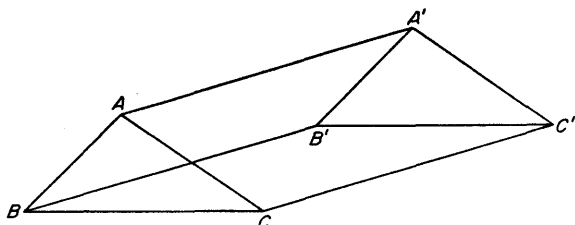


Fig. 104

Let ABC be a given triangle. It may be moved bodily in any given direction (without "rotation", which we shall discuss later) to a new position $A'B'C'$ by moving each of the vertices an agreed distance in that direction; thus

$$AA' \parallel BB' \parallel CC' \text{ and } AA' = BB' = CC'.$$

Definition. Such a movement from position ABC to position $A'B'C'$ is called a *translation*.

It is an immediate consequence of the theorems of Chapter 1 that

$$B'C' = BC, C'A' = CA, A'B' = AB,$$

so that

$$\triangle A'B'C' \equiv \triangle ABC;$$

that is, *the triangle in the new position under a translation is congruent to the triangle in the old position.* The point to be emphasized, though, is that all this arises from our instinctive belief that space itself has such a property—space, so to speak, does not "crinkle".

2. Rotation

Take the triangle ABC as before, and select a point O in its plane. Rotate the triangle about O through an angle θ to a position $A'B'C'$: then

$$OA' = OA, OB' = OB, OC' = OC, \\ \angle AOA' = \angle BOB' = \angle COC' = \theta.$$

Definition. The movement from position ABC to position $A'B'C'$ is called a *rotation with centre O* .

When a position $A'B'C'$ is *known* to be obtainable by rotation from a position ABC , the centre O is easily located: *it lies at the intersection of the perpendicular bisectors of AA' and BB' .*

Here, again, the result of congruence holds:

$$\triangle A'B'C' \equiv \triangle ABC.$$

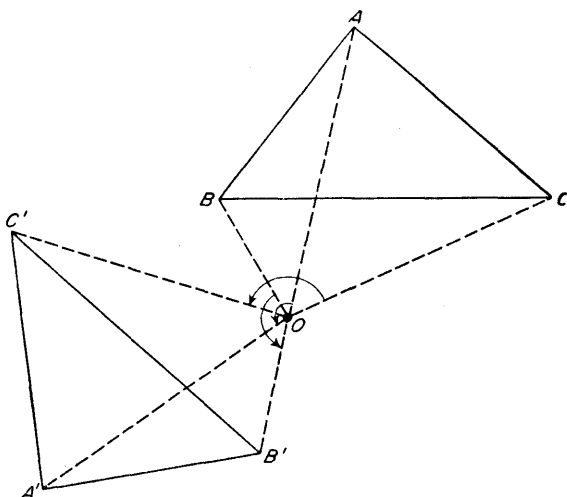


Fig. 105

3. Direct and Inverse Congruence

Experience with left-handed and right-handed gloves will already have convinced the reader that there are two kinds of congruence: one in which a triangle $A'B'C'$ can be moved continuously to a congruent triangle ABC so as to lie on it point for point (A' on A , B' on B , C' on C), and the other in which such superposition is not possible until the triangle has been “turned over” first.

The two cases may be called *direct congruence* and *inverse congruence* respectively.

The “turning over” of the triangle may be described more scientifically as a rotation of 180° about a line l lying in its plane.

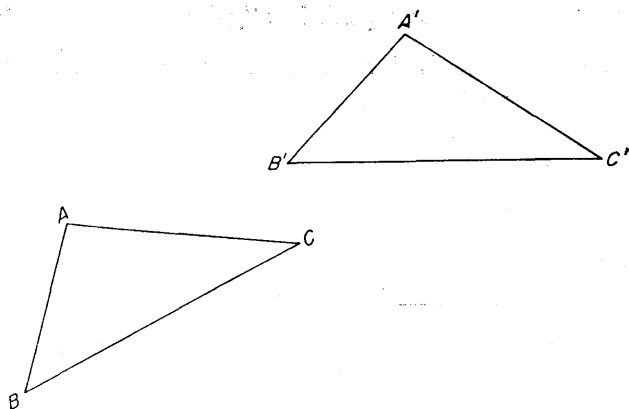


Fig. 106

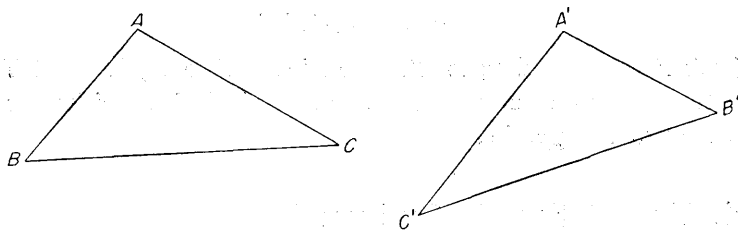


Fig. 107

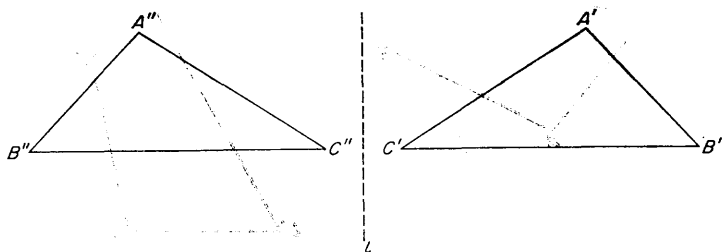


Fig. 108

G

Note. A point may be regarded as “going round” the sides of a triangle if it is conceived as moving along the sides \vec{BC} , \vec{CA} , \vec{AB} in the sense implied by the arrows.

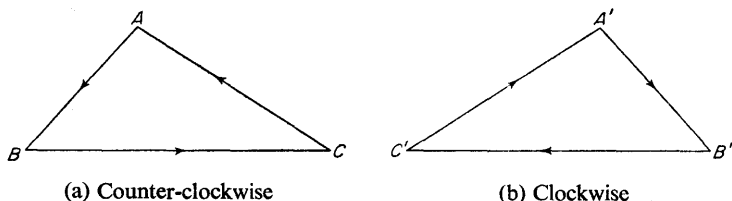


Fig. 109

In Fig. 109 (a) the sense may be called *counterclockwise*, and in Fig. 109 (b) it may be called *clockwise*. Then the congruence

$$\triangle ABC \equiv \triangle A'B'C'$$

(with A corresponding to A' , B to B' and C to C') is direct if both senses are counterclockwise or both clockwise, and inverse if the two are opposite.

4. The Rotation of a Configuration

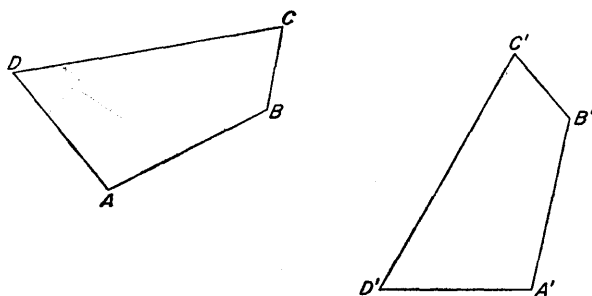


Fig. 110

It is a problem of interest to decide whether two congruent configurations can be rotated the one to coincide with the other. This will certainly not be possible unless the congruence is *direct*.

On the other hand, if the congruence is *direct*, then it is sufficient to consider only two corresponding lines, say $AB, A'B'$; for if $A'B'$ is brought to coincidence with AB , the other points will automatically fall into position.

Suppose, then, that we are given two equal lines $AB, A'B'$. If they *can* be brought to coincidence by rotation, the centre O , being equidistant from A, A' and from B, B' , must lie on the perpendicular bisectors of AA' and BB' .

Let these perpendicular bisectors be constructed, as in the diagram. [We pass over a point of intrinsic difficulty about the relative senses in which the angles $OAB, OA'B'$ turn. The diagram is correct, but it is not easy to prove that it *must* be. For a discussion of the problems involved, see E. A. Maxwell, *Fallacies in Mathematics*, Cambridge University Press (1959) p. 34. The significance of the dotted lines in the diagram will appear later.]

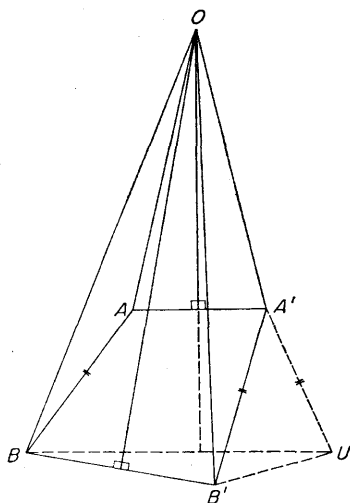


Fig. 111

Now

$$\begin{aligned}
 &O \text{ on perpendicular bisectors of } AA', BB' \\
 &\Rightarrow OA = OA', OB = OB'
 \end{aligned}$$

and

$$\begin{aligned} OA &= OA', OB = OB', AB = A'B' \\ \Rightarrow \triangle OAB &\equiv \triangle OA'B' \\ \Rightarrow \angle BOA &= \angle B'OA' \\ \Rightarrow \angle BOA + \angle AOB' &= \angle AOB' + \angle B'OA' \\ \Rightarrow \angle BOB' &= \angle AOA'. \end{aligned}$$

Hence rotation about O through an angle AOA' (or BOB') carries A to A' and B to B' , so that AB can be rotated† to the position $A'B'$.

Note. The difficulty in this proof is to make sure that the triangles OAB , $OA'B'$ lie on the correct sides of OA , OA' for the additions

$$\begin{aligned} \angle BOA + \angle AOB' &= \angle AOB' + \angle B'OA' \\ \Rightarrow \angle BOB' &= \angle AOA' \end{aligned}$$

to be legitimate. If, however, $\triangle OA'B'$ is rotated about OA' over to the position $OA'U$, then it can be shown that $BU \parallel AA'$, the figure $AA'UB$ being symmetrical about the perpendicular bisector of AA' . Thus U lies "out from" OA' just as B lies "out from" OA ; and B' is therefore in the position shown. But this is hard for a reader at the level of study envisaged and may perhaps be accepted for the present.

5. Expansion; Homothetic figures

Let ABC be a given triangle and O a point in its plane (inside or outside the triangle). Take points A' , B' , C' on \vec{OA} , \vec{OB} , \vec{OC} so that

$$OA' = kOA, OB' = kOB, OC' = kOC.$$

Then it is an immediate exercise in similar triangles that

$$\triangle A'B'C' \sim \triangle ABC,$$

† If, exceptionally, $AB \parallel A'B'$, the argument breaks down.

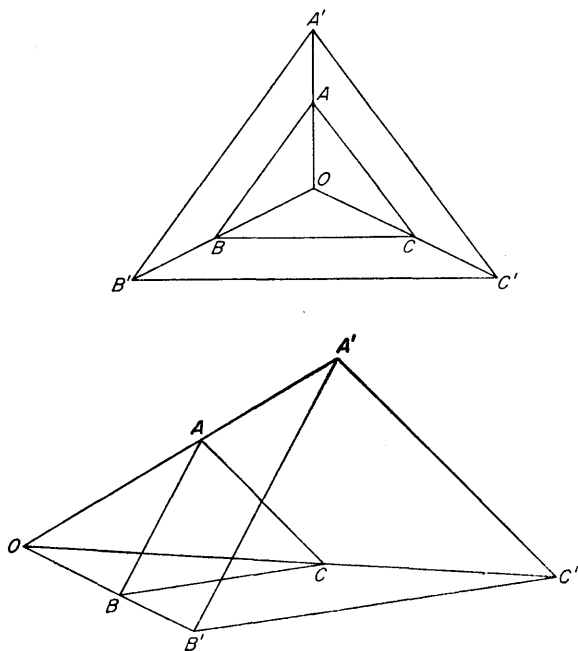


Fig. 112

with

$$B'C' \parallel BC, C'A' \parallel CA, A'B' \parallel AB,$$

$$B'C' = kBC, C'A' = kCA, A'B' = kAB.$$

The triangle $A'B'C'$ is an *expansion* ($k > 1$) or *contraction* ($k < 1$) of $\triangle ABC$.

Definition. Two figures related in this way are said to be *homothetic* or *similar and similarly situated*. They may be called *directly similar* in that the senses of description $\vec{BC}, \vec{CA}, \vec{AB}$ and $\vec{B'C'}, \vec{C'A'}, \vec{A'B'}$ are the same.

Suppose, next, that A', B', C' are taken on $\vec{AO}, \vec{BO}, \vec{CO}$ so that

$$OA' = kAO, OB' = kBO, OC' = kCO.$$

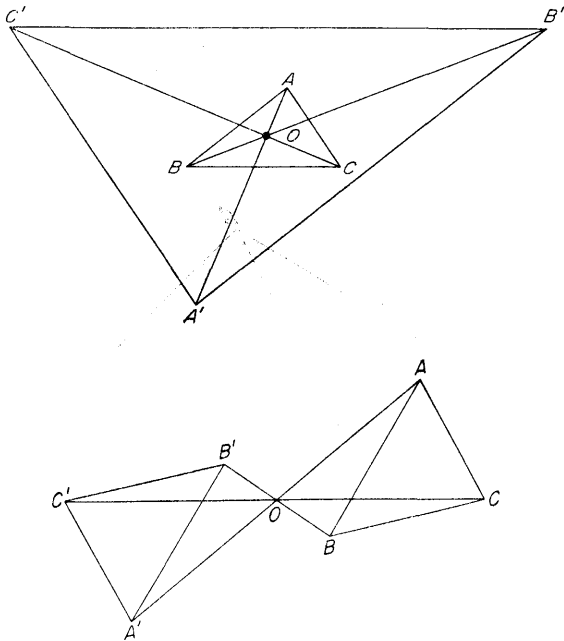


Fig. 113

Then, once again,

$$\triangle A'B'C' \sim \triangle ABC,$$

with

$$B'C' \parallel BC, C'A' \parallel CA, A'B' \parallel AB,$$

$$B'C' = kBC, C'A' = kCA, A'B' = kAB.$$

The triangles may be called *inversely similar* in that the senses of description $\vec{BC}, \vec{CA}, \vec{AB}$ and $\vec{B'C'}, \vec{C'A'}, \vec{A'B'}$ are opposite.

6. Symmetry

Definition. A figure is said to be *symmetric about a line l* if corresponding to each point P of the figure there is another point Q of the figure such that the line PQ is perpendicular to l and bisected by l . The line l is called the *axis of symmetry*.

Typical symmetrical figures are the rectangle, the isosceles triangle, the circle, as indicated in the diagram (Fig. 115).

A figure may have more than one axis of symmetry. For

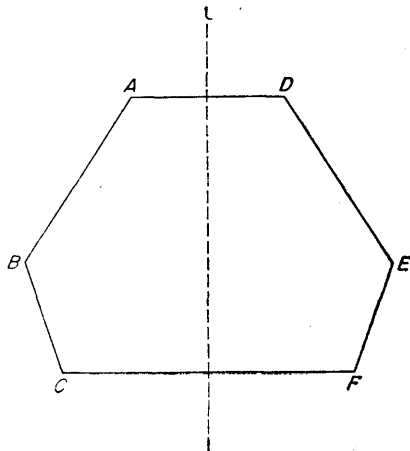


Fig. 114

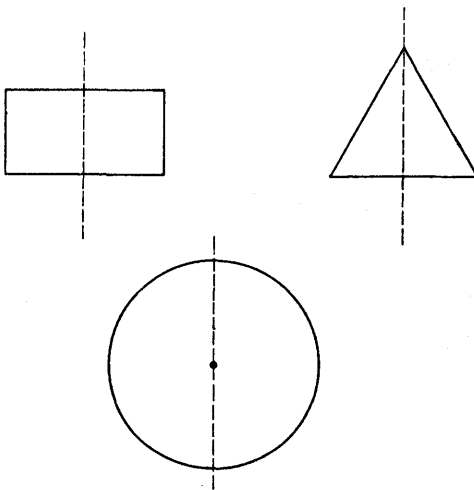


Fig. 115

example, a rectangle has two, an equilateral triangle three and a square four. A circle is symmetrical about every diameter.

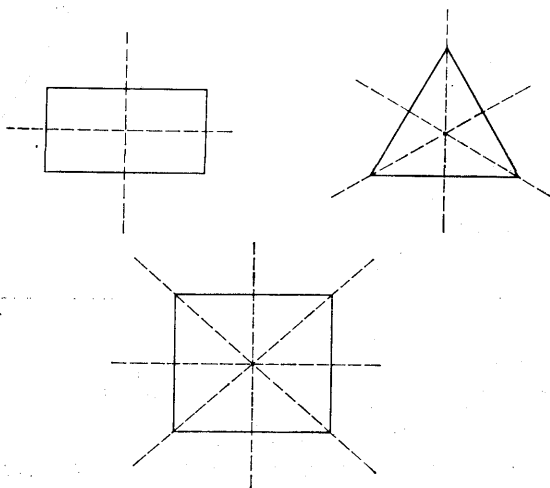


Fig. 116

A figure may also be *symmetric about a point O* (the *centre of symmetry*), when corresponding to each point P of the figure there is another point Q of the figure such that O is the middle point of PQ . Thus a rectangle is symmetric about its centre, and a circle is symmetric about its centre.

Theorem

1. If a figure is symmetric about a line l and a line m , it is also symmetric about the point $O = l \cap m$.

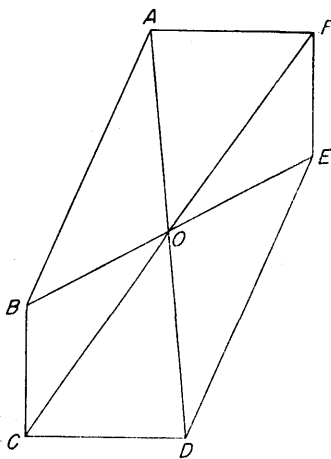


Fig. 117

TWELVE

Transformations

THERE ARE many problems in geometry where a figure F can be brought into close relationship with another figure F' which, at first sight, is very dissimilar. The advantages are twofold: on the one hand, an unexpected unity can be brought to the subject when it is realized that apparently distinct configurations are, basically, just different aspects of one another; on the other hand, it often happens that the properties of one aspect are particularly simple, or particularly familiar, and then the corresponding properties of a more complicated alternative can be "read off" with comparative ease.

1. Orthogonal Projection

Definition. The foot of the perpendicular from a point A on to a given plane π is known as the *orthogonal projection* of A on π .

If A, B are two points whose orthogonal projections on π are A', B' , then it follows directly from the work of Chapter 6 that the orthogonal projection of every point of AB lies on $A'B'$. That is, if $C \in AB$, then $C' \in A'B'$.

More generally, if a figure F lies in a plane α , then the corresponding figure F' in π obtained by taking the orthogonal projections of all the points of F is called the *orthogonal projection* of F on π .

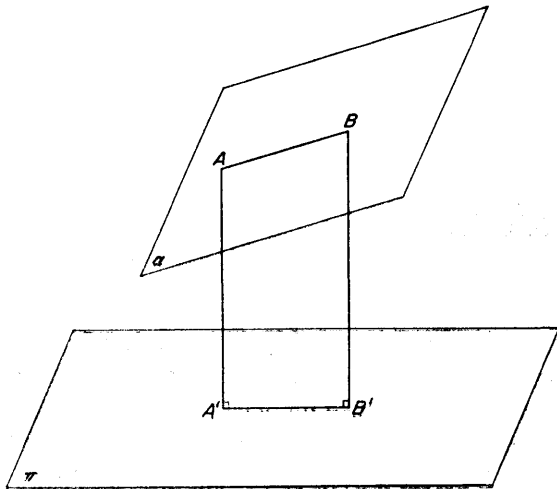


Fig. 118

Theorems

[The points A', B', C', \dots are the orthogonal projections on π of points A, B, C, \dots in α .]

1. A, B, C collinear $\Leftrightarrow A', B', C'$ collinear.
2. $AB|BC = A'B'|B'C'$.
3. If $L = AB \cap CD$, then $L' = A'B' \cap C'D'$.
4. If $AB \parallel CD$, then $A'B' \parallel C'D'$.
5. If $ABCD$ is a parallelogram, then $A'B'C'D'$ is a parallelogram.
6. A harmonic range (AB, CD) projects orthogonally into a harmonic range $(A'B', C'D')$.
7. Let the planes α, π meet at an angle θ , and denote by l their line of intersection. Then
 - (i) the length of the projection $A'B'$ of a line AB parallel to l is unchanged, so that $A'B' = AB$;
 - (ii) the length of the projection $C'D'$ of a line CD perpendicular to l is reduced in the ratio $\cos \theta$, so that $C'D' = CD \cos \theta$.
8. If a triangle ABC in the plane α is projected into a triangle $A'B'C'$ in π , then $\text{area } A'B'C' = \text{area } ABC \times \cos \theta$.

Problems

1. A line AB lies in the plane α and makes an angle ϕ with the line of intersection of α, π . Its orthogonal projection on π is $A'B'$, and θ is the angle between the planes α, π . Prove that

$$\left(\frac{A'B'}{AB}\right)^2 = \cos^2 \phi + \sin^2 \phi \cos^2 \theta.$$

Prove also that, if ψ is the angle between $A'B'$ and the line of intersection of the planes, then

$$\tan \psi = \tan \phi \cos \theta.$$

2. Prove that, in general, if $ABCD$ is a rectangle, then $A'B'C'D'$ is not a rectangle.
3. Show how to project (a) a rectangle, (b) a rhombus orthogonally into a square.

2. Conical Projection

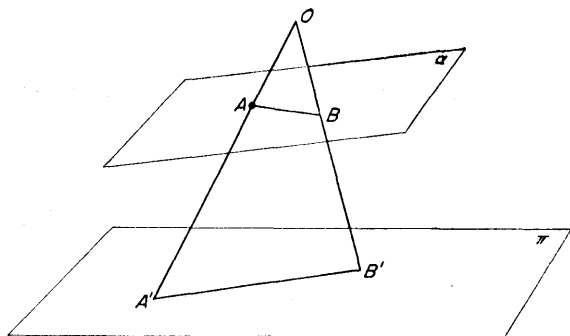


Fig. 119

Definition. Let O be a given point (the *vertex of projection*) and π a given plane (the *plane of projection*). If A is any point in general position in space, then the point where the line OA meets π is called the *projection* (or *conical projection*) of A on π ; it is usually denoted by the name A' .

If A, B are two given points and A', B' their projections, then the projection of any point of AB lies on $A'B'$.

(i) The vanishing plane

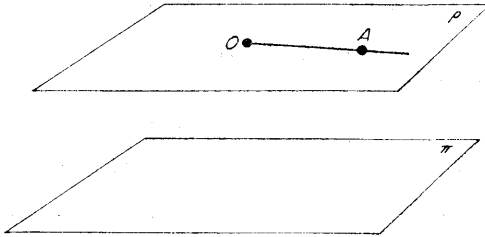


Fig. 120

Definition. The plane ρ through O parallel to π is called the *vanishing plane* for the projection. The reason for the name is simple: if A is any point in the plane, the line OA cannot meet π and so the projection A' is non-existent.

For a figure lying in a general plane α , the intersection of α

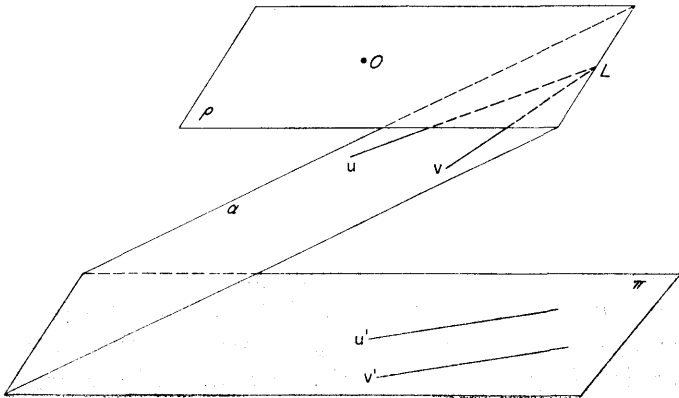


Fig. 121

with the vanishing plane ρ is called the *vanishing line in the plane α for the projection on π by means of the vertex O .*

In particular, two lines u, v of α which meet on the vanishing line project into two parallel lines u', v' in π :—

For an intersection L' of u', v' would arise from the intersection L of u, v . But, by definition of the plane ρ , L' cannot exist, and so u', v' are parallel.

(ii) THE PROBLEM. It is required to project a given quadrangle $ABCD$ into a parallelogram $A'B'C'D'$.

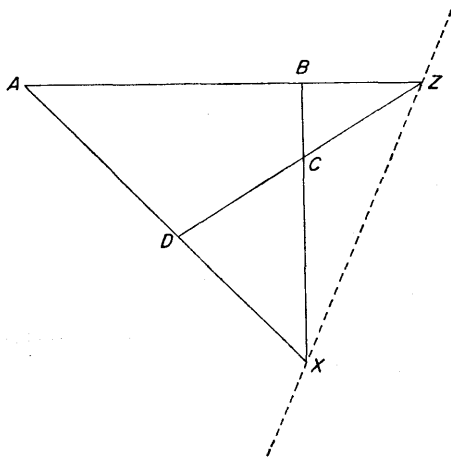


Fig. 122

THE DISCUSSION. Write $X = BC \cap AD$, $Z = AB \cap CD$.

Through XZ draw any plane ρ , and select any point O in it. Take any plane π parallel to ρ . Then, by what we have just done,

$$B'C' \parallel A'D', A'B' \parallel C'D',$$

so that $A'B'C'D'$ is a parallelogram.

(iii) THE PROBLEM. Given two lines u, v , intersecting in A , lying in the plane α and meeting the vanishing line in points U, V , it is required to prove that the angle between u' and v' is equal to $\angle UOV$.

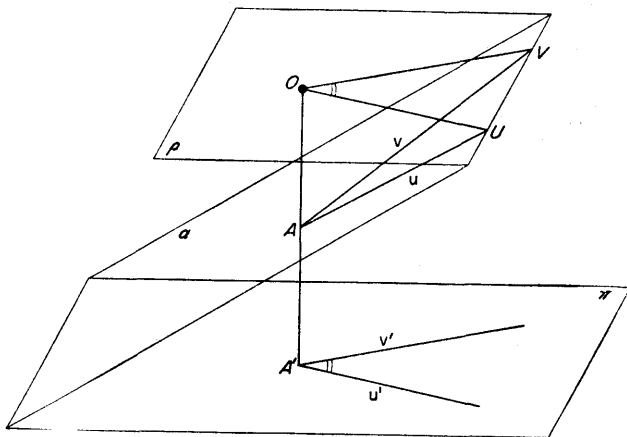


Fig. 123

THE DISCUSSION. By definition of conical projection, OAA' , u, u' are coplanar, so that

$$OU \parallel u'$$

and OAA', v, v' are coplanar, so that

$$OV \parallel v'.$$

Hence the angle between u', v' is $\angle UOV$.

(iv)]To project a given quadrangle into a square

THE PROBLEM. Given a quadrangle $ABCD$, it is required to find a vertex and plane of projection such that $A'B'C'D'$ is a square.

THE DISCUSSION. A parallelogram is a square if one angle, say $D'A'B'$, is a right angle and if the angle $B'U'C'$ between the diagonals is also a right angle.

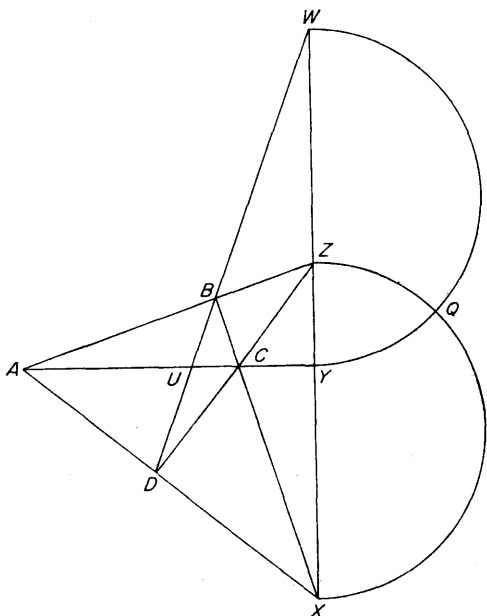


Fig. 124

Let $X = BC \cap AD$, $Y = CA \cap BD$, $Z = AB \cap CD$, $W = BD \cap YZ$. Following on (iii), take XYZ as vanishing line. In order to get the right angles, let Q be one of the points of intersection of the circles on XZ , YW as diameters. Rotate the lines QX , QY , QZ , QW about XYZ out of the plane α of $ABCD$ so that Q now assumes a position O , the plane OXW being the vanishing plane ρ . Any plane π parallel to ρ can then be taken as the plane of projection.

The proof is immediate, since

$$\angle D'A'B' = XOZ = \angle XQZ = 90^\circ,$$

and

$$\angle B'U'C' = \angle WOY = \angle WQY = 90^\circ.$$

Theorem

1. A harmonic range (AB, CD) projects conically into a harmonic range $(A'B', C'D')$.

If, however, A is on the vanishing line for the projection, then B' is the middle point of $C'D'$.

Problems

1. Prove by the conical projection of a quadrangle $ABCD$, with the side XZ of the diagonal triangle as vanishing line, that the diagonals of a parallelogram $A'B'C'D'$ bisect each other.

2. (AB, UP) and (AC, VQ) are harmonic ranges. Prove that UV, BC meet on PQ .

By conical projection of this figure with PQ as vanishing line, prove that the line joining the middle points of the sides of a triangle is parallel to the base.

By conical projection making $UVCB$ a parallelogram, prove that the line joining the middle points of one pair of opposite sides of a parallelogram is parallel to the other pair.

3. Two triangles ABC, PQR are so related (in *perspective*) that AP, BQ, CR have a common point O ; $L = BC \cap QR, M = CA \cap RP, N = AB \cap PQ$. Prove that, in a projection with MN as vanishing line, the projections $B'C', Q'R'$ of BC, QR are parallel. Deduce that $L \in MN$ (Theorem of *Desargues*).

4. A, B, C and P, Q, R are two sets of collinear points on distinct lines; $L = BR \cap CQ, M = CP \cap AR, N = AQ \cap BP$. Prove that, in a projection with MN as vanishing line, the projections $B'R', C'Q'$ of BR, CQ are parallel. Deduce that $L \in MN$ (Theorem of *Pappus*).

5. Establish the equivalence of the following theorems:

(a) $ABCD, A'B'C'D'$ are two quadrangles so related that $BC \cap AD = B'C' \cap A'D' = X, AB \cap CD = A'B' \cap C'D' = Z$.

Then, if $BD \cap B'D' \in XZ$, it follows that $AC \cap A'C' \in XZ$.

(b) $ABCD, A'B'C'D'$ are two parallelograms so related that $AB \parallel CD \parallel A'B' \parallel C'D'$ and $AD \parallel BC \parallel A'B' \parallel C'D'$.

Then, if $BD \parallel B'D'$, it follows that $AC \parallel A'C'$.

Give independent proofs of each of the results.

6. $ABCD$ is a quadrangle; $X = BC \cap AD$, $Y = CA \cap BD$, $Z = AB \cap CD$. A line through Y meets AB in P , CD in R ; another line through Y meets BC in Q , AD in S . By means of a projection with XZ as vanishing line, prove that $PQ \cap RS \in XZ$, $PS \cap QR \in XZ$.

7. ABC is a triangle. Parallel lines AP , BQ , CR meet BC , CA , AB in P , Q , R ; $L = QR \cap BC$, $M = RP \cap CA$, $N = PQ \cap AB$. Prove that $L \in MN$.

8. A , B , O are three non-collinear points, and parallel lines AU , BV are drawn; $P = OA \cap BV$, $Q = OB \cap AU$, R is the point where the line through O parallel to AU and BV meets AB ; $L = QR \cap BV$, $M = RP \cap AU$, $N = PQ \cap AB$. Prove that $L \in MN$.

9. A line LMN meets the sides BC , CA , AB of $\triangle ABC$ in L , M , N . Points A' , B' , C' are chosen on BC , CA , AB so that harm. $(A'L, BC)$, harm. $(B'M, CA)$, harm. $(C'N, AB)$. By means of a projection with LMN as vanishing line, prove that AA' , BB' , CC' are concurrent.

10. Establish the following interpretation of the problem stated in converse in question 9:

$B'CC'B$ is a parallelogram whose diagonals BC , $B'C'$ meet in L ; a line through L meets $B'C$ in M and BC' in N . If the line is chosen so that $NB = BC'$, then $MC = CB'$.

3. Inversion

Definition. Recall, first, the definition of inverse points. Given a circle Ω of centre O and radius a , the *inverse* of a point A with respect to Ω is the point A' (on the same side of O as A) such that $OA \cdot OA' = a^2$.

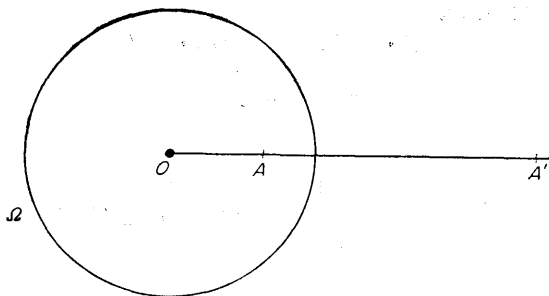


Fig. 125

In many problems the actual circle Ω is irrelevant so long as its centre O is known. We then speak of *inversion with respect to O* .

If A moves on some such curve as a straight line or a circle, the locus of A' is called the *inverse curve* of the given curve with respect to Ω .

(i) The magnification theorem

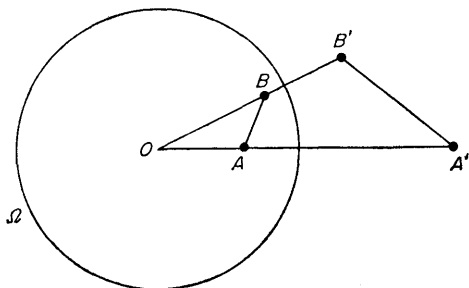


Fig. 126

THE PROBLEM. Let A, B be two given points, with inverses A', B' . The *magnification* of AB under the inversion may be defined as the ratio $A'B'/AB$. It is required to *prove that*

$$\frac{A'B'}{AB} = \frac{OB'}{OA} = \frac{OA'}{OB}$$

THE DISCUSSION.

$$OA \cdot OA' = a^2 = OB \cdot OB'$$

$$\Rightarrow \frac{OA}{OB} = \frac{OB'}{OA'}$$

and

$$\left. \begin{array}{l} \frac{OA}{OB} = \frac{OB'}{OA'} \\ \angle AOB = \angle B'OA' \end{array} \right\} \Rightarrow \triangle AOB \sim \triangle B'OA'$$

$$\Rightarrow \frac{AB}{B'A'} = \frac{OA}{OB'} = \frac{OB}{OA'}$$

Thus

$$A'B' = \frac{OB'}{OA} \cdot AB = \frac{OA'}{OB} \cdot AB.$$

(ii) We can now find the inverses of straight lines and circles:

(a) To prove that the inverse of a straight line not through O is a circle through O .

Let l be the given line, O the centre of the circle of inversion Ω , A the foot of the perpendicular from O to l and A' the inverse of A .

Take an arbitrary point P on l , and let P' be its inverse. Then

$$OA \cdot OA' = a^2 = OP \cdot OP' \\ \Rightarrow A, A', P, P' \text{ concyclic,}$$

and

$$\angle A'AP = 90^\circ \Rightarrow A'P \text{ subtends a right angle on the circle } AA'P'P \\ \Rightarrow \angle A'P'P = 90^\circ \\ \Rightarrow \angle OP'A' = 90^\circ.$$

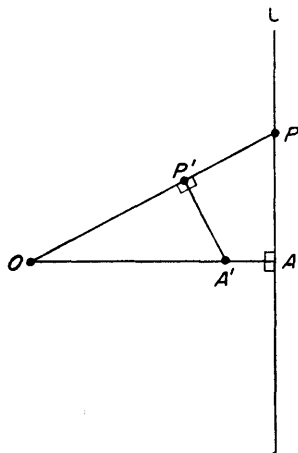


Fig. 127

But O, A' are fixed points, and so P' lies on the circle on OA' as diameter.

(b) To prove that the inverse of a circle through O is a straight line not through O .

Let m be the given circle, O the centre of the circle of inversion Ω , A the other end of the diameter of m through O , and A' the inverse of A .

Take an arbitrary point P on m , and let P' be its inverse. Then

$$OA \cdot OA' = a^2 = OP \cdot OP' \\ \Rightarrow A, A', P, P' \text{ concyclic,}$$

and

$$\begin{aligned}
 OA \text{ diameter} &\Rightarrow \angle OPA = 90^\circ \\
 &\Rightarrow \angle OA'P' = 90^\circ.
 \end{aligned}$$

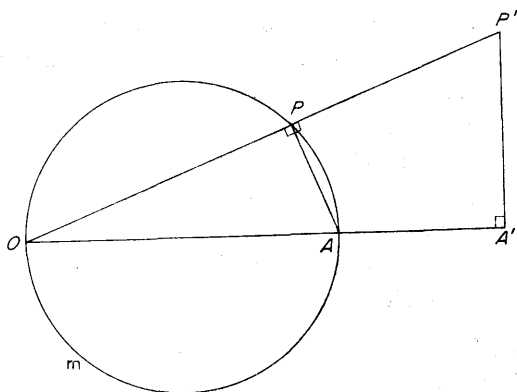


Fig. 128

But O, A' are fixed points, and so P' lies on the straight line through A' perpendicular to OA' .

Problems

1. Two circles cut orthogonally at O . Prove that their inverses with respect to O are two perpendicular lines.
2. Two circles touch at O . Prove that their inverses with respect to O are two parallel lines.
3. Two lines l, m are parallel and $O \in l$. Prove that the inverse with respect to O of l is l itself, and that the inverse of m is a circle touching l at O .

(c) To prove that the inverse of a circle not through O is a circle not through O .

Let m be the given circle, O the centre of the circle of inversion Ω , B the inverse of O with respect to the circle m , and B' the inverse of B with respect to Ω .

What we shall prove is that the inverse of m is a circle m' whose centre is B' , so that the method of proof locates the centre of m' at the same time.

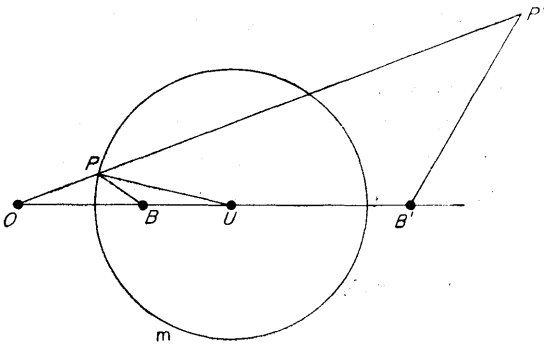


Fig. 129

Take an arbitrary point P on m , and let P' be its inverse with respect to Ω . Then, by the magnification theorem,

$$\begin{aligned} B'P' &= \frac{OB'}{OP} \cdot BP \\ &= \frac{BP}{OP} \cdot OB' \end{aligned}$$

Suppose now that U is the centre of the circle m . Then
 O, B inverse with respect to m

$$\begin{aligned} \Rightarrow UP^2 &= UB \cdot UO \Rightarrow UP/UB = UO/UP \\ \Rightarrow \triangle UPB &\sim \triangle UOP \\ \Rightarrow \frac{PB}{OP} &= \frac{UP}{UO} \end{aligned}$$

Hence

$$\begin{aligned} B'P' &= \frac{UP}{UO} \cdot OB' \\ &= \text{constant,} \end{aligned}$$

since UP is the radius of m and U, O, B' are fixed points, and so P' lies on the circle of centre B' and radius $B'P'$.

(iii) The angle between two curves

In order to make full use of the technique of inversion, we must give a short account of the angle between two curves at a common point. The basic ideas will be familiar from work on calculus.

(a) The tangent at a point.

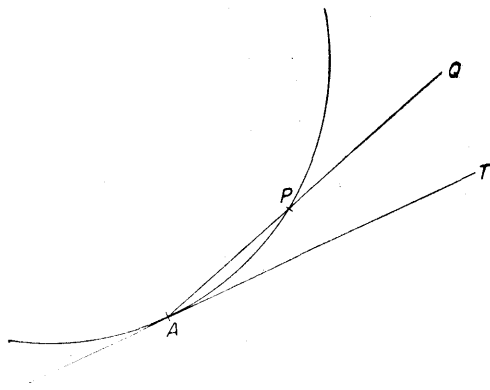


Fig. 130

Let A be a point on a given curve. We seek to *define the tangent to the curve at A* . Let P be any point of the curve, fairly near to A , and consider the line AQ through P . The point P may be supposed to approach more and more closely to A , and the line AQ will

then (in normal cases) take up a limiting positive AT known as the *tangent at A* to the curve.

(b) *The angle between two curves.*

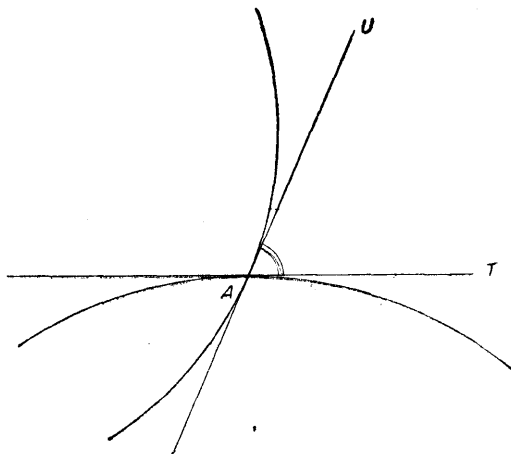


Fig. 131

If two curves cut at a point A , the *angle between the curves at A* is defined to be the angle between the tangents there.

(c) *The fundamental theorem:*

The angle between two curves is equal to the angle between their inverses.

Let m, n be two given curves intersecting at A . Take a point P on m , fairly near to A , and let OP meet n in Q . The inverses P', Q' on the inverse curves m', n' also lie on the line OPQ . Suppose, finally, that A' is the inverse of A ; the two curves m', n' then necessarily intersect at A' .

Now

$$\begin{aligned}
 OP \cdot OP' &= OQ \cdot OQ' = OA \cdot OA' \\
 \Rightarrow APP'A' &\text{ cyclic and } AQQ'A' \text{ cyclic} \\
 \Rightarrow \angle OAP &= \angle OP'A', \quad \angle OAQ = \angle OQ'A' \\
 \Rightarrow \angle OAQ - \angle OAP &= \angle OQ'A' - \angle OP'A' \\
 \Rightarrow \angle PAQ &= \angle P'A'Q'.
 \end{aligned}$$

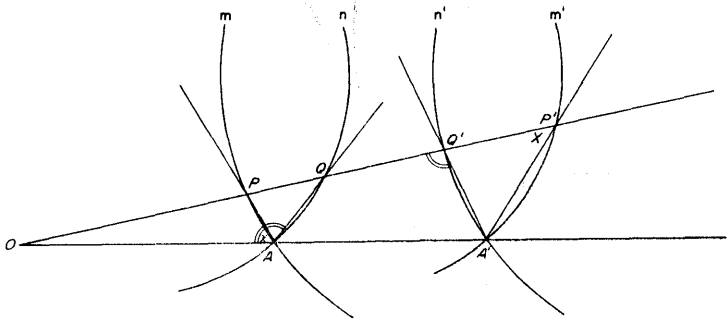


Fig. 132

Thus, however close P is to A (so that Q is also close to A), the angle between the chords $A'P'$ and $A'Q'$ is equal to the angle between the chords AP and AQ . In the limit, then, the angle between the tangents at A is equal to the angle between the tangents at A' .

INTERPRETATION. (a) Two curves which have the same tangent at A are said to *touch* at A . Their inverse curves will also touch at A —but note that the tangent AT at A will, in general, NOT invert into the tangent $A'T'$.

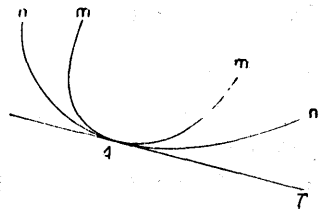


Fig. 133

at A' , since straight lines usually do not invert into straight lines but into circles.

When A , where the curves touch, is also the centre of inversion, more complicated problems arise. For straight lines and circles, the rules are:

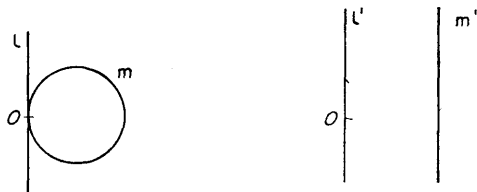


Fig. 134

(i) When a line l touches a circle m at O , the inverses are a line l' which is the same as l and a line m' which is parallel to l' .

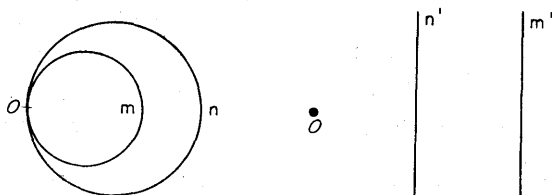


Fig. 135

(ii) When two circles m, n touch at O , the inverses are two parallel straight lines m', n' .

(b) Two curves are called *orthogonal* when the angles between them is a right angle. In particular, when a line l is orthogonal to a circle m , then l is a diameter of m .

This interpretation for line and circle is very important in problems involving inversion.

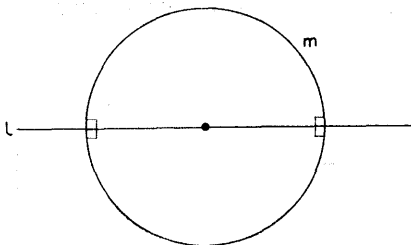


Fig. 136

Theorem

1. A circle and a diameter invert, with respect to a point on the circle, into "a diameter and a circle".

Problems

- Two circles α, β touch at O . A circle Ω through O cuts them again at A, B respectively. Verify (what is in any case obvious at O) that the circles α, Ω cut at A at the same angle that the circles β, Ω cut at B . (Invert with respect to O .)
- A point O lies on a circle having BC as diameter and A is a point on BC . By inversion with respect to O , prove that the circles OAB, OAC cut orthogonally.
- Establish the equivalence of the two following theorems:
 - Two circles $OAPX, OBQX$ meet in O, X and PXQ, AOB are straight lines. Then $AP \parallel BQ$.
 - Two straight lines APX, BQX meet in X and O is a point on AB ; the points O, P, X, Q are concyclic. Then the circles OAP, OBQ touch at O .
- Given a circle l and two points A, O not on it, prove that a unique circle can be drawn through A, O to cut l orthogonally.
- A, B, C are three collinear points and O a point not on the line ABC . Prove that there is a common point H to each of the circles passing through O and one of the points A, B, C and cutting orthogonally the circle through O and the two others of the points A, B, C .

6. Establish the equivalence of the following theorems:

(a) Given three non-collinear points A, B, C , the circles on AB, AC as diameters meet on BC .

(b) Given three non-collinear points A, B, C , the line through B perpendicular to AB meets the line through C perpendicular to AC on $\odot ABC$.

(c) Given three non-collinear points A, B, C , the line through A perpendicular to AB cuts the circle through A, C cutting $\odot ABC$ orthogonally in a point on BC .

7. Four points A, B, C, O lie on a given circle. A circle through B, C touches a circle through A, O at a point P . Prove that, as the touching circles vary, the locus of P is a circle.

8. Given three fixed points O, A, B and a variable point P on either the line AB or a fixed circle through A, B , prove that the circles OPA, OPB cut at a constant angle.

9. Establish the equivalence of the following theorems:

(a) Two circles p, q touch at O and two circles r, s touch at O . If one of the pairs of circles $(p, r), (p, s), (q, s), (q, r)$ is orthogonal, so are the other three pairs.

(b) If one angle of a parallelogram is a right angle, so are the other three.

10. In question 9(a), the further intersections of the four pairs of circles, in that order, are A, B, C, D . Prove that, if a circle can be drawn to touch each of the circles p, q, r, s , then the circles OAC, OBD are orthogonal.



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