N. BOURBAKI

ELEMENTS OF MATHEMATICS

Functions of a Real Variable

Elementary Theory

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To the reader

1. The Elements of Mathematics Series takes up mathematics at the beginning, and gives complete proofs. In principle, it requires no particular knowledge of mathematics on the readers' part, but only a certain familiarity with mathematical reasoning and a certain capacity for abstract thought. Nevertheless it is directed especially to those who have a good knowledge of at least the content of the first year or two of a university mathematics course.

2. The method of exposition we have chosen is axiomatic, and normally proceeds from the general to the particular. The demands of proof impose a rigorously fixed order on the subject matter. It follows that the utility of certain considerations may not be immediately apparent to the reader until later chapters unless he has already a fairly extended knowledge of mathematics.

3. The series is divided into Books and each Book into chapters. The Books already published, either in whole or in part, in the French edition, are listed below. When an English translation is available, the corresponding English title is mentioned between parentheses. Throughout the volume a reference indicates the English edition, when available, and the French edition otherwise.

Théorie des Ensembles (Theory of Sets)	designated by	E	(*Set Theory*)
Algèbre (Algebra) [1]	—	A	(*Alg*)
Topologie Générale (General Topology)	—	TG	(*Gen. Top.*)
Fonctions d'une Variable Réelle (Functions of a Real Variable) [2]	—	FVR	(*FRV*)
Espaces Vectoriels Topologiques (Topological Vector Spaces)	—	EVT	(*Top. Vect. Sp.*)
Intégration	—	INT	
Algèbre Commutative (Commutative Algebra) [3]	—	AC	(*Comm. Alg.*)
Variétes Différentielles et Analytiques	—	VAR	
Groupes et Algèbres de Lie (Lie Groups and Lie Algebras) [4]	—	LIE	(LIE)
Théories Spectrales		TS	

In the first six Books (according to the above order), every statement in the text assumes as known only those results which have already discussed in the same

[1] So far, chapters I to VII only have been translated.
[2] This volume!
[3] So far, chapters I to VII only have been translated.
[4] So far, chapters I to III only have been translated.

chapter, or in the previous chapters ordered as follows: E ; A, chapters I to III ; TG, chapters I to III ; A, from chapter IV on ; TG, from chapter IV on ; FVR ; EVT ; INT.

From the seventh Book on, the reader will usually find a precise indication of its logical relationship to the other Books (the first six Books being always assumed to be known).

4. However, we have sometimes inserted examples in the text which refer to facts which the reader may already know but which have not yet been discussed in the Series. Such examples are placed between two asterisks : $^*\ldots_*$. Most readers will undoubtedly find that these examples will help them to understand the text. In other cases, the passages between $^*\ldots_*$ refer to results which are discussed elsewhere in the Series. We hope the reader will be able to verify the absence of any vicious circle.

5. The logical framework of each chapter consists of the *definitions,* the *axioms,* and the *theorems* of the chapter. These are the parts that have mainly to be borne in mind for subsequent use. Less important results and those which can easily be deduced from the theorems are labelled as "propositions", "lemmas", "corollaries", "remarks", etc. Those which may be omitted at a first reading are printed in small type. A commentary on a particularly important theorem appears occasionally under the name of "scholium".

To avoid tedious repetitions it is sometimes convenient to introduce notation or abbreviations which are in force only within a certain chapter or a certain section of a chapter (for example, in a chapter which is concerned only with commutative rings, the word "ring" would always signify "commutative ring"). Such conventions are always explicitly mentioned, generally at the beginning of the *chapter* in which they occur.

6. Some passages are designed to forewarn the reader against serious errors. These passages are signposted in the margin with the sign (“dangerous bend”).

7. The Exercises are designed both to enable the reader to satisfy himself that he has digested the text and to bring to his notice results which have no place in the text but which are nonetheless of interest. The most difficult exercises bear the sign ¶.

8. In general we have adhered to the commonly accepted terminology, *except where there appeared to be good reasons for deviating from it.*

9. We have made a particular effort always to use rigorously correct language, without sacrificing simplicity. As far as possible we have drawn attention in the text to *abuses of language,* without which any mathematical text runs the risk of pedantry, not to say unreadability.

10. Since in principle the text consists of a dogmatic exposition of a theory, it contains in general no references to the literature. Bibliographical are gathered together in *Historical Notes*. The bibliography which follows each historical note contains in general only those books and original memoirs which have been of the greatest importance in the evolution of the theory under discussion. It makes no sort of pretence to completeness.

As to the exercises, we have not thought it worthwhile in general to indicate their origins, since they have been taken from many different sources (original papers, textbooks, collections of exercises).

11. In the present Book, references to theorems, axioms, definitions, . . . are given by quoting successively:

– the Book (using the abbreviation listed in Section 3), chapter and page, where they can be found ;

– the chapter and page only when referring to the present Book.

The *Summaries of Results* are quoted by to the letter R; thus *Set Theory,* R signifies "*Summary of Results of the Theory of Sets*".

CONTENTS

CHAPTER VII THE GAMMA FUNCTION

INTRODUCTION

The purpose of this Book is the elementary study of the infinitesimal properties of *one* real variable; the extension of these properties to functions of *several* real variables, or, all the more, to functions defined on more general spaces, will be treated only in later Books.

The results which we shall demonstrate will be useful above all in relation to (finite) real-valued functions of a real variable; but most of them extend without further argument to functions of a real variable taking values in a *topological vector space* over $\mathbf{R}$ (see below); as these functions occur frequently in Analysis we shall state for them all results which are not specific to real-valued functions.

The notion of a topological vector space, of which we have just spoken, is defined and studied in detail in Book V of this Series; but we do not need *any* of the results of Book V in this Book; some definitions, however, are needed, and we shall reproduce them below for the convenience of the reader.

We shall not repeat the definition of a *vector space* over a (*commutative*) *field* K (*Alg*., II, p. 193). [1] A *topological vector space* E over a *topological field* K is a vector space over K endowed with a topology such that the functions $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}t$ are *continuous* on $\mathrm{E}\times\mathrm{E}$ and $\mathrm{E}\times\mathrm{K}$ respectively; in particular, such a topology is compatible with the structure of the additive group of E. All topological vector spaces considered in this Book are implicitly assumed to be Hausdorff. When the topological group E is complete one says that the topological vector space E is *complete*. Every *normed* vector space over a *valued field* K (*Gen. Top*., IX, p. 169) [2] is a topological vector space over K.

Let E be a vector space (with or without a topology) over the real field $\mathbf{R}$; if $\mathbf{x}$, $\mathbf{y}$ are arbitrary points in E the set of points $\mathbf{x}t+\mathbf{y}(1-t)$ where t runs through the closed

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[1] The elements (or *vectors*) of a vector space E over a commutative field K will usually be denoted in this chapter by thick minuscules, and scalars by roman minuscules; most often we shall place the scalar t to the *right* in the product of a vector $\mathbf{x}$ by t, writing the product as $\mathbf{x}t$; on occasion we will allow ourselves to use the left notation $t\mathbf{x}$ in certain cases where it is more convenient; also, sometimes we shall write the product of the scalar $1/t$ ($t \neq 0$) and the vector $\mathbf{x}$ in the form $\mathbf{x}/t$.

[2] We recall that a *norm* on E is a real function $\|\mathbf{x}\|$ defined on E, taking finite non-negative values, such that the relation $\|\mathbf{x}\|=0$ is equivalent to $\mathbf{x}=0$ and such that

$$\|\mathbf{x}+\mathbf{y}\| \leqslant \|\mathbf{x}\|+\|\mathbf{y}\| \quad \text{and} \quad \|\mathbf{x}t\| = \|\mathbf{x}\|\,.\,|t|$$

for all $t \in \mathrm{K}$ ($|t|$ being the absolute value of t in K).

segment $[0,\ 1]$ of $\mathbf{R}$ is called the *closed segment* with endpoints $\mathbf{x}$, $\mathbf{y}$. One says that a subset A of E is *convex* if for any $\mathbf{x}$, $\mathbf{y}$ in A the closed segment with endpoints $\mathbf{x}$ and $\mathbf{y}$ is contained in A. For example, an affine linear variety is convex; so is any closed segment; in $\mathbf{R}^n$ any parallelotope (*Gen. Top.*, VI, p. 34) is convex. Every intersection of convex sets is convex.

We say that a topological vector space E over the field $\mathbf{R}$ is *locally convex* if the origin (and thus any point of E) has a fundamental system of *convex* neighbourhoods. Every *normed* space is locally convex; indeed, the balls $\|\mathbf{x}\| \leqslant r$ $(r > 0)$ form a fundamental system of neighbourhoods of 0 in E, and each of these is convex, for the relations $\|\mathbf{x}\| \leqslant r$, $\|\mathbf{y}\| \leqslant r$ imply that

$$\|\mathbf{x}t + \mathbf{y}(1-t)\| \leqslant \|\mathbf{x}\|\, t + \|\mathbf{y}\|\,(1-t) \leqslant r$$

for $0 \leqslant t \leqslant 1$.

Finally, a *topological algebra* A over a (commutative) *topological field* K is an algebra over K endowed with a topology for which the functions $\mathbf{x}+\mathbf{y}$, $\mathbf{xy}$ and $\mathbf{x}t$ are continuous on $\mathrm{A} \times \mathrm{A}$, $\mathrm{A} \times \mathrm{A}$ and $\mathrm{A} \times \mathrm{K}$ respectively; when one endows A only with its topology and vector space structure over K then A is a topological vector space. Every *normed algebra* over a *valued field* K (*Gen. Top.*, IX, p. 175) is a topological algebra over K.

CHAPTER I
Derivatives

§ 1. FIRST DERIVATIVE

As was said in the Introduction, in this chapter and the next we shall study the infinitesimal properties of functions which are defined on a subset of the real field **R** and take their values in a *Hausdorff topological vector space* E over the field **R**; for brevity we shall say that such a function is a *vector function of a real variable*. The most important case is that where $E = \mathbf{R}$ (real-valued functions of a real variable). When $E = \mathbf{R}^n$, consideration of a vector function with values in E reduces to the simultaneous consideration of n finite real functions.

Many of the definitions and properties stated in chapter I extend to functions which are defined on a subset of the field **C** of complex numbers and take their values in a topological vector space over **C** (vector functions of a complex variable). Some of these definitions and properties extend even to functions which are defined on a subset of an arbitrary commutative *topological field* K and take their values in a topological vector space over K.

We shall indicate these generalizations in passing (see in particular I, p. 10, *Remark 2*), emphasising above all the case of functions of a complex variable, which are by far the most important, together with functions of a real variable, and will be studied in greater depth in a later Book.

1. DERIVATIVE OF A VECTOR FUNCTION

DEFINITION 1. *Let* $\mathbf{f}$ *be a vector function defined on an interval* $I \subset \mathbf{R}$ *which does not reduce to a single point. We say that* $\mathbf{f}$ *is differentiable at a point* $x_0 \in I$ *if* $\lim\limits_{x \to x_0, x \in I, x \neq x_0} \dfrac{\mathbf{f}(x) - \mathbf{f}(x_0)}{x - x_0}$ *exists* (*in the vector space where* $\mathbf{f}$ *takes its values*)*; the value of this limit is called the first derivative* (or simply the *derivative*) *of* $\mathbf{f}$ *at the point* x_0*, and it is denoted by* $\mathbf{f}'(x_0)$ *or* $\mathrm{D}\mathbf{f}(x_0)$.

If $\mathbf{f}$ is differentiable at the point x_0, so is the *restriction* of $\mathbf{f}$ to any interval $J \subset I$ which does not reduce to a single point and such that $x_0 \in J$; and the derivative of this restriction is equal to $\mathbf{f}'(x_0)$. Conversely, let J be an interval contained in I and containing a neighbourhood of x_0 relative to I; if the restriction of $\mathbf{f}$ to J admits a derivative at the point x_0, then so does $\mathbf{f}$.

We summarise these properties by saying that the concept of derivative is a *local* concept.

Remarks. *1) In Kinematics, if the point $\mathbf{f}(t)$ is the position of a moving point in the space $\mathbf{R}^3$ at time t, then $\dfrac{\mathbf{f}(t)-\mathbf{f}(t_0)}{t-t_0}$ is termed the *average velocity* between the instants t_0 and t, and its limit $\mathbf{f}'(t_0)$ is the *instantaneous velocity* (or simply *velocity*) at the time t_0 (when this limit exists).$_*$

2) If a function $\mathbf{f}$, defined on I, is differentiable at a point $x_0 \in \mathrm{I}$, it is necessarily *continuous relative to* I at this point.

DEFINITION 2. *Let* $\mathbf{f}$ *be a vector function defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *and let* x_0 *be a point of* I *such that the interval* $\mathrm{I} \cap [x_0, +\infty[$ (resp. $\mathrm{I} \cap\,]-\infty, x_0]$) *does not reduce to a single point. We say that* $\mathbf{f}$ *is differentiable on the right* (resp. *on the left*) *at the point* x_0 *if the restriction of* $\mathbf{f}$ *to the interval* $\mathrm{I} \cap [x_0, +\infty[$ (resp. $\mathrm{I} \cap\,]-\infty, x_0]$) *is differentiable at the point* x_0; *the value of the derivative of this restriction at the point* x_0 *is called the right* (resp. *left*) *derivative of* $\mathbf{f}$ *at the point* x_0 *and is denoted by* $\mathbf{f}'_d(x_0)$ (resp. $\mathbf{f}'_g(x_0)$).

Let $\mathbf{f}$ be a vector function defined on I, and x_0 an *interior* point of I such that $\mathbf{f}$ is continuous at this point; it follows from defs. 1 and 2 that for $\mathbf{f}$ to be differentiable at x_0 it is necessary and sufficient that $\mathbf{f}$ admit both a right and a left derivative at this point, and that these derivatives be *equal;* and then

$$\mathbf{f}'(x_0) = \mathbf{f}'_d(x_0) = \mathbf{f}'_g(x_0).$$

Examples. 1) A *constant* function has zero derivative at every point.

2) An affine linear function $x \mapsto \mathbf{a}x + \mathbf{b}$ has derivative equal to $\mathbf{a}$ at every point.

3) The real function $1/x$ (defined for $x \neq 0$) is differentiable at each point $x_0 \neq 0$, for we have $\left(\dfrac{1}{x} - \dfrac{1}{x_0}\right) \Big/ (x - x_0) = -\dfrac{1}{xx_0}$, and, since $1/x$ is continuous at x_0, the limit of the preceding expression is $-1/x_0^2$.

4) The scalar function $|x|$, defined on $\mathbf{R}$, has right derivative $+1$ and left derivative -1 at $x = 0$; it is not differentiable at this point.

*5) The real function equal to 0 for $x = 0$, and to $x \sin 1/x$ for $x \neq 0$, is defined and continuous on $\mathbf{R}$, but has neither right nor left derivative at the point $x \neq 0$.$_*$ One can give examples of functions which are continuous on an interval and fail to have a derivative at *every* point of the interval (I, p. 35, exerc. 2 and 3).

DEFINITION 3. *We say that a vector function* $\mathbf{f}$ *defined on an interval* $\mathrm{I} \subset \mathbf{R}$ *is differentiable* (resp. *right differentiable, left differentiable*) *on* I *if it is differentiable* (resp. *right differentiable, left differentiable*) *at each point of* I; *the function* $x \mapsto \mathbf{f}'(x)$ (resp. $x \mapsto \mathbf{f}'_d(x)$, $x \mapsto \mathbf{f}'_g(x)$) *defined on* I, *is called the derived function, or* (by abuse of language) *the derivative* (resp. *right derivative, left derivative*) *of* $\mathbf{f}$, *and is denoted by* $\mathbf{f}'$ *or* $\mathrm{D}\mathbf{f}$ *or* $d\mathbf{f}/dx$ (resp. $\mathbf{f}'_d$, $\mathbf{f}'_g$).

Remark. A function may be differentiable on an interval without its derivative being continuous at every point of the interval (*cf.* I, p. 36, exerc. 5); *this is shown by the example of the function equal to 0 for $x = 0$ and to $x^2 \sin 1/x$ for $x \neq 0$; it has a derivative everywhere, but this derivative is discontinuous at the point $x = 0$.$_*$

2. LINEARITY OF DIFFERENTIATION

PROPOSITION 1. *The set of vector functions defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *taking values in a given topological vector space* E, *and differentiable at the point* x_0, *is a vector space over* $\mathbf{R}$, *and the map* $\mathbf{f} \mapsto \mathrm{D}\mathbf{f}(x_0)$ *is a linear mapping of this space into* E.

In other words, if $\mathbf{f}$ and $\mathbf{g}$ are defined on I and differentiable at the point x_0, then $\mathbf{f} + \mathbf{g}$ and $\mathbf{f}a$ (a an arbitrary scalar) are differentiable at x_0 and their derivatives there are $\mathbf{f}'(x_0) + \mathbf{g}'(x_0)$ and $\mathbf{f}'(x_0)a$ respectively. This follows immediately from the continuity of $\mathbf{x} + \mathbf{y}$ and of $\mathbf{x}a$ on $\mathrm{E} \times \mathrm{E}$ and E respectively.

COROLLARY. *The set of vector functions defined on an interval* I, *taking values in a given topological vector space* E, *and differentiable on* I, *is a vector space over* $\mathbf{R}$, *and the map* $\mathbf{f} \mapsto \mathrm{D}\mathbf{f}$ *is a linear mapping of this space into the vector space of mappings from* I *into* E.

Remark. If one endows the vector space of mappings from I into E and its subspace of differentiable mappings (*cf. Gen. Top.*, X, p. 277) with the topology of simple convergence (or the topology of uniform convergence), the linear mapping $\mathbf{f} \mapsto \mathrm{D}\mathbf{f}$ *is not continuous* (in general) *for example, the sequence of functions $\mathbf{f}_n(x) = \sin n^2x/n$ converges uniformly to 0 on $\mathbf{R}$, but the sequence of derivatives $\mathbf{f}_n'(x) = n \cos n^2x$ does not converge even simply to 0.$_*$

PROPOSITION 2. *Let* E *and* F *be two topological vector spaces over* $\mathbf{R}$, *and* $\mathbf{u}$ *a continuous linear map from* E *into* F. *If* $\mathbf{f}$ *is a vector function defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *taking values in* E, *and differentiable at the point* $x_0 \in \mathrm{I}$, *then the composite function* $\mathbf{u} \circ \mathbf{f}$ *has a derivative equal to* $\mathbf{u}(\mathbf{f}'(x_0))$ *at* x_0.

Indeed, since $\dfrac{\mathbf{u}(\mathbf{f}(x)) - \mathbf{u}(\mathbf{f}(x_0))}{x - x_0} = \mathbf{u}\left(\dfrac{\mathbf{f}(x) - \mathbf{f}(x_0)}{x - x_0}\right)$, this follows from the continuity of $\mathbf{u}$.

COROLLARY. *If* φ *is a continuous linear form on* E, *then the real function* $\varphi \circ \mathbf{f}$ *has a derivative equal to* $\varphi(\mathbf{f}'(x_0))$ *at the point* x_0.

Examples. 1) Let $\mathbf{f} = (f_i)_{1 \leqslant i \leqslant n}$ be a function with values in $\mathbf{R}^n$, defined on an interval $\mathrm{I} \subset \mathbf{R}$; each real function f_i is none other than the composite function $\mathrm{pr}_i \circ \mathbf{f}$, so is differentiable at the point x_0 if $\mathbf{f}$ is, and, if so, $\mathbf{f}'(x_0) = (f_i'(x_0))_{1 \leqslant i \leqslant n}$.

*2) In Kinematics, if $\mathbf{f}(t)$ is the position of a moving point M at time t, if $\mathbf{g}(t)$ is the position at the same instant of the projection M′ of M onto a plane P (resp. a line D) with kernel a line (resp. a plane) not parallel to P (resp. D), then $\mathbf{g}$ is the composition of the projection $\mathbf{u}$ of $\mathbf{R}^3$ onto P (resp. D) and of $\mathbf{f}$; since $\mathbf{u}$ is a (continuous) linear mapping one sees that the projection of the velocity of a moving point onto a plane (resp. a line) is equal to the velocity of the projection of the moving point onto the plane (resp. line).$_*$

3) Let f be a complex-valued function defined on an interval $\mathrm{I} \subset \mathbf{R}$, and let a be an arbitrary complex number; prop. 2 shows that if f is differentiable at a point x_0 then so is af, and the derivative of this function at x_0 is equal to $af'(x_0)$.

3. DERIVATIVE OF A PRODUCT

Let us now consider p topological vector spaces E_i $(1 \leqslant i \leqslant p)$ over $\mathbf{R}$, and a continuous multilinear [1] map (which we shall denote by

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \mapsto [\mathbf{x}_1.\mathbf{x}_2 \ldots \mathbf{x}_p])$$

of $\mathrm{E}_1 \times \mathrm{E}_2 \times \cdots \times \mathrm{E}_p$ into a topological vector space F over $\mathbf{R}$.

PROPOSITION 3 . *For each index i $(1 \leqslant i \leqslant p)$ let $\mathbf{f}_i$ be a function defined on an interval $\mathrm{I} \subset \mathbf{R}$, taking values in E_i, and differentiable at the point $x_0 \in \mathrm{I}$. Then the function*

$$x \mapsto [\mathbf{f}_1(x).\mathbf{f}_2(x) \ldots \mathbf{f}_p(x)]$$

defined on I *with values in* F *has a derivative equal to*

$$\sum_{i=1}^{p} \big[\mathbf{f}_1(x_0) \ldots \mathbf{f}_{i-1}(x_0).\mathbf{f}_i'(x_0).\mathbf{f}_{i+1}(x_0) \ldots \mathbf{f}_p(x_0)\big] \tag{1}$$

at x_0.

Let us put $\mathbf{h}(x) = [\mathbf{f}_1(x).\mathbf{f}_2(x) \ldots \mathbf{f}_p(x)]$; then, by the identity

$$\big[\mathbf{b}_1.\mathbf{b}_2 \ldots \mathbf{b}_p\big] - \big[\mathbf{a}_1.\mathbf{a}_2 \ldots \mathbf{a}_p\big] = \sum_{i=1}^{p} \big[\mathbf{b}_1 \ldots \mathbf{b}_{i-1}.(\mathbf{b}_i - \mathbf{a}_i).\mathbf{a}_{i+1} \ldots \mathbf{a}_p\big],$$

we can write

$$\mathbf{h}(x) - \mathbf{h}(x_0) = \sum_{i=1}^{p} \big[\mathbf{f}_1(x) \ldots \mathbf{f}_{i-1}(x).(\mathbf{f}_i(x) - \mathbf{f}_i(x_0)).\mathbf{f}_{i+1}(x_0) \ldots \mathbf{f}_p(x_0)\big].$$

On multiplying both sides by $\dfrac{1}{x - x_0}$ and letting x approach x_0 in I, we obtain the expression (1), since both the map

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \mapsto [\mathbf{x}_1.\mathbf{x}_2 \ldots \mathbf{x}_p]$$

and addition in F are continuous.

[1] Recall (*Alg.*, II, p. 265) that a map $\mathbf{f}$ of $\mathrm{E}_1 \times \mathrm{E}_2 \times \cdots \times \mathrm{E}_p$ into F is said to be *multilinear* if each partial mapping

$$\mathbf{x}_i \mapsto \mathbf{f}(\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{x}_i, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_p)$$

from E_i into F $(1 \leqslant i \leqslant p)$ is a *linear* map, the $\mathbf{a}_j$ for indices $j \neq i$ being arbitrary in E_j. We note that if the E_i are *finite* dimensional over $\mathbf{R}$ then every multilinear map of $\mathrm{E}_1 \times \mathrm{E}_2 \times \cdots \times \mathrm{E}_p$ into F is necessarily *continuous;* this need not be so if some of these spaces are topological vector spaces of infinite dimension.

When some of the functions $\mathbf{f}_i$ are *constant,* the terms in the expression (1) containing their derivatives $\mathbf{f}_i'(x_0)$ are zero.

Let us consider in detail the particular case $p = 2$, the most important in applications: if $(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}.\mathbf{y}]$ is a *continuous bilinear* map of $\mathrm{E} \times \mathrm{F}$ into G, $\big($E, F, G being topological vector spaces over $\mathbf{R}\big)$, and $\mathbf{f}$ and $\mathbf{g}$ are two vector functions, differentiable at x_0, with values in E and F respectively, then the vector function $x \mapsto [\mathbf{f}(x).\mathbf{g}(x)]$ (which we denote by $[\mathbf{f}.\mathbf{g}]$) has a derivative equal to $[\mathbf{f}'(x_0).\mathbf{g}(x_0)] + [\mathbf{f}(x_0).\mathbf{g}'(x_0)]$ at x_0. In particular, if $\mathbf{a}$ is a constant vector, then $[\mathbf{a}.\mathbf{f}]$ (resp. $[\mathbf{f}.\mathbf{a}]$) has a derivative equal to $[\mathbf{a}.\mathbf{f}'(x_0)]$ (resp. $[\mathbf{f}'(x_0).\mathbf{a}]$) at x_0.

If $\mathbf{f}$ and $\mathbf{g}$ are both differentiable on I then so is $[\mathbf{f}.\mathbf{g}]$, and we have

$$[\mathbf{f}.\mathbf{g}]' = [\mathbf{f}'.\mathbf{g}] + [\mathbf{f}.\mathbf{g}']. \tag{2}$$

Examples. 1) Let f be a real function, $\mathbf{g}$ a vector function, both differentiable at a point x_0; the function $\mathbf{g}f$ has a derivative equal to $\mathbf{g}'(x_0)f(x_0) + \mathbf{g}(x_0)f'(x_0)$ at x_0. In particular, if $\mathbf{a}$ is constant, then $\mathbf{a}f$ has derivative $\mathbf{a}f'(x_0)$. This last remark, in conjunction with example 1 of I, p. 5, proves that if $\mathbf{f} = (f_i)_{1 \leqslant i \leqslant n}$ is a vector function with values in $\mathbf{R}^n$, then for $\mathbf{f}$ to be differentiable at the point x_0 it is necessary and sufficient that each of the real functions f_i $(1 \leqslant i \leqslant n)$ be differentiable there: for, if $(\mathbf{e}_i)_{1 \leqslant i \leqslant n}$ is the canonical basis of $\mathbf{R}^n$, we can write $\mathbf{f} = \sum_{i=1}^{n} \mathbf{e}_i f_i$.

2) The real function x^n arises from the multilinear function

$$(x_1, x_2, \ldots, x_n) \mapsto x_1 x_2 \ldots x_n$$

defined on $\mathbf{R}^n$, by substituting x for each of the x_i; so prop. 3 shows that x^n is differentiable on $\mathbf{R}$ and has derivative nx^{n-1}. As a result the polynomial function $\mathbf{a}_0 x^n + \mathbf{a}_1 x^{n-1} + \cdots + \mathbf{a}_{n-1} x + \mathbf{a}_n$ (the $\mathbf{a}_i$ being constant vectors) has derivative

$$n\mathbf{a}_0 x^{n-1} + (n-1)\mathbf{a}_1 x^{n-2} + \cdots + \mathbf{a}_{n-1};$$

when the $\mathbf{a}_i$ are real numbers this function coincides with the derivative of a polynomial function as defined in Algebra (A, IV).

3) The euclidean *scalar product* $(\mathbf{x} \,|\, \mathbf{y})$ (*Gen. Top.*, VI, p. 40) is a bilinear map (necessarily continuous) of $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{R}$. If $\mathbf{f}$ and $\mathbf{g}$ are two vector functions with values in $\mathbf{R}^n$, and differentiable at the point x_0, then the real function $x \mapsto (\mathbf{f}(x) \,|\, \mathbf{g}(x))$ has a derivative equal to $(\mathbf{f}'(x_0) \,|\, \mathbf{g}(x_0)) + (\mathbf{f}(x_0) \,|\, \mathbf{g}'(x_0))$ at the point x_0. There is an analogous result for the hermitian scalar product on $\mathbf{C}^n$, this space being considered as a vector space *over* $\mathbf{R}$.

Let us consider in particular the case where the euclidean norm $\|\mathbf{f}(x)\|$ is *constant*, so that $(\mathbf{f}(x) \,|\, \mathbf{f}(x)) = \|\mathbf{f}(x)\|^2$ is also constant; on writing that the derivative of $(\mathbf{f}(x) \,|\, \mathbf{f}(x))$ vanishes at x_0 we obtain $(\mathbf{f}(x_0) \,|\, \mathbf{f}'(x_0)) = 0$; in other words, $\mathbf{f}'(x_0)$ is *orthogonal* to $\mathbf{f}(x_0)$.

4) If E is a *topological algebra* over $\mathbf{R}$ (*cf.* Introduction), the product $\mathbf{xy}$ of two elements of E is a continuous bilinear function of $(\mathbf{x}, \mathbf{y})$; if $\mathbf{f}$ and $\mathbf{g}$ have their values in E and are differentiable at the point x_0, then the function $x \mapsto \mathbf{f}(x)\mathbf{g}(x)$ has a derivative equal to $\mathbf{f}'(x_0)\mathbf{g}(x_0) + \mathbf{f}(x_0)\mathbf{g}'(x_0)$ at x_0. In particular, if $U(x) = (\alpha_{ij}(x))$ and $V(x) = (\beta_{ij}(x))$ are two *square matrices* of order n, differentiable at x_0, their product UV has a derivative equal to $U'(x_0)V(x_0) + U(x_0)V'(x_0)$ at x_0 (where $U'(x) = (\alpha'_{ij}(x))$ and $V'(x) = (\beta'_{ij}(x))$).

5) The *determinant* $\det(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ of n vectors $\mathbf{x}_i = (x_{ij})_{1 \leqslant j \leqslant n}$ from the space $\mathbf{R}^n$ (*Alg.*, III, p. 522) being a (continuous) multilinear function of the $\mathbf{x}_i$, one sees that if the

n^2 real functions f_{ij} are differentiable at x_0, then their determinant $g(x) = \det(f_{ij}(x))$ has a derivative equal to

$$\sum_{i=1}^{n} \Big[\mathbf{f}_1(x_0), \ldots, \mathbf{f}_{i-1}(x_0), \mathbf{f}_i'(x_0), \mathbf{f}_{i+1}(x_0), \ldots, \mathbf{f}_n(x_0)\Big]$$

at x_0, where $\mathbf{f}_i(x) = (f_{ij}(x))_{1 \leqslant j \leqslant n}$; in other words, one obtains the derivative of a determinant of order n by taking the sum of the n determinants formed by replacing, for each i, the terms of the i^{th} column by their derivatives.

Remark. If $U(x)$ is a square matrix which is differentiable and invertible at the point x_0, then the derivative of its determinant $\Delta(x) = \det(U(x))$ can be expressed through the derivative of $U(x)$ by the formula

$$\Delta'(x_0) = \Delta(x_0).\mathrm{Tr}(U'(x_0)U^{-1}(x_0)). \tag{3}$$

Indeed, let us put $U(x_0+h) = U(x_0)+hV$; then, by definition, V tends to $U'(x_0)$ when h tends to 0. One can write

$$\Delta(x_0 + h) = \Delta(x_0).\det(I + hVU^{-1}(x_0)).$$

Now $\det(I + hX) = 1 + h\mathrm{Tr}(X) + \sum_{k=2}^{n} \lambda_k h^k$, the λ_k ($k \geqslant 2$) being polynomials in the elements of the matrix X; since the elements of $VU^{-1}(x_0)$ have a limit when h tends to 0, we indeed obtain the formula (3).

4. DERIVATIVE OF THE INVERSE OF A FUNCTION

PROPOSITION 4. *Let* E *be a complete normed algebra with a unit element over* **R** *and let* **f** *be a function defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *taking values in* E, *and differentiable at the point* $x_0 \in \mathrm{I}$. *If* $\mathbf{y}_0 = \mathbf{f}(x_0)$ *is invertible* [2] *in* E, *then the function* $x \mapsto (\mathbf{f}(x))^{-1}$ *is defined on a neighbourhood of* x_0 *(relative to* I), *and has a derivative equal to* $-\big(\mathbf{f}(x_0)\big)^{-1}\mathbf{f}'(x_0)\big(\mathbf{f}(x_0)\big)^{-1}$ *at* x_0.

Indeed, the set of invertible elements in E is an open set on which the function $\mathbf{y} \mapsto \mathbf{y}^{-1}$ is continuous (*Gen. Top.*, IX, p. 178); since **f** is continuous (relative to I) at x_0, $\big(\mathbf{f}(x)\big)^{-1}$ is defined on a neighbourhood of x_0, and we have

$$\big(\mathbf{f}(x)\big)^{-1} - \big(\mathbf{f}(x_0)\big)^{-1} = \big(\mathbf{f}(x)\big)^{-1}\Big(\mathbf{f}(x_0) - \mathbf{f}(x)\Big)\big(\mathbf{f}(x_0)\big)^{-1}.$$

The proposition thus follows from the continuity of $\mathbf{y}^{-1}$ on a neighbourhood of $\mathbf{y}_0$ and the continuity of $\mathbf{xy}$ on $\mathrm{E} \times \mathrm{E}$.

[2] Recall from (*Alg.*, I, p. 15) that an element $\mathbf{z} \in \mathrm{E}$ is said to be *invertible* if there exists an element of E, denoted by $\mathbf{z}^{-1}$, such that $\mathbf{z}\mathbf{z}^{-1} = \mathbf{z}^{-1}\mathbf{z} = \mathbf{e}$ ($\mathbf{e}$ being the unit element of E).

Examples. 1) The most important particular case is that where E is one of the fields **R** or **C** : if f is a function with real or complex values, differentiable at the point x_0, and such that $f(x_0) \neq 0$, then $1/f$ has derivative equal to $-f'(x_0)/(f(x_0))^2$ at x_0.

2) If $U = (\alpha_{ij}(x))$ is a square matrix of order n, differentiable at x_0 and invertible at this point, then U^{-1} has derivative equal to $-U^{-1}U'U^{-1}$ at x_0.

5. DERIVATIVE OF A COMPOSITE FUNCTION

PROPOSITION 5. *Let f be a real function defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *and* $\mathbf{g}$ *a vector function defined on an interval of* $\mathbf{R}$ *containing* $f(\mathrm{I})$. *If f is differentiable at the point x_0 and* $\mathbf{g}$ *is differentiable at the point $f(x_0)$ then the composite function* $\mathbf{g} \circ f$ *has a derivative equal to* $\mathbf{g}'(f(x_0))f'(x_0)$ *at* x_0.

Let us put $\mathbf{h} = \mathbf{g} \circ f$; for $x \neq x_0$ we can write

$$\frac{\mathbf{h}(x) - \mathbf{h}(x_0)}{x - x_0} = \mathbf{u}(x)\,\frac{f(x) - f(x_0)}{x - x_0}$$

where we set $\mathbf{u}(x) = \dfrac{\mathbf{g}(f(x)) - \mathbf{g}(f(x_0))}{f(x) - f(x_0)}$ if $f(x) \neq f(x_0)$, and $\mathbf{u}(x) = \mathbf{g}'(f(x_0))$ otherwise. Now $f(x)$ has limit $f(x_0)$ when x tends to x_0, so $\mathbf{u}(x)$ has limit $\mathbf{g}'(f(x_0))$, from which the proposition follows in view of the continuity of the function $\mathbf{y}x$ on $\mathrm{E} \times \mathbf{R}$.

6. DERIVATIVE OF AN INVERSE FUNCTION

PROPOSITION 6. *Let f be a homeomorphism of an interval* $\mathrm{I} \subset \mathbf{R}$ *onto an interval* $\mathrm{J} = f(\mathrm{I}) \subset \mathbf{R}$, *and let g be the inverse homeomorphism*[3]. *If f is differentiable at the point* $x_0 \in \mathrm{I}$, *and if* $f'(x_0) \neq 0$, *then g has a derivative equal to* $1/f'(x_0)$ *at* $y_0 = f(x_0)$.

For each $y \in \mathrm{J}$ we have $g(y) \in \mathrm{I}$ and $u = f(g(y))$; we thus can write $\dfrac{g(y) - g(y_0)}{y - y_0} = \dfrac{g(y) - x_0}{f(g(y)) - f(x_0)}$, for $y \neq y_0$. When y tends to y_0 while remaining in J and $\neq y_0$, then $g(y)$ tends to x_0 remaining in I and $\neq x_0$, and the right-hand side in the preceding formula thus has limit $1/f'(x_0)$, since by hypothesis $f'(x_0) \neq 0$.

COROLLARY. *If f is differentiable on* I *and if* $f'(x) \neq 0$ *on* I, *then g is differentiable on* J *and its derivative at each point* $y \in \mathrm{J}$ *is* $1/f'(g(y))$.

For example, for each integer $n > 0$, the function $x^{1/n}$ is a homeomorphism of $\mathbf{R}_+$ onto itself, is the inverse of x^n, and has derivative $\frac{1}{n}x^{\frac{1}{n}-1}$ at each $x > 0$.

One deduces easily, from prop. 5, that for every rational number $r = p/q > 0$ the function $x^r = \left(x^{1/q}\right)^p$ has derivative rx^{r-1} at every $x > 0$.

[3] For f to be a homeomorphism of I onto a subset of **R** we know that it is necessary and sufficient that f be continuous and strictly monotone on I (*Gen. Top.*, IV, p. 338, th. 5).

Remarks. 1) All the preceding propositions, stated for functions differentiable at a point x_0, immediately yield propositions for functions which are right (resp. left) differentiable at x_0, when, instead of the functions themselves, one considers their restrictions to the intersection of their intervals of definition with the interval $[x_0, +\infty[$ (resp. $]-\infty, x_0]$); we leave it to the reader to state them.

2) The preceding definitions and propositions (except for those concerning right and left derivatives) extend easily to the case where one replaces $\mathbf{R}$ by an arbitrary *commutative non-discrete topological field* K, and the topological vector spaces (resp. topological algebras) over $\mathbf{R}$ by topological vector spaces (resp. topological algebras) over K. In def. 1 and props. 1, 2 and 3 it is enough to replace I by a *neighbourhood* of x_0 in K; in prop. 4 one must assume further that the map $\mathbf{y} \mapsto \mathbf{y}^{-1}$ is defined and continuous on a neighbourhood of $\mathbf{f}(x_0)$ in E. Prop. 5 generalizes in the following manner: let K′ be a non-discrete subfield of the topological field K, let E be a topological vector space *over* K; let f be a function defined on a neighbourhood $V \subset \mathrm{K}'$ of $x_0 \in \mathrm{K}'$, with values in K (considered as a topological vector space over K′), differentiable at x_0, and let $\mathbf{g}$ be a function defined on a neighbourhood of $f(x_0) \in \mathrm{K}$, with values in E, and differentiable at the point $f(x_0)$; then the map $\mathbf{g} \circ f$ is differentiable at x_0 and has derivative $\mathbf{g}'(f(x_0))f'(x_0)$ there (E being then considered as a topological vector space *over* K′).

With the same notation, let $\mathbf{f}$ be a function defined on a neighbourhood V of $a \in \mathrm{K}$, with values in E, and differentiable at the point a; if $a \in \mathrm{K}'$, then the *restriction* of $\mathbf{f}$ to $\mathrm{V} \cap \mathrm{K}'$ is differentiable at a, and has derivative $\mathbf{f}'(a)$ there. These considerations apply above all, in practice, to the case where $\mathrm{K} = \mathbf{C}$ and $\mathrm{K}' = \mathbf{R}$.

Finally, prop. 6 extends to the case where one replaces I by a neighbourhood of $x_0 \in \mathrm{K}$, and f by a homeomorphism of I onto a neighbourhood $\mathrm{J} = f(\mathrm{I})$ of $y_0 = f(x_0)$ in K.

7. DERIVATIVES OF REAL-VALUED FUNCTIONS

The preceding definitions and propositions may be augmented in several respects when we deal with *real-valued* functions of a real variable.

In the first place, if f is such a function, defined on an interval $\mathrm{I} \subset \mathbf{R}$, and continuous relative to I at a point $x_0 \in \mathrm{I}$, it can happen that when x tends to x_0 while remaining in I and $\neq x_0$, that $\dfrac{f(x) - f(x_0)}{x - x_0}$ has a limit equal to $+\infty$ or to $-\infty$; one then says that f is differentiable at x_0 and has derivative $+\infty$ (resp. $-\infty$) there; if the function f has a derivative $f'(x)$ (finite or infinite) at every point x of I, then the function f' (with values in $\overline{\mathbf{R}}$) is again called the derived function (or simply the derivative) of f. One generalizes the definitions of right and left derivative similarly.

Example. At the point $x = 0$ the function $x^{1/3}$ (the inverse function of x^3, a homeomorphism of $\mathbf{R}$ onto itself) has a derivative, equal to $+\infty$; at $x = 0$ the function $|x|^{1/3}$ has right derivative $+\infty$ and left derivative $-\infty$.

The formulae for the derivative of a sum, of a product of differentiable real functions, and for the inverse of a differentiable function (props. 1, 3 and 4), as well as for the derivative of a (real-valued) composition of functions (prop. 5) remain valid when the derivatives that occur are infinite, so long as all the expressions that occur in these formulae make sense (*Gen. Top.*, IV, p. 345–346). In fact, if in prop. 6 one supposes that f is strictly increasing (resp. strictly decreasing) and continuous on I, and if $f'(x_0) = 0$, then the inverse function g has a derivative equal to $+\infty$

(resp. $-\infty$) at the point $y_0 = f(x_0)$; if $f'(x_0) = +\infty$ (resp. $-\infty$), then g has derivative 0. There are similar results for right and left derivatives, which we leave to the care of the reader.

Let C be the *graph* or *representing curve* of a finite real function f, the subset of the plane $\mathbf{R}^2$ formed by the points $(x, f(x))$ where x runs through the set where f is defined. If the function f has a finite right derivative at a point $x_0 \in \mathrm{I}$, then the half-line with origin at the point $\mathrm{M}_{x_0} = (x_0, f(x_0))$ of C, and direction numbers $(1, f'_d(x_0))$ is called the *right half-tangent* to the curve C at the point M_{x_0}; when $f'_d(x_0) = +\infty$ (resp. $f'_d(x_0) = -\infty$) we use the same terminology for the half-line with origin M_{x_0} and direction numbers $(0, 1)$ (resp. $(0, -1)$). In the same way one defines the *left half-tangent* at M_{x_0} when $f'_g(x_0)$ exists. With these definitions one can verify quickly that the angle which the right (resp. left) half-tangent makes with the abscissa is the *limit* of the angle made by this axis with the half-line originating at M_{x_0} and passing through the point $\mathrm{M}_x = (x, f(x))$ of C, as x tends to x_0 while remaining $> x_0$ (resp. $< x_0$).

One can also say that the right (resp. left) half-tangent is the *limit* of the half-line originating at M_{x_0} passing through M_x, on endowing the set of half-lines having the same origin with the quotient space topology $\mathrm{C}^*/\mathbf{R}_+^*$ (*Gen. Top.*, VIII, p. 107).

If the two half-tangents exist at a point M_{x_0} of C, they are in opposite directions only when f has a *derivative* (finite or not) at the point x_0 (assumed interior to I); they are identical only when $f'_d(x_0)$ and $f'_g(x_0)$ are infinite and of opposite sign. In these two cases we say that the line containing these two half-tangents is the *tangent* to C at the point M_{x_0}.

When the tangent at M_{x_0} exists it is the *limit* of the line passing through M_{x_0} and M_x as x tends to x_0 remaining $\neq x_0$, the topology on the set of lines which pass through a given fixed point being that of the quotient space $\mathrm{C}^*/\mathbf{R}^*$ (*Gen. Top.*, VIII, p. 114).

The concepts of tangent and half-tangent to a graph are particular cases of general concepts which will be defined in the part of this Series devoted to differentiable varieties.

DEFINITION 4. *We say that a real function f, defined on a subset* A *of a topological space* E, *has a relative maximum* (resp. *strict relative maximum, relative minimum, strict relative minimum*) *at a point $x_0 \in$* A, *relative to* A, *if there is a neighbourhood* V *of x_0 in* E *such that at every point $x \in \mathrm{V} \cap \mathrm{A}$ distinct from x_0 one has $f(x) \leqslant f(x_0)$* (resp. $f(x) < f(x_0)$, $f(x) \geqslant f(x_0)$, $f(x) > f(x_0)$).

It is clear that if f attains its least upper bound (resp. greatest lower bound) over A at a point of A, then it has a relative maximum (resp. relative minimum) relative to A at this point; the converse is of course incorrect.

Note that if $\mathrm{B} \subset \mathrm{A}$, and if f admits (for example) a relative maximum at a point $x_0 \in \mathrm{B}$ *relative to* B, then f does not necessarily have a relative maximum *relative to* A at this point.

PROPOSITION 7. *Let f be a finite real function, defined on an interval* $\mathrm{I} \subset \mathbf{R}$. *If f admits a relative maximum* (resp. *relative minimum*) *at a point x_0 interior to* I, *and has both right and left derivatives at this point, then one has* $f'_d(x_0) \leqslant 0$ *and* $f'_g(x_0) \geqslant 0$ (resp. $f'_d(x_0) \geqslant 0$ *and* $f'_g(x_0) \leqslant 0$); *in particular, if* f *is differentiable at the point* x_0, *then* $f'(x_0) = 0$.

The proposition follows trivially from the definitions.

We can say further that if at a point x_0 interior to I the function f is differentiable and admits a relative maximum or minimum, then the tangent to its graph is *parallel to the abscissa*. The converse is incorrect, as is shown by the example of the function x^3 which has zero derivative at the point $x = 0$, but has neither relative maximum nor minimum at this point.

§ 2. THE MEAN VALUE THEOREM

The hypotheses and conclusions demonstrated in § 1 are *local* in character: they concern the properties of the functions under consideration only on an *arbitrarily small* neighbourhood of a fixed point. In contrast, the questions which we treat in this section involve the properties of a function on *all of an interval*.

1. ROLLE'S THEOREM

PROPOSITION 1 ("Rolle's theorem"). *Let* f *be a real function which is finite and continuous on a closed interval* $\mathrm{I} = [a, b]$ *(where* $a < b$), *has a derivative* (finite or not) *at every point of* $]a, b[$, *and is such that* $f(a) = f(b)$. *Then there exists a point* c *of* $]a, b[$ *such that* $f'(c) = 0$.

The proposition is evident if f is constant: if not, f takes, for example, values $> f(a)$, and so attains its least upper bound at a point c *interior* to I (*Gen. Top.*, IV, p. 359, th. 1). Since f has a relative maximum at this point we have $f'(c) = 0$ (I, p. 20, prop. 7).

COROLLARY. *Let* f *be a real function which is finite and continuous on* $[a, b]$ *(where* $a < b$), *and has a derivative* (finite or not) *at every point. Then there exists a point* c *of* $]a, b[$ *such that* $f(b) - f(a) = f'(c)(b - a)$.

We need only apply prop. 1 to the function $f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a)$.

This corollary signifies that there is a point $\mathrm{M}_c = (c, f(c))$ on the graph C of f such that $a < c < b$ and such that the tangent to C at this point is *parallel* to the line joining the points $\mathrm{M}_a = (a, f(a))$ and $\mathrm{M}_b = (b, f(b))$.

2. THE MEAN VALUE THEOREM FOR REAL-VALUED FUNCTIONS

The following important result is a consequence of the corollary to prop. 1: if one has $m \leqslant f'(x) \leqslant \mathrm{M}$ on $]a,\ b[$, then also $m \leqslant \dfrac{f(b)-f(a)}{b-a} \leqslant \mathrm{M}$. In other words, a *bound for the derivative of* f' on the whole interval with endpoints a, b implies the *same bound* for $\dfrac{f(b)-f(a)}{b-a}$ (the ratio of the "increment" of the function to the "increment" of the variable on the interval). We shall make this fundamental result more precise, and generalize it, in the sequel.

PROPOSITION 2. *Let f be a real function which is finite and continuous on the closed bounded interval* $\mathrm{I} = [a,\ b]$ *(where $a < b$) and has a right derivative* (finite or not) *at all the points of the relative complement in* $[a,\ b)$ *of a* countable *subset* A *of this interval. If $f_d'(x) \geqslant 0$ at every point of $[a,\ b[$ not belonging to* A, *then one has $f(b) \geqslant f(a)$; if, further, $f_d'(x) > 0$ for at least one point of $[a,\ b[$, then $f(b) > f(a)$.*

Let $\varepsilon > 0$ be arbitrary, and denote by $(a_n)_{n\geqslant 1}$ a sequence obtained by listing the countable set A. Let J be the set of points $y \in \mathrm{I}$ such that one has

$$f(x) - f(a) \geqslant -\varepsilon(x-a) - \varepsilon \sum_{a_n < x} \frac{1}{2^n} \tag{1}$$

for all x with $a \leqslant x \leqslant y$, the sum in the second term of the right-hand side being taken over all indices n for which $a_n < x$. We shall show that if $f_d'(x) \geqslant 0$ at every point of $[a,\ b[$ distinct from the a_n, then $\mathrm{J} = \mathrm{I}$.

It is clear that J is not empty, since $a \in \mathrm{J}$; moreover the definition of this set shows that if $y \in \mathrm{J}$ one has $x \in \mathrm{J}$ for $a \leqslant x \leqslant y$, so J is an *interval* with left-hand endpoint a (*Gen. Top.*, IV, p. 336, prop. 1); let c be its right-hand endpoint. One has $c \in \mathrm{J}$; this is clear if $c = a$; if not, for every $x < c$ we have the inequality (1), and *a fortiori*

$$f(x) - f(a) \geqslant -\varepsilon(c-a) - \varepsilon \sum_{a_n < c} \frac{1}{2^n}$$

from which it follows, on letting x tend to c in this inequality (since f is continuous), that c satisfies (1).

This being so, we shall see that we must have $c = b$. Indeed, if one had $c < b$, then certainly one would have $c \notin \mathrm{A}$; now $f_d'(c)$ exists, and since $f_d'(c) \geqslant 0$ by hypothesis, there exists a y such that $c < y \leqslant b$ and such that for $c \leqslant x \leqslant y$ one has

$$f(x) - f(c) \geqslant -\varepsilon(x-c)$$

from which, taking account of (1), where x is replaced by c,

$$f(x) - f(a) \geqslant -\varepsilon(x-a) - \varepsilon \sum_{a_n < c} \frac{1}{2^n} \geqslant -\varepsilon(x-a) - \varepsilon \sum_{a_n < x} \frac{1}{2^n}$$

which signifies that $y \in \mathrm{J}$, contradicting the definition of c. Thus we have $c = a_k$ for some index k; since f is continuous at the point a_k there is a y such that $c < y \leqslant b$ and such that for $c < x \leqslant y$ one has

$$f(x) - f(c) \geqslant -\frac{\varepsilon}{2^k}$$

from which, taking account of (1), where x is replaced by c,

$$f(x) - f(a) \geqslant -\varepsilon(c-a) - \varepsilon \sum_{a_n<x} \frac{1}{2^n} \geqslant -\varepsilon(x-a) - \varepsilon \sum_{a_n<x} \frac{1}{2^n}$$

which again leads to a contradiction; we thus have $c = b$, and in consequence

$$f(b) - f(a) \geqslant -\varepsilon(b-a) - \varepsilon \sum_{a_n<b} \frac{1}{2^n} \geqslant -\varepsilon(b-a) - \varepsilon. \qquad (2)$$

Since $\varepsilon > 0$ is arbitrary we deduce from (2) that $f(b) \geqslant f(a)$, which demonstrates the first part of the proposition.

We remark now that this result applied to an interval $[x, y]$ where $a \leqslant x < y \leqslant b$ proves that f is *increasing* on I; if one had $f(b) = f(a)$ one could deduce that f is *constant* on I, and then that $f'_d(x) = 0$ at every point of $[a, b[$; the second part follows from this.

COROLLARY. *Let f be a finite continuous real function on $[a, b]$ (where $a < b$) and have a right derivative at all points of the complement in $[a, b[$ of a countable subset* A *of this interval. For f to be increasing on* I *it is necessary and sufficient that $f'_d(x) \geqslant 0$ at every point of $[a, b[$ that does not belong to* A; *for f to be strictly increasing it is necessary and sufficient that that the preceding condition holds, and further that the set of points x where $f'_d(x) > 0$ be dense in $[a, b]$.*

Remarks. 1) Prop. 2 remains true when one replaces the interval $[a, b[$ by $]a, b]$ and the words "right derivative" by "left derivative".

2) The hypothesis of *continuity* on f on the closed interval I (and not just *right continuity* [4] at every point of $[a, b[$) is essential for the validity of prop. 2 (*cf.* I, p. 36, exerc. 8).

3) The conclusion of prop. 2 is not guaranteed if one merely supposes that the set A of "exceptional" points is nowhere dense in I, but not countable (*cf.* I, p. 37, exerc. 3).

Prop. 2 entails the following fundamental theorem (which appears to be more general):

THEOREM 1 (mean value theorem). *Let f and g be two finite continuous real-valued functions defined on a closed bounded interval* $\mathrm{I} = [a, b]$ *and having a*

[4] A function defined on an interval $\mathrm{I} \subset \mathbf{R}$ is said to be *right continuous* at a point $x_0 \in \mathrm{I}$ if its restriction to the interval $\mathrm{I} \cap [x_0, +\infty[$ is continuous at the point x_0 relative to this interval; it comes to the same to say that the right limit of this function exists at this point and is equal to the value of the function at this point.

right derivative (finite or not) *at all points of the relative complement in* $[a,\ b[$ *of a countable subset of this interval. Suppose further that* $f'_d(x)$ *and* $g'_r(x)$ *are not simultaneously infinite except at the points of a countable subset of* I *and that there are finite numbers* m, M *such that*

$$m g'_r(x) \leqslant f'_d(x) \leqslant \mathrm{M} g'_r(x) \tag{3}$$

except at the points of a countable subset of I (replacing $\mathrm{M}g'_r(x)$ (resp. $mg'_r(x)$) by 0 if $\mathrm{M} = 0$ (resp. $m = 0$) and $g'_r(x) = \pm\infty$). *Under these conditions one has*

$$m\,(g(b) - g(a)) < f(b) - f(a) < \mathrm{M}\,(g(b) - f(a)) \tag{4}$$

except when one has $f(x) = \mathrm{M}g(x) + k$, *or* $f(x) = mg(x) + k$ (k *constant) for all* $x \in \mathrm{I}$.

It suffices to apply prop. 2 to the functions $\mathrm{M}g - f$ and $f - mg$, which, under our hypotheses, have a positive right derivative except at the points of a countable subset of I.

Remark. Th. 1 fails if one allows f'_d and g'_r to be simultaneously infinite on an uncountable subset of I (*cf.* I, p. 37, exerc. 3).

COROLLARY. *Let* f *be a finite continuous function on* $[a,\ b]$ *(where* $a < b$) *and have a right derivative* (finite or not) *at all points of the relative complement* B *in* $[a,\ b[$ *of a countable subset of this interval. If* m *and* M *are the greatest lower and least upper bounds of* f'_d *on* B *then one has*

$$m(b - a) < f(b) - f(a) < \mathrm{M}(b - a) \tag{5}$$

if f *is not an affine linear function; if* f *is affine linear one has*

$$m = \mathrm{M} = \frac{f(b) - f(a)}{b - a}.$$

The inequalities (5) are consequences of (4) when m and M are finite; the case when one or the other of these numbers is infinite is trivial.

Remark. The inequalities (5) prove that a continuous function cannot have right derivative equal to $+\infty$ at all points of an interval (*cf.* I, p. 38, exerc. 6).

3. THE MEAN VALUE THEOREM FOR VECTOR FUNCTIONS

THEOREM 2. *Let* $\mathbf{f}$ *be a vector function defined and continuous on a closed bounded interval* $\mathrm{I} = [a,\ b]$ *of* $\mathbf{R}$ *(where* $a < b$) *and taking values in a normed space* E *over* $\mathbf{R}$; *let* g *be a continuous increasing real function on* I. *Suppose that* $\mathbf{f}$ *and* g *have right derivatives at all points of the relative complement in* $[a,\ b[$ *of a countable* *subset* A *of this interval* (allowing $g'_r(x)$ to be infinite at some of the points $x \notin \mathrm{A}$), *and suppose that at each of these points we have*

$$\left\|\mathbf{f}'_d(x)\right\| \leqslant g'_r(x). \tag{6}$$

Under these hypotheses one has

$$\|\mathbf{f}(b) - \mathbf{f}(a)\| \leqslant g(b) - g(a). \tag{7}$$

The proof proceeds similarly to that of prop. 2. Let $\varepsilon > 0$ be arbitrary, and (a_n) the sequence obtained by enumerating A in some order. Let J be the set of points $y \in \mathrm{I}$ such that, for all x such that $a \leqslant x \leqslant y$ one has

$$\|\mathbf{f}(x) - \mathbf{f}(a)\| \leqslant g(x) - g(a) + \varepsilon(x - a) + \varepsilon \sum_{a_n < x} \frac{1}{2^n}; \tag{8}$$

we shall show that $\mathrm{J} = \mathrm{I}$. One sees immediately, as in prop. 2, that J is an interval with left-hand endpoint a; if c is its right-hand endpoint then $c \in \mathrm{J}$; indeed, for all $x < c$ one has (8), and *a fortiori*

$$\|\mathbf{f}(x) - \mathbf{f}(a)\| \leqslant g(c) - g(a) + \varepsilon(c - a) + \varepsilon \sum_{a_n < c} \frac{1}{2^n}$$

from which, letting x tend to c in this inequality, it follows from the continuity of $\mathbf{f}$ that c satisfies (8).

Let us show that we must have $c = b$. So suppose that $c < b$ and that moreover $c \notin \mathrm{A}$: then $\mathbf{f}'_d(c)$ and $g'_r(c)$ exist and satisfy (6); suppose in the first place that $g'_r(c)$ (which is necessarily $\geqslant 0$) is finite; then one can always write $\mathbf{f}'_d(c) = \mathbf{u}g'_r(c)$, with $\|\mathbf{u}\| \leqslant 1$; since the function $\mathbf{f}(x) - \mathbf{u}g(x)$ has zero right derivative at the point c there must exist a y such that $c < y \leqslant b$ and such that for $c \leqslant x \leqslant y$ one has

$$\|\mathbf{f}(x) - \mathbf{f}(c) - \mathbf{u}(g(x) - g(c))\| \leqslant \varepsilon(x - c)$$

from which

$$\|\mathbf{f}(x) - \mathbf{f}(c)\| \leqslant g(x) - g(c) + \varepsilon(x - c)$$

and, taking account of (8), in which x is replaced by c,

$$\begin{aligned}\|\mathbf{f}(x) - \mathbf{f}(a)\| &\leqslant g(x) - g(a) + \varepsilon(x - a) + \varepsilon \sum_{a_n < c} \frac{1}{2^n} \\ &\leqslant g(x) - g(a) + \varepsilon(x - a) + \varepsilon \sum_{a_n < x} \frac{1}{2^n}.\end{aligned}$$

Thus one has $y \in \mathrm{J}$, which is a contradiction. Suppose next that $c \notin \mathrm{A}$ and that $g'_r(c) = +\infty$; then there is a y such that $c < y \leqslant b$ and such that for $c \leqslant x \leqslant y$ one has on the one hand

$$\|\mathbf{f}(x)-\mathbf{f}(c)\| \leqslant \left(\left\|\mathbf{f}'_d(c)\right\|+1\right)(x-c)$$

while on the other hand

$$g(x)-g(c) \geqslant \left(\left\|\mathbf{f}'_d(c)\right\|+1\right)(x-c)$$

from which

$$\|\mathbf{f}(x)-\mathbf{f}(c)\| \leqslant g(x)-g(c)$$

and one concludes as above. Finally, if one has $c = a_k$, then there is a y such that $c < y \leqslant b$, and such that for $c < x \leqslant y$ one has

$$\|\mathbf{f}(x)-\mathbf{f}(c)\| \leqslant \frac{\varepsilon}{2^k}$$

from which, taking account of (8), with x replaced by c,

$$\begin{aligned}\|\mathbf{f}(x)-\mathbf{f}(a)\| &\leqslant g(c)-g(a)+\varepsilon(c-a)+\varepsilon\sum_{a_n<x}\frac{1}{2^n}\\ &\leqslant g(x)-g(a)+\varepsilon(x-a)+\varepsilon\sum_{a_n<x}\frac{1}{2^n}\end{aligned}$$

which again entails a contradiction. The proof finishes as that of prop. 2.

Q.E.D.

Remarks. 1) Here again, in the statement of th. 2 one can replace the interval $[a, b[$ by $]a, b]$ and "right derivative" by "left derivative".

2) We shall show later how to identify the case of equality in (7), and also how to generalize th. 2 to the case where E is an arbitrary locally convex space, with the help of another method of proof which allows one to deduce th. 2 from th. 1.

COROLLARY. *For a continuous vector function on an interval* $\mathrm{I} \subset \mathbf{R}$, *with values in a normed space* E *over* $\mathbf{R}$, *to be constant on* I *it suffices that it have zero right derivative at all points of the complement* (relative to I) *of a countable subset of* I.

Remark. The proofs of ths. 1 and 2 rely in an essential manner on the special topological properties of the field $\mathbf{R}$; one can give examples of valued fields K for which there are nonconstant linear maps of K to itself with zero derivative at every point (*cf.* I, p. 37, exerc. 2).

PROPOSITION 3. *Let* $\mathbf{f}$ *be a vector function with values in a normed space* E *over* $\mathbf{R}$, *defined and continuous on an interval* $\mathrm{I} \subset \mathbf{R}$, *and right differentiable on the complement* B (relative to I) *of a countable subset of* I; *then for all points* $x_0 \in \mathrm{B}$, $x \in \mathrm{I}$, $y \in \mathrm{I}$, *one has* (supposing that $x < y$, for example)

$$\left\|\mathbf{f}(y)-\mathbf{f}(x)-\mathbf{f}'_d(x_0)(y-x)\right\| \leqslant (y-x)\sup_{z\in\mathrm{B},\ x<z<y}\left\|\mathbf{f}'_d(z)-\mathbf{f}'_d(x_0)\right\|. \qquad (9)$$

Indeed it suffices to apply th. 2, replacing $\mathbf{f}$ by the function

$$\mathbf{f}(z)-\mathbf{f}'_d(x_0)z,$$

and g by the linear function whose derivative is $\sup\limits_{z\in\mathrm{B},\ x<z<y}\left\|\mathbf{f}'_d(z)-\mathbf{f}'_d(x_0)\right\|$.

Theorem 2 extends to vector functions of a *complex* variable:

PROPOSITION 4. *Let* $\mathbf{f}$ *be a continuous differentiable function of a complex variable defined on a convex open subset* A *of the field* $\mathbf{C}$, *with values in a normed space* E *over the field* $\mathbf{C}$. *If one has* $\|\mathbf{f}'(z)\| \leqslant m$ *for all* $z \in$ A, *then one has* $\|\mathbf{f}(b) - \mathbf{f}(a)\| \leqslant m\,|b-a|$ *for every pair of points* a, b *of* A.

We put $\mathbf{g}(t) = \dfrac{1}{b-a}\,\mathbf{f}(a + t(b-a))$ for $0 \leqslant t \leqslant 1$; since $\mathbf{g}'(t) = \mathbf{f}'(a + t(b-a))$, applying th. 2 to the function $\mathbf{g}$ yields the proposition immediately.

COROLLARY. *For a vector function* $\mathbf{f}$ *of a complex variable, defined and continuous on an open set* $\mathrm{A} \subset \mathbf{C}$, *and with values in a normed space over* $\mathbf{C}$, *to be constant, it suffices that it have zero derivative at every point of* A.

Indeed, let a be an arbitrary point of A; the set B of points z of A where $\mathbf{f}(z) = \mathbf{f}(a)$ is *closed* because $\mathbf{f}$ is continuous; it is also *open,* as is shown by applying prop. 4 (with $m = 0$) to a convex open neighbourhood, contained in A, of an arbitrary point of B; so is identical to A.

PROPOSITION 5. *Let* $\mathbf{f}$ *be a vector function of a complex variable, defined, continuous and differentiable on a convex open set* $\mathrm{A} \subset \mathbf{C}$, *taking values in a normed space over the field* $\mathbf{C}$; *then, no matter what the points* x_0, x *and* y *in* A, *one has*

$$\left\|\mathbf{f}(y) - \mathbf{f}(x) - \mathbf{f}'_d(x_0)(y-x)\right\| \leqslant |y-x| \sup_{z \in \mathrm{A}} \left\|\mathbf{f}'(z) - \mathbf{f}'(x_0)\right\|. \tag{10}$$

It suffices to apply th. 2 to the function

$$\mathbf{g}(t) = \mathbf{f}(x + t(y-x)) - \mathbf{f}'(x_0)(y-x)t$$

on the interval $[0,\ 1]$.

4. CONTINUITY OF DERIVATIVES

PROPOSITION 6. *Let* I *be an open interval in* $\mathbf{R}$, *let* x_0 *be one of the endpoints of* I, *and* $\mathbf{f}$ *a vector function defined and continuous on* I, *with values in a complete normed space* E *over* $\mathbf{R}$; *suppose that* $\mathbf{f}$ *has a right derivative at the points of the complement* B *in* I *of a countable subset of* I. *Then for* $\mathbf{f}'_d(x)$ *to have a limit as* x *tends to* x_0 *while remaining in* B *and* $\neq x_0$ *it is necessary and sufficient that* $\dfrac{\mathbf{f}(y) - \mathbf{f}(x)}{y-x}$ *have a limit* $\mathbf{c}$ *as* (x, y) *tends to* (x_0, x_0) *subject to* $x \in \mathrm{I}, y \in \mathrm{I}, x \neq x_0, y \neq x_0$ *and* $x \neq y$. *Under these conditions* $\mathbf{f}$ *extends by continuity to the point* x_0, *the right* *derivative* $\mathbf{f}'_d(x)$ *tends to* $\mathbf{c}$ *as* x *tends to* x_0 (while remaining in B) *and the function* $\mathbf{f}$ *extended* (defined on $\mathrm{I} \cup \{x_0\}$) *has derivative at* x_0 *equal to* $\mathbf{c}$.

Suppose for example that x_0 is the right-hand endpoint of I. Let us first show that if $\mathbf{f}'_d(x)$ tends to $\mathbf{c}$ as x tends to x_0 while remaining in B and $\neq x_0$, then $\dfrac{\mathbf{f}(y) - \mathbf{f}(x)}{y - x}$ tends to $\mathbf{c}$; this follows immediately from th. 2 applied to the function $\mathbf{f}(z) - \mathbf{c}z$, which yields

$$\|\mathbf{f}(y) - \mathbf{f}(x) - \mathbf{c}(y - x)\| \leqslant (y - x) \sup_{z \in \mathrm{B},\, x<z<y} \left\|\mathbf{f}'_d(z) - \mathbf{c}\right\|$$

for $x < y < x_0$. Conversely, if $\dfrac{\mathbf{f}(y) - \mathbf{f}(x)}{y - x}$ tends to $\mathbf{c}$, then for every $\varepsilon > 0$ there exists an $h > 0$ such that the conditions $|x - x_0| < h$, $|y - x_0| < h$ $(x \neq x_0, y \neq x_0)$ imply

$$\|\mathbf{f}(y) - \mathbf{f}(x) - \mathbf{c}(y - x)\| \leqslant \varepsilon\, |y - x|\,. \tag{11}$$

But for all $x \in \mathrm{B}$ and $\neq x_0$ such that $|x - x_0| < h$ there exists a $k > 0$ (depending on x) such that the relation $x < y < x + k$ entails

$$\left\|\mathbf{f}(y) - \mathbf{f}(x) - \mathbf{f}'_d(x)(y - x)\right\| \leqslant \varepsilon\, |y - x| \tag{12}$$

from which, considering (11):

$$\left\|\mathbf{f}'_d(x) - \mathbf{c}\right\| \leqslant 2\varepsilon$$

for $|x - x_0| < h, x \in \mathrm{B}$ and $x \neq x_0$, which proves that $\mathbf{f}'_d(x)$ tends to $\mathbf{c}$. Moreover, from the relation (11) one has immediately that

$$\|\mathbf{f}(y) - \mathbf{f}(x)\| \leqslant (\|\mathbf{c}\| + \varepsilon)\, |y - x|\,,$$

which proves (by Cauchy's criterion) that $\mathbf{f}$ has a limit $\mathbf{d}$ at the point x_0 as x tends to this point while remaining in I and $\neq x_0$; now, letting x approach x_0 in (11), for $y \in \mathrm{I}$, $y \neq x_0$ and $|y - x_0| \leqslant h$, we have

$$\left\|\frac{\mathbf{f}(y) - \mathbf{d}}{y - x_0} - \mathbf{c}\right\| \leqslant \varepsilon$$

which proves that $\mathbf{c}$ is the derivative at the point x_0 of the function $\mathbf{f}$ extended by continuity to $\mathrm{I} \cup \{x_0\}$.

Remark. A similar argument, based on th. 1, shows that if f is a real function such that $f'_d(x)$ tends to $+\infty$ at the point x_0 then the ratio

$$(f(y) - f(x))/(y - x)$$

also tends to $+\infty$, and conversely; if moreover f has a finite limit at the point x_0 (which is not a consequence of the present hypothesis), then the function f extended by continuity to x_0 has a derivative equal to $+\infty$ at this point.

§ 3. DERIVATIVES OF HIGHER ORDER

1. DERIVATIVES OF ORDER *n*

Let $\mathbf{f}$ be a vector function of a real variable, defined, continuous and differentiable on an interval I. If the derivative $\mathbf{f}'$ exists on a neighbourhood (with respect to I) of a point $x_0 \in \mathrm{I}$, and is differentiable at the point x_0, then its derivative is called the *second derivative* of $\mathbf{f}$ at the point x_0, and is denoted by $\mathbf{f}''(x_0)$ or $\mathrm{D}^2\mathbf{f}(x_0)$. If this second derivative exists at every point of I (which implies that $\mathbf{f}'$ exists and is continuous on I), then $x \mapsto \mathbf{f}''(x)$ is a vector function which one denotes by $\mathbf{f}''$ or $\mathrm{D}^2\mathbf{f}$. We define, in the same way, recursively, the n^{th} *derivative* (or *derivative of order* n) of $\mathbf{f}$, and denote it by $\mathbf{f}^{(n)}$ or $\mathrm{D}^n\mathbf{f}$; by definition, its value at the point $x_0 \in \mathrm{I}$ is the derivative of the function $\mathbf{f}^{(n-1)}$ at the point x_0 : this definition presupposes the existence of *all* the derivatives $\mathbf{f}^{(k)}$ of order $k \leqslant n-1$ on a *neighbourhood* of x_0 relative to I, and the differentiability of $\mathbf{f}^{(n-1)}$ at the point x_0.

We will say that $\mathbf{f}$ is *n times differentiable* at the point x_0 (resp. in an interval) if it admits an n^{th} derivative at this point (resp. in this interval). One says that $\mathbf{f}$ is *indefinitely differentiable* on I if for each integer $n > 0$ it admits a derivative of order n on I.

By induction on m one sees that

$$\mathrm{D}^m(D^n\mathbf{f}) = \mathrm{D}^{m+n}\mathbf{f}. \tag{1}$$

More precisely, when one of the two terms in (1) is defined, then so is the other, and is equal to it.

PROPOSITION 1. *The set of vector functions defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *taking values in a given topological vector space* E, *and having an* n^{th} *derivative on* I, *is a vector space over* $\mathbf{R}$, *and* $\mathbf{f} \mapsto \mathrm{D}^n\mathbf{f}$ *is a linear mapping of this space into the vector space of linear mappings from* I *into* E.

One proves the formulae

$$\mathrm{D}^n(\mathbf{f}+\mathbf{g}) = \mathrm{D}^n\mathbf{f} + \mathrm{D}^n\mathbf{g} \tag{2}$$

$$\mathrm{D}^n(\mathbf{f}a) = \mathrm{D}^n\mathbf{f}.a \tag{3}$$

by induction on n when $\mathbf{f}$ and $\mathbf{g}$ have an n^{th} derivative on I (a being constant).

PROPOSITION 2 ("Leibniz' formula"). *Let* E, F, G *be three topological vector spaces over* $\mathbf{R}$, *and* $(\mathbf{x},\mathbf{y}) \mapsto [\mathbf{x}.\mathbf{y}]$ *a continuous bilinear mapping of* $\mathrm{E}\times\mathrm{F}$ *into* G. *If* $\mathbf{f}$ (resp. $\mathbf{g}$) *is defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *takes its values in* E (resp. F) *and has an* n^{th} *derivative on* I, *then* $[\mathbf{f}.\mathbf{g}]$ *has an* n^{th} *derivative on* I, *given by the formula*

$$\mathrm{D}^n[\mathbf{f}.\mathbf{g}] = [\mathbf{f}^{(n)}.\mathbf{g}] + \binom{n}{1}[\mathbf{f}^{(n-1)}.\mathbf{g}'] + \cdots + \binom{n}{p}[\mathbf{f}^{(n-p)}.\mathbf{g}^{(p)}] + \cdots + [\mathbf{f}.\mathbf{g}^{(n)}]. \tag{4}$$

Formula (4) is proved by induction on n (using the relation $\binom{n}{p} = \binom{n-1}{p} + \binom{n-1}{p-1}$ for the binomial coefficients).

In the same way one can verify the following formula (where the hypotheses are the same as in prop. 2):

$$[\mathbf{f}^{(n)}.\mathbf{g}] + (-1)^{n-1}[\mathbf{f}.\mathbf{g}^{(n)}] = \mathrm{D}\big([\mathbf{f}^{(n-1)}.\mathbf{g}] - [\mathbf{f}^{(n-2)}.\mathbf{g}'] + \cdots + (-1)^{n-1}[\mathbf{f}.\mathbf{g}^{(n-1)}]\big). \tag{5}$$

The preceding propositions have been stated for functions that are n times differentiable on an interval; we leave it to the reader to formulate the analogous propositions for functions that are n times differentiable at a point.

2. TAYLOR'S FORMULA

Let $\mathbf{f}$ be a vector function defined on an interval $\mathrm{I} \subset \mathbf{R}$, with values in a *normed* space E over $\mathbf{R}$; to say that $\mathbf{f}$ has a derivative at a point $a \in \mathrm{I}$ signifies that

$$\lim_{x\to a,\, x\in\mathrm{I},\, x\neq a} \frac{\mathbf{f}(x) - \mathbf{f}(a) - \mathbf{f}'(a)(x-a)}{x-a} = 0; \tag{6}$$

or, otherwise, that $\mathbf{f}$ is "approximately equal" to the *linear* function $\mathbf{f}(a)+\mathbf{f}'(a)(x-a)$ on a neighbourhood of a (*cf.* chap. V, where this concept is developed in a general manner). We shall see that the existence of the n^{th} order derivative of $\mathbf{f}$ at the point a entails in the same way that $\mathbf{f}$ is "approximately equal" to a *polynomial of degree n in x*, with coefficients in E (*Gen. Top.*, X, p. 315) on a neighbourhood of a. To be precise:

THEOREM 1. *If the function $\mathbf{f}$ has an n^{th} derivative at the point a then*

$$\lim_{x\to a,\, x\in\mathrm{I},\, x\neq a} \frac{\mathbf{f}(x) - \mathbf{f}(a) - \mathbf{f}'(a)\frac{(x-a)}{1!} - \cdots - \mathbf{f}^{(n)}(a)\frac{(x-a)^n}{n!}}{(x-a)^n} = 0. \tag{7}$$

We proceed by induction on n. The theorem holds for $n = 1$. For arbitrary n one can, by the induction hypothesis, apply it to the derivative $\mathbf{f}'$ of $\mathbf{f}$: for any $\varepsilon > 0$ there is an $h > 0$ such that, if one puts

$$\mathbf{g}(x) = \mathbf{f}(x) - \mathbf{f}(a) - \mathbf{f}'(a)\frac{(x-a)}{1!} - \mathbf{f}''(a)\frac{(x-a)^2}{2!} - \cdots - \mathbf{f}^{(n)}(a)\frac{(x-a)^n}{n!}$$

one has, for $|y-a| \leqslant h$ and $y \in \mathrm{I}$,

$$\begin{aligned}\|\mathbf{g}'(y)\| &= \left\|\mathbf{f}'(y) - \mathbf{f}'(a) - \mathbf{f}''(a)\frac{(y-a)}{1!} - \cdots - \mathbf{f}^{(n)}(a)\frac{(y-a)^{n-1}}{(n-1)!}\right\| \\ &\leqslant \varepsilon\ |y-a|^{n-1}.\end{aligned}$$

We apply the mean value theorem (I, p. 15, th. 2) on the interval with endpoints a, x (with $|x-a| \leqslant h$) to the vector function $\mathbf{g}$ and to the real increasing function equal to $\varepsilon\,|y-a|^n/n$ if $x > a$, and to $-\varepsilon\,|y-a|^n/n$ if $x < a$; it follows that $\|\mathbf{g}(x)\| \leqslant \varepsilon\,|x-a|^n/n$, which proves the theorem.

We thus can write

$$\begin{aligned}\mathbf{f}(x) = \mathbf{f}(a) + \mathbf{f}'(a)\frac{(x-a)}{1!} + \mathbf{f}''(a)\frac{(x-a)^2}{2!} + \cdots \\ + \mathbf{f}^{(n)}(a)\frac{(x-a)^n}{n!} + \mathbf{u}(x)\frac{(x-a)^n}{n!}\end{aligned} \tag{8}$$

where $\mathbf{u}(x)$ approaches 0 as x approaches a while remaining in I; this formula is called *Taylor's formula of order n* at the point a, and the right-hand side of (8) is called the *Taylor expansion of order n* of the function $\mathbf{f}$ at the point a. The last term $\mathbf{r}_n(x) = \mathbf{u}(x)(x-a)^n/n!$ is called the *remainder* in the Taylor formula of order n.

When $\mathbf{f}$ has a *derivative of order* $n+1$ on I, one can estimate $\|\mathbf{r}_n(x)\|$ in terms of this $n+1^{th}$ derivative, on all of I, and not just on an unspecified neighbourhood of a :

PROPOSITION 3. *If* $\left\|\mathbf{f}^{(n+1)}(x)\right\| \leqslant \mathrm{M}$ *on* I, *then we have*

$$\|\mathbf{r}_n(x)\| \leqslant \mathrm{M}\,\frac{|x-a|^{n+1}}{(n+1)!} \tag{9}$$

on I.

Indeed, the formula holds for $n=0$, by I, p. 15, th. 2. Let us prove it by induction on n : by the induction hypothesis applied to $\mathbf{f}'$, one has

$$\left\|\mathbf{r}'_n(y)\right\| \leqslant \mathrm{M}\frac{|y-a|^n}{n!}$$

from which the formula (9) follows by the mean value theorem (I, p. 23, th. 2).

COROLLARY. *If f is a finite real function with a derivative of order $n+1$ on* I, *and if $m \leqslant f^{(n+1)}(x) \leqslant \mathrm{M}$ on* I, *then for all $x \geqslant a$ in* I *one has*

$$m\,\frac{(x-a)^{n+1}}{(n+1)!} \leqslant r_n(x) \leqslant \mathrm{M}\,\frac{(x-a)^{n+1}}{(n+1)!} \tag{10}$$

and the second term cannot be equal to the first (resp. *to the third*) *unless $f^{(n+1)}$ is constant and equal to m* (resp. M) *on the interval* $[a, x]$.

The proof proceeds in the same way, but applying th. 1 of I, p. 14.

Remarks. 1) We have already noticed in the proof of th. 1 that if $\mathbf{f}$ has a derivative of order n on I, and if

$$\mathbf{f}(x) = \mathbf{a}_0 + \mathbf{a}_1(x-a) + \mathbf{a}_2(x-a)^2 + \cdots + \mathbf{a}_n(x-a)^n + \mathbf{r}_n(x) \tag{11}$$

is its Taylor expansion of order n at the point a, then the Taylor expansion of order $n-1$ for $\mathbf{f}'$ at the point a is

$$\mathbf{f}'(x) = \mathbf{a}_1 + 2\mathbf{a}_2(x-a) + \cdots + n\mathbf{a}_n(x-a)^{n-1} + \mathbf{r}'_n(x). \tag{12}$$

We say that it is obtained from the expansion (11) of $\mathbf{f}$ by *differentiating term-by-term*.

2) With the same hypotheses, the coefficients $\mathbf{a}_i$ in (11) are determined recursively by the relations

$$\begin{aligned}
\mathbf{a}_0 &= \mathbf{f}(a)\\
\mathbf{a}_1 &= \lim_{x\to a} \frac{\mathbf{f}(x)-\mathbf{f}(a)}{x-a}\\
\mathbf{a}_2 &= \lim_{x\to a} \frac{\mathbf{f}(x)-\mathbf{f}(a)-\mathbf{a}_1(x-a)}{(x-a)^2}\\
&\dots\\
\mathbf{a}_n &= \lim_{x\to a} \frac{\mathbf{f}(x)-\mathbf{f}(a)-\mathbf{a}_1(x-a)-\cdots-\mathbf{a}_{n-1}(x-a)^{n-1}}{(x-a)^n}.
\end{aligned}$$

In the case $a = 0$ one concludes, in particular, that if $\mathbf{f}(x^p)$ (p an integer > 0) has a derivative of order pn on a neighbourhood of 0 then the Taylor expansion of order pn of this function is simply

$$\mathbf{f}(x^p) = \mathbf{a}_0 + \mathbf{a}_1 x^p + \mathbf{a}_2 x^{2p} + \cdots + \mathbf{a}_n x^{np} + \mathbf{r}_n(x^p) \tag{13}$$

where $\mathbf{r}_n(x^p)$ is the remainder in the expansion (*cf.* V, p. 222).

3) The definition of the derivative of order n and the preceding results generalize immediately to functions of a complex variable; we shall not pursue this topic further here; it will be treated in detail in a later Book in this Series.

§ 4. CONVEX FUNCTIONS OF A REAL VARIABLE

Let H be a subset of $\mathbf{R}$, f a finite real function defined on H, and let G be the *graph* or representative set of the function f in $\mathbf{R}\times\mathbf{R} = \mathbf{R}^2$, the set of points $\mathrm{M}_x = (x, f(x))$, where x runs through H. It is convenient to say that a point (a, b) of $\mathbf{R}^2$ such that $a \in H$ lies *above* (resp. *strictly above, below, strictly below*) G if one has $b \geqslant f(a)$ (resp. $b > f(a), b \leqslant f(a), b < f(a)$). If $\mathrm{A} = (a, a')$ and $\mathrm{B} = (b, b')$ are two points of $\mathbf{R}^2$ we denote by AB the closed segment with endpoints A and B; if $a < b$ then AB is the graph of the linear function $a' + \dfrac{b'-a'}{b-a}(x-a)$ defined on $[a, b]$; we denote the slope $\dfrac{b'-a'}{b-a}$ of this segment by $p(\mathrm{AB})$, and will make use of the following lemma, whose verification is immediate:

Lemma. *Let* $\mathrm{A} = (a, a')$, $\mathrm{B} = (b, b')$, $\mathrm{C} = (c, c')$ *be three points in* $\mathbf{R}^2$ *such that* $a < b < c$. *The following statements are equivalent:*

a) B *is below* AC*;*

b) C *lies above the line passing through* A *and* B*;*

c) A *is above the line passing through* B *and* C;

d) $p(\mathrm{AB}) \leqslant p(\mathrm{AC})$;

e) $p(\mathrm{AC}) \leqslant p(\mathrm{BC})$.

The lemma still holds when one replaces "above" (resp. "below") by "strictly above" (resp. "strictly below") and the sign $\leqslant$ by $<$ (fig. 1).

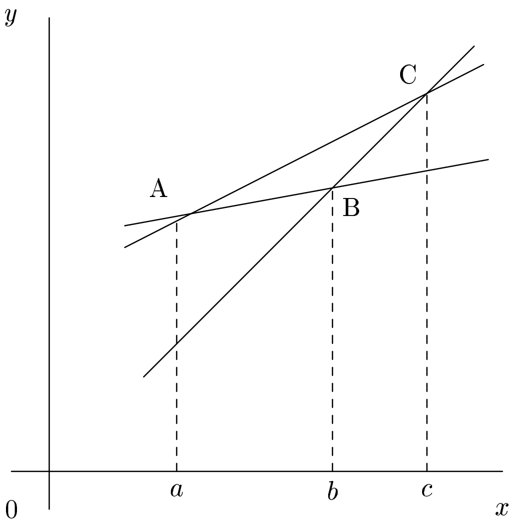

Fig. 1

1. DEFINITION OF A CONVEX FUNCTION

DEFINITION 1. *We say that a finite numerical function* f, *defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *is convex on* I *if, no matter what the points* x, x' *of* I, $(x < x')$, *every point* M_z *of the graph* G *of* f *such that* $x \leqslant z \leqslant x'$ *lies below the segment* $\mathrm{M}_x\mathrm{M}_{x'}$ (or, what comes to the same, if every point of this segment lies above G) (fig. 2).

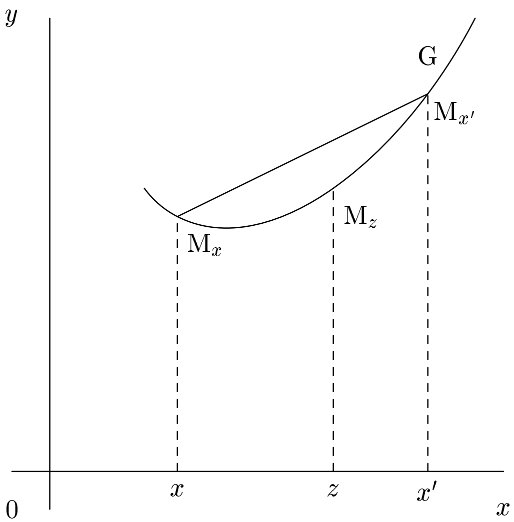

Fig. 2

Taking account of the parametric representation of a segment (*Gen. Top.*, VI, p. 35), the condition for f to be convex on I is that one has the inequality

$$f(\lambda x + (1-\lambda)x') \leqslant \lambda f(x) + (1-\lambda) f(x') \tag{1}$$

for each pair of points (x, x') of I and every $\lambda \in [0,\ 1]$.

Definition 1 is equivalent to the following: *the set of points in* $\mathbf{R}^2$ *lying above the graph* G *of* f *is convex*. Indeed, this condition is clearly sufficient for f to be convex on I; it is also necessary, for if f is convex on I, and if (x, y), (x', y') are two points lying above G, then one has $y \geqslant f(x)$, $y' \geqslant f(x')$, from which, for $0 \leqslant \lambda \leqslant 1$,

$$\lambda y + (1-\lambda)y' \geqslant \lambda f(x) + (1-\lambda) f(x') \geqslant f(\lambda x + (1-\lambda)x')$$

by (1), which shows that every point of the segment with endpoints (x, y) and (x', y') lies above G.

Remark. On sees in the same way that the set of points lying *strictly above* G is convex. Conversely, if this set is convex one has

$$\lambda y + (1-\lambda)y' > f(\lambda x + (1-\lambda)x')$$

for $0 \leqslant \lambda \leqslant 1$ and $y > f(x)$, $y' > f(x')$; on letting y tend to $f(x)$ and y' approach $f(x')$ in this formula it follows that f is convex.

Examples. 1) Every (real) affine linear function $ax + b$ is convex on $\mathbf{R}$.

2) The function x^2 is convex on $\mathbf{R}$, since one has

$$\lambda x^2 + (1-\lambda)x'^2 - \Big(\lambda x + (1-\lambda)x'\Big)^2 = \lambda(1-\lambda)(x - x')^2 \geqslant 0$$

for $0 \leqslant \lambda \leqslant 1$.

3) The function $|x|$ is convex on $\mathbf{R}$, since

$$\left|\lambda x + (1-\lambda)x'\right| \leqslant \lambda\,|x| + (1-\lambda)\left|x'\right|$$

for $0 \leqslant \lambda \leqslant 1$.

It is clear that if f is convex on I, then its restriction to any interval $\mathrm{J} \subset \mathrm{I}$ is convex on J.

Let f be a convex function on I, and x, x' two points of I such that $x < x'$; if $z \in \mathrm{I}$ is *exterior* to $[x, x']$ then M_z lies *above* the line D joining M_x and $\mathrm{M}_{x'}$; this is an immediate consequence of the lemma.

One deduces from this that if z is a point such that $x < z < x'$ and such that M_z lies *on* the segment $\mathrm{M}_x\mathrm{M}_{x'}$, then, for *every other point* z' such that $x < z' < x'$ the point $\mathrm{M}_{z'}$ also lies *on* the segment $\mathrm{M}_x\mathrm{M}_{x'}$, for it follows from the above that $\mathrm{M}_{z'}$ is at the same time both above and below this segment; in other words, f is then equal to an *affine linear* function on $[x, x']$.

DEFINITION 2. *We say that a finite real function* f *defined on an interval* $\mathrm{I} \subset \mathbf{R}$ *is strictly convex on* I *if, for any points* x, x' *of* I $(x < x')$, *every point* M_z *of the graph* G *of* f *such that* $x < z < x'$ *lies strictly below the segment* $\mathrm{M}_x\mathrm{M}_{x'}$ (or, what comes to the same, if every point of the segment, apart from the endpoints, lies strictly above G).

In other words, we must have the inequality

$$f(\lambda x + (1-\lambda)x') < \lambda f(x) + (1-\lambda) f(x') \tag{2}$$

for every pair of distinct points (x, x') of I and every λ such that $0 < \lambda < 1$.

The remarks that precede def. 2 show that for a convex function f to be strictly convex on I it is necessary and sufficient that there be no interval contained in I (not reducing to a single point) such that the restriction of f to this interval is *affine linear.*

Of the examples above, the first and third are not strictly convex; on the other hand, x^2 is strictly convex on $\mathbf{R}$; a similar calculation shows that $1/x$ is strictly convex on $]0, +\infty[$.

PROPOSITION 1. *Let f be a finite real function, convex* (resp. *strictly convex*) *on an interval* $\mathrm{I} \subset \mathbf{R}$. *For every family $(x_i)_{1 \leqslant i \leqslant p}$ of $p \geqslant 2$ distinct points of* I, *and every family $(\lambda_i)_{1 \leqslant i \leqslant p}$ of p real numbers such that $0 < \lambda_i < 1$ and $\sum_{i=1}^{p} \lambda_i = 1$, we have*

$$f\left(\sum_{i=1}^{p} \lambda_i x_i\right) \leqslant \sum_{i=1}^{p} \lambda_i f(x_i) \tag{3}$$

(resp.

$$f\left(\sum_{i=1}^{p} \lambda_i x_i\right) < \sum_{i=1}^{p} \lambda_i f(x_i)). \tag{4}$$

Since the proposition (for convex functions) reduces to the inequality (1) for $p = 2$ we argue by induction for $p > 2$. The number $\mu = \sum_{i=1}^{p-1} \lambda_i$ is > 0; it is immediate that if a and b are the smallest and largest of the x_i then $a \leqslant \sum_{i=1}^{p-1} \lambda_i x_i \Big/ \sum_{i=1}^{p-1} \lambda_i \leqslant b$; in other words, the point $x = \frac{1}{\mu} \sum_{i=1}^{p-1} \lambda_i x_i$ belongs to I, and the induction hypothesis implies that $\mu f(x) \leqslant \sum_{i=1}^{p-1} \lambda_i f(x_i)$; moreover we have, from (1), that

$$f\left(\sum_{i=1}^{p} \lambda_i x_i\right) = f(\mu x + (1-\mu)x_p) \leqslant \mu f(x) + (1-\mu) f(x_p) \leqslant \sum_{i=1}^{p} \lambda_i f(x_i).$$

One argues in the same way for strictly convex functions, starting from the inequality (2).

We say that a finite real function f is *concave* (resp. *strictly concave*) on I if $-f$ is convex (resp. strictly convex) on I. It comes to the same to say that for every pair (x, x') of distinct points of I and every λ such that $0 < \lambda < 1$ one has

$$f(\lambda x + (1-\lambda)x') \geqslant \lambda f(x) + (1-\lambda) f(x')$$
$$\text{(resp. } f(\lambda x + (1-\lambda)x') > \lambda f(x) + (1-\lambda) f(x')).$$

2. FAMILIES OF CONVEX FUNCTIONS

PROPOSITION 2. *Let* f_i $(1 \leqslant i \leqslant p)$ *be* p *convex functions on an interval* $\mathrm{I} \subset \mathbf{R}$, *and* c_i $(1 \leqslant i \leqslant p)$ *be* p *arbitrary positive numbers; then the function* $f = \sum_{i=1}^{p} c_i f_i$ *is convex on* I. *Further, if for at least one index* j *the function* f_j *is strictly convex on* I, *and* $c_j > 0$, *then* f *is strictly convex on* I.

This follows immediately by applying the inequality (1) (resp. (2)) to each of the f_i, multiplying the inequality for f_i by c_i, and then adding term-by-term.

PROPOSITION 3. *Let* (f_α) *be a family of convex functions on an interval* $\mathrm{I} \subset \mathbf{R}$; *if the upper envelope* g *of this family is finite at every point of* I *then* g *is convex on* I.

Indeed, the set of points $(x, y) \in \mathbf{R}^2$ lying above the graph of g is the intersection of the convex sets formed by the points lying above the graph of each of the functions f_α; so it is convex.

PROPOSITION 4. *Let* H *be a set of convex functions on an interval* $\mathrm{I} \subset \mathbf{R}$; *if* $\mathfrak{F}$ *is a filter on* H *which converges pointwise on* I *to a finite real function* f_0, *then this function is convex on* I.

To see this it suffices to pass to the limit along $\mathfrak{F}$ in the inequality (1).

3. CONTINUITY AND DIFFERENTIABILITY OF CONVEX FUNCTIONS

PROPOSITION 5. *For a real finite function* f *to be convex* (resp. *strictly convex*) *on an interval* I *it is necessary and sufficient that for all* $a \in \mathrm{I}$ *the gradient*

$$p(\mathrm{M}_a\mathrm{M}_x) = \frac{f(x) - f(a)}{x - a}$$

be an increasing (resp. *strictly increasing*) *function of* x *on* $\mathrm{I} \cap \complement\{a\}$.

This proposition is an immediate consequence of definitions 1 and 2 and of the lemma of I, p. 23.

PROPOSITION 6. *Let* f *be a finite real function convex on an interval* $\mathrm{I} \subset \mathbf{R}$. *Then at every* interior *point* a *of* I *the function* f *is continuous, has finite right and left derivatives, and* $f'_g(a) \leqslant f'_d(a)$.

Indeed, for $x \in \mathrm{I}$ and $x > a$ the function $x \mapsto \dfrac{f(x) - f(a)}{x - a}$ is increasing (prop. 5) and bounded below, since if $y < a$ and $y \in \mathrm{I}$ we have

$$\frac{f(y)-f(a)}{y-a} \leqslant \frac{f(x)-f(a)}{x-a} \tag{5}$$

by prop. 5; this function therefore has a finite right limit at the point a; in other words, $f'_d(a)$ exists and is finite; further, letting x approach a $(x > a)$ in (5), it follows that

$$\frac{f(y)-f(a)}{y-a} \leqslant f'_d(a) \tag{6}$$

for all $y < a$ belonging to I. In the same way one shows that $f'_g(a)$ exists and that

$$f'_d(a) \leqslant \frac{f(x)-f(a)}{x-a} \tag{7}$$

for $x \in \mathrm{I}$ and $x > a$. On letting x approach a $(x > a)$ in this last inequality we obtain $f'_g(a) \leqslant f'_d(a)$. The existence of the left and right derivatives at the point a clearly ensures the continuity of f at this point.

COROLLARY 1. *Let f be a convex* (resp. *strictly convex*) *function on* I; *if a and b are two interior points of* I *such that $a < b$ one has* (fig. 3)

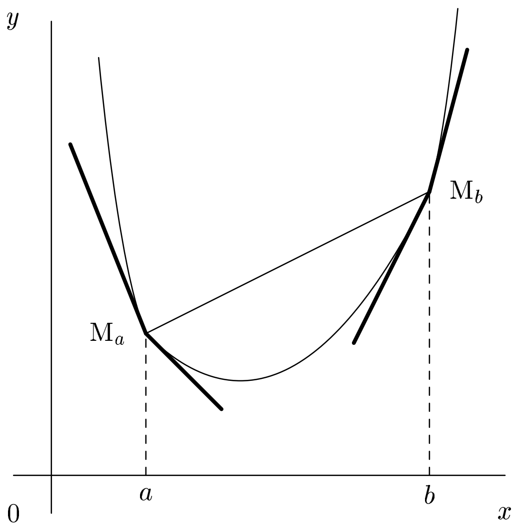

Fig. 3

$$f'_d(a) \leqslant \frac{f(b)-f(a)}{b-a} \leqslant f'_g(b) \tag{8}$$

(resp.

$$f'_d(a) < \frac{f(b)-f(a)}{b-a} < f'_g(b)\,). \tag{9}$$

The double inequality (8) results from (6) and (7) by a simple change of notation. On the other hand, if f is strictly convex and c is such that $a < c < b$ one has, from (8) and prop. 5,

$$f'_d(a) \leqslant \frac{f(c)-f(a)}{c-a} < \frac{f(b)-f(a)}{b-a} < \frac{f(b)-f(c)}{b-c} \leqslant f'_g(b)$$

from which (9).

COROLLARY 2. *If f is convex* (resp. *strictly convex*) *on* I *then f'_d and f'_g are increasing* (resp. *strictly increasing*) *on the interior of* I*; the set of points in* I *at which f is not differentiable is countable, and f'_d and f'_g are continuous at every point where f is differentiable.*

The first part follows immediately from (8) (resp. (9)) and the inequality

$$f'_g(a) \leqslant f'_d(a).$$

On the other hand, let E be the set of interior points x of I where f is not differentiable (that is $f'_g(x) < f'_d(x)$). For each $x \in \mathrm{E}$ let J_x be the open interval $]f'_g(x), f'_d(x)[$; it follows from (8) that if x and y are two points of E such that $x < y$, then $u < v$ for all $u \in \mathrm{J}_x$ and all $v \in \mathrm{J}_y$; in other words, as x runs through E the open nonempty intervals J_x are pairwise disjoint; the set of such intervals is thus countable, and hence so is E. Finally, f'_d (resp. f'_g) being increasing, it has a right limit and a left limit at every interior point x of I; prop. 6 of I, p. 18 now shows that the right limit of f'_d (resp. f'_g) at the point x is equal to $f'_d(x)$, and its left limit is $f'_g(x)$; from which we have the last part of the corollary.

Let f be a convex function on I, a an interior point of I, and D a line passing through the point M_a, with equation $y - f(a) = \alpha(x-a)$. It follows from the inequalities (8) that if $f'_g(a) \leqslant \alpha \leqslant f'_d(a)$ then every point of the graph G lies *above* D, and, if f is strictly convex, M_a is the only point common to D and G; one says that D is a *support line* to G at the point M_a. Conversely, if G lies above D, one has $f(x) - f(a) \geqslant \alpha(x-a)$ for every $x \in \mathrm{I}$, from which $\dfrac{f(x)-f(a)}{x-a} \geqslant \alpha$ for $x > a$, and $\dfrac{f(x)-f(a)}{x-a} \leqslant \alpha$ for $x < a$; on letting x tend to a in these inequalities it follows that $f'_g(a) \leqslant \alpha \leqslant f'_d(a)$.

In particular, if f is differentiable at the point a there is *only one* supporting line to G at the point M_a, the *tangent* to G at M_a.

Remark. If f is a strictly convex function on an open interval I then f'_d is strictly increasing on I, so there are three possible cases, according to prop. 2 of I, p. 13:

1° f is strictly decreasing on I;

2° f is strictly increasing on I;

3° there is an $a \in \mathrm{I}$ such that f is strictly decreasing for $x \leqslant a$, and is strictly increasing for $x \geqslant a$.

When f is convex on I, but not strictly convex, f can be constant on an interval contained in I; let $\mathrm{J} =]a, b[$ be the largest open interval on which f is constant (that is to say, the interior of the interval where $f'_d(x) = 0$); then f is strictly decreasing on the interval formed by the points $x \in \mathrm{I}$ such that $x \leqslant a$ (if it exists), strictly increasing on the interval formed by the points $x \in \mathrm{I}$ such that $x \geqslant b$ (if it exists).

In all cases one sees that f possesses a *right limit* at the left-hand endpoint of I (in $\overline{\mathbf{R}}$), and a *left limit* at the right-hand endpoint; these limits may be finite or infinite (*cf.* I, p. 46, exerc. 5, 6 and 7). By abuse of language one sometimes says that the continuous function (with values in $\overline{\mathbf{R}}$), equal to f on the interior of I, and extended by continuity to the endpoints of I, is *convex on* $\bar{\mathrm{I}}$.

4. CRITERIA FOR CONVEXITY

PROPOSITION 7. *Let f be a finite real function defined on an interval $\mathrm{I} \subset \mathbf{R}$. For f to be convex on I it is necessary and sufficient that for every pair of numbers a, b of I such that $a < b$, and for every real number μ, the function $f(x) + \mu x$ attains its supremum on $[a, b]$ at one of the points a, b.*

The condition is *necessary;* indeed, since μx is convex on $\mathbf{R}$, the function $f(x) + \mu x$ is convex on I; one can therefore restrict oneself to the case $\mu = 0$. Then, for

$$x = \lambda a + (1 - \lambda) b \quad (0 \leqslant \lambda \leqslant 1),$$

one has

$$f(x) \leqslant \lambda f(a) + (1 - \lambda) f(b) \leqslant \mathrm{Max}(f(a), f(b)).$$

The condition is *sufficient*. Let us take $\mu = -\dfrac{f(b) - f(a)}{b - a}$ and let $g(x) = f(x) + \mu x$; one has $g(a) = g(b)$ and therefore $g(x) \leqslant g(a)$ for all $x \in [a, b]$, and one can check immediately that this inequality is equivalent to the inequality (1) where one replaces z by a and x' by b.

PROPOSITION 8. *For a finite real function f to be convex* (resp. *strictly convex*) *on an open interval $\mathrm{I} \subset \mathbf{R}$ it is necessary and sufficient that it be continuous on* I, *have a derivative at every point of the complement* B *relative to* I *of a countable subset of this interval, and that the derivative be increasing* (resp. *strictly increasing*) *on* B.

The condition is necessary, from prop. 6 and its corollary 2 (I, p. 27); let us show that it is sufficient. Suppose, therefore, that f' is increasing on B, and that f is not convex; there then exist (I, p. 27, prop. 5) three points a, b, c of I, such that $a < c < b$, and $\dfrac{f(c) - f(a)}{c - a} > \dfrac{f(b) - f(c)}{b - c}$; but from the mean value theorem (I, p. 14, th. 1) one has

$$\frac{f(c) - f(a)}{c - a} \leqslant \sup_{x \in \mathrm{B},\ a < x < c} f'(x) \qquad \text{and} \qquad \frac{f(b) - f(c)}{b - c} \geqslant \inf_{x \in \mathrm{B},\ c < x < b} f'(x).$$

One thus has $\sup_{x \in \mathrm{B},\ a<x<c} f'(x) > \inf_{x \in \mathrm{B},\ c<x<b} f'(x)$, contrary to the hypothesis that f' is increasing on B.

If now we assume that f' is strictly increasing on B, then f is convex and cannot be equal to an affine linear function on any open interval contained in I, for then f' would be constant on this interval, contrary to the hypothesis.

COROLLARY. *Let f be a finite real function, continuous and twice differentiable on an interval* $\mathrm{I} \subset \mathbf{R}$; *for f to be convex on* I *it is necessary and sufficient that $f''(x) \geqslant 0$ for all $x \in \mathrm{I}$; for f to be strictly convex on* I *it is necessary and sufficient that the previous condition be satisfied and further that the set of points $x \in \mathrm{I}$ where $f''(x) > 0$ be dense in* I.

This follows immediately from the preceding proposition, and from the corollary at I, p. 14.

Example. *On the interval $]0, +\infty[$ the function x^r (r any real number) has a second derivative equal to $r(r-1)x^{r-2}$; thus it is strictly convex if $r > 1$ or $r < 0$, and strictly concave if $0 < r < 1$.*

In order to be able to formulate another criterion for convexity we make the following definition: given the graph G of a finite real function defined on an interval $\mathrm{I} \subset \mathbf{R}$ and an interior point a of I, we shall say that a line D passing through $\mathrm{M}_a = (a, f(a))$ is *locally above* (resp. *locally below*) G if there exists a neighbourhood $\mathrm{V} \subset \mathrm{I}$ of a such that every point of D contained in $\mathrm{V} \times \mathbf{R}$ is above (resp. below) G; we shall say that D is *locally on* G at the point M_a if there is a neighbourhood $\mathrm{V} \subset \mathrm{I}$ of a such that the intersection of D and $\mathrm{V} \times \mathbf{R}$ is identical to that of G and $\mathrm{V} \times \mathbf{R}$ (in other words, if D is simultaneously locally above and locally below G).

PROPOSITION 9. *Let f be a real finite function which is upper semi-continuous on an open interval* $\mathrm{I} \subset \mathbf{R}$. *For f to be convex on* I *it is necessary and sufficient that for every point* M_x *of the graph* G *of f every line locally above* G *at this point should be locally on* G (at the point M_x).

The condition is *necessary:* indeed, if f is convex on I then at every point M_a of the graph G of f there exists a *support line* Δ to G; now Δ is below G, so *a fortiori* locally below G (I, p. 29); if a line D is locally above G at the point M_a it is locally above Δ, so must coincide with Δ, and consequently is locally on G at the point M_a.

The condition is *sufficient*. Indeed, suppose it is satisfied, and suppose that f is not convex on I; then there are two points a, b of I ($a < b$) such that there are points M_x of G strictly above the segment $\mathrm{M}_a\mathrm{M}_b$ (fig. 4). In other words, the function $g(x) = f(x) - f(a) - \dfrac{f(b)-f(a)}{b-a}(x-a)$ takes values > 0 on $[a, b]$; since g is finite and upper semi-continuous on this compact interval its least upper bound k on $[a, b]$ is finite and > 0, and

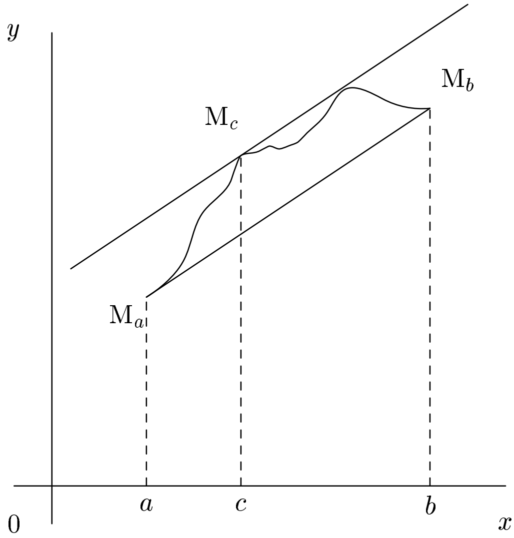

Fig. 4

the set $\overset{-1}{g}(k)$ is closed and not empty (*Gen. Top.*, IV, p. 361, th. 3 and prop. 1). Let c be the greatest lower bound of $\overset{-1}{g}(k)$; we have $a < c < b$, and at the point M_c the line D with equation $y = f(c) + \dfrac{f(b) - f(a)}{b - a}(x - c)$ lies locally above G; but it cannot be locally on G at this point, since, for $a < x < c$, one has $g(x) < k$, which signifies that M_x is strictly below D. This has led us to a contradiction, which establishes the proposition.

COROLLARY 1. *For a real finite function f defined on an open interval* $\mathrm{I} \subset \mathbf{R}$ *and upper semi-continuous on* I *to be convex on* I *it is necessary and sufficient that for all* $x \in \mathrm{I}$ *there should exist an* $\varepsilon > 0$ *such that the relation* $|h| \leqslant \varepsilon$ *entails*

$$f(x) \leqslant \frac{1}{2}\,(f(x + h) + f(x - h))\,.$$

We have only to show that the condition is *sufficient*. Indeed, if at a point M_a of the graph G of f a line D is locally above G, then it is locally on G at this point; for, in the opposite case, for example, a point M_{a+h} would be strictly below D, while a point M_{a-h} would be below D; the mid-point of the segment $M_{a-h}M_{a+h}$ would thus be strictly above D (fig. 5), and, in virtue of the hypothesis, M_a would *a fortiori* be strictly below D, which is absurd.

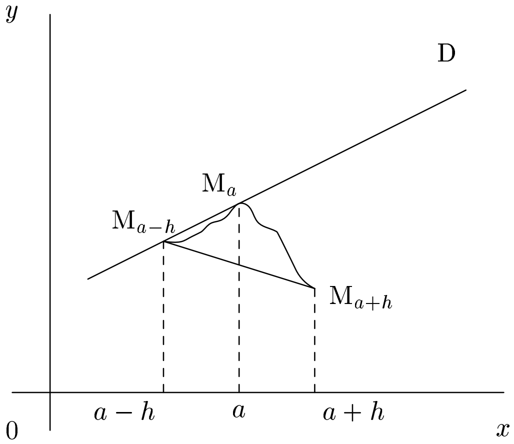

Fig. 5

COROLLARY 2. *Let f be a finite real function defined on an open interval* $\mathrm{I} \subset \mathbf{R}$. *If for every point $x \in \mathrm{I}$ there is an open interval $\mathrm{J}_x \subset \mathrm{I}$ containing x and such that the restriction of f to J_x is convex on J_x, then f is convex on* I.

It is clear that f satisfies the criterion of prop. 9.

EXERCISES

§ 1.

1) Let f be a vector function of a real variable, defined on an interval $\mathrm{I} \subset \mathbf{R}$ and differentiable at a point x_0 interior to I. Show that the quotient

$$\frac{f(x_0+h)-f(x_0-k)}{h+k}$$

tends to $f'(x_0)$ as h and k tend to 0 *through values* > 0. Converse.

*Show that the function f equal to $x^2 \sin 1/x$ for $x \neq 0$, and to 0 for $x = 0$, is everywhere differentiable, but that $(f(y)-f(z))/(y-z)$ does not approach $f'(0)$ as y and z tend to 0, while remaining distinct and > 0.$_*$

2) On the interval $\mathrm{I} = [0,\ 1]$ we define a sequence of continuous real functions (f_n) inductively as follows: We take $f_0(x) = x$; for each integer $n \geqslant 1$ the function f_n is affine linear on each of the 3^n intervals $\left[\frac{k}{3^n}, \frac{k+1}{3^n}\right]$ for $0 \leqslant k \leqslant 3^n - 1$; further, we take

$$f_{n+1}\left(\frac{k}{3^n}\right) = f_n\left(\frac{k}{3^n}\right)$$

$$f_{n+1}\left(\frac{k}{3^n}+\frac{1}{3^{n+1}}\right) = f_n\left(\frac{k}{3^n}+\frac{2}{3^{n+1}}\right),\quad f_{n+1}\left(\frac{k}{3^n}+\frac{2}{3^{n+1}}\right) = f_n\left(\frac{k}{3^n}+\frac{1}{3^{n+1}}\right).$$

Show that the sequence (f_n) converges uniformly on I to a continuous function which has no derivative (finite or infinite) at any point of the interval $]0, 1[$ (use exerc. 1).

3) Let $\mathscr{C}(\mathrm{I})$ be the complete space of continuous finite real functions defined on the compact interval $\mathrm{I} = [a, b]$ of $\mathbf{R}$, and endow $\mathscr{C}(\mathrm{I})$ with the topology of uniform convergence (*Gen. Top.*, X, p. 277). Let A be the subset of $\mathscr{C}(\mathrm{I})$ formed by the functions x such that for *at least one* point $t \in [a, b[$ (depending on the function x) the function x has a *finite* right derivative. Show that A is a *meagre* set in $\mathscr{C}(\mathrm{I})$ (*Gen. Top.*, IX, p. 192), and hence its complement, that is, the set of continuous functions on I not having a finite right derivative at *any point* of $[a, b[$ is a Baire subspace of $\mathscr{C}(\mathrm{I})$ (*Gen. Top.*, IX, p. 192). (Let A_n be the set of functions $x \in \mathscr{C}(\mathrm{I})$ such that for at least one value of t satisfying $a \leqslant t \leqslant b - 1/n$ (and depending on x) one has $|x(t') - x(t)| \leqslant n\,|t' - t|$ for all t' such that $t \leqslant t' \leqslant t + 1/n$. Show that each A_n is a closed *nowhere dense* set in $\mathscr{C}(\mathrm{I})$: remark that in $\mathscr{C}(\mathrm{I})$ each ball contains a function having bounded right derivative on $[a, b[$; on the other hand, for every $\varepsilon > 0$ and every integer $m > 0$ there exists on I a continuous function having at every point of $[a, b[$ a finite right derivative such that, for all $t \in [a, b[$ one has $|y(t)| \leqslant \varepsilon$ and $\left|y_r'(t)\right| \geqslant m$.)

4) Let E be a topological vector space over **R** and **f** a continuous vector function defined on an open interval $I \subset \mathbf{R}$, and having a right derivative and a left derivative at every point of I.

a) Let U be a nonempty open set in E, and A the subset of I formed by the points x such that $\mathbf{f}'_d(x) \in U$. Given a number $\alpha > 0$ let B be the subset of I formed by the points x such that there exists at least one $y \in I$ satisfying the conditions $x - \alpha \leqslant y < x$ and $(\mathbf{f}(x) - \mathbf{f}(y))/(x - y) \in U$; show that the set B is open and that $A \cap \complement B$ is countable (remark that this last set is formed by the left-hand endpoints of intervals contiguous to $\complement B$). Deduce that the set of points $x \in A$ such that $\mathbf{f}'_g(x) \notin \overline{U}$ is countable.

b) Suppose that E is a *normed* space; the image $\mathbf{f}(I)$ is then a metric space having a countable base, and the same is true for the closed vector subspace F of E generated by $\mathbf{f}(I)$, a subspace which contains $\mathbf{f}'_d(I)$ and $\mathbf{f}'_g(I)$. Deduce from *a*) that the set of points $x \in I$ such that $\mathbf{f}'_d(x) \neq \mathbf{f}'_g(x)$ is *countable*. (If (U_m) is a countable base for the topology of F note that for two distinct points a, b of F there exist two disjoint sets U_p, U_q such that $a \in U_p$ and $b \in U_q$.)

c) Take for E the product $\mathbf{R}^I$ (the space of mappings from I into **R**, endowed with the topology of simple convergence), and for each $x \in I$ denote by $\mathbf{g}(x)$ the map $t \mapsto |x - t|$ of I into **R**. Show that **g** is continuous and that, for every $x \in I$, one has $\mathbf{g}'_r(x) \neq \mathbf{g}'_l(x)$.

5) Let **f** be a continuous vector function defined on an open interval $I \subset \mathbf{R}$ with values in a normed space E over **R**, and admitting a right derivative at every point of I.

a) Show that the set of points $x \in I$ such that $\mathbf{f}'_d$ is bounded on a neighbourhood of x is an open set dense in I (use th. 2 of *Gen. Top.*, IX, p. 194).

b) Show that the set of points of I where $\mathbf{f}'_d$ is continuous is the complement of a *meagre* subset of I (*cf. Gen. Top.*, IX, p. 255, exerc. 21).

6) Let (r_n) be the sequence formed by the rational numbers in $[0, 1]$, arranged in a certain order. Show that the function $f(x) = \sum_{n=0}^{\infty} 2^{-n}(x - r_n)^{1/3}$ is continuous and differentiable at every point of **R**, and has an infinite derivative at every point r_n. (To see that f is differentiable at a point x distinct from the r_n, distinguish two cases, according to whether the series with general term $2^{-n}(x - r_n)^{-2/3}$ has sum $+\infty$ or converges; in the second case, note for all $x \neq 0$ and all $y \neq x$, one has

$$0 \leqslant (y^{1/3} - x^{1/3})/(y - x) \leqslant 4/3x^{2/3}).$$

7) Let f be a real function defined on an interval $I \subset \mathbf{R}$, admitting a right derivative $f'_d(x_0) = 0$ at a point of I, and let **g** be a vector function defined on a neighbourhood of $y_0 = f(x_0)$, having a right derivative and a left derivative (not necessarily equal) at this point. Show that $\mathbf{g} \circ f$ has a right derivative equal to 0 at the point x_0.

8) Let f be a mapping from **R** to itself such that the set C of points of **R** where f is continuous is dense in **R**, and such that the complement A of C is also dense. Show that the set D of points of C where f is right differentiable is meagre. (For each integer n, let E_n be the set of points $a \in \mathbf{R}$ such that there exist two points x, y such that $0 < x - a < 1/n$, $0 < y - a < 1/n$ and

$$\frac{f(x) - f(a)}{x - a} - \frac{f(y) - f(a)}{y - a} > 1.$$

Show that the interior of E_n is dense in **R**. For this, note that for every open nonempty interval I in **R** there is a point $b \in I \cap A$; show that for $a < b$ and $b - a$ sufficiently small one has $a \in E_n$.)

9) Let $\mathcal{B}(\mathbf{N})$ be the space of bounded sequences $\mathbf{x} = (x_n)_{n\in\mathrm{N}}$ of real numbers, endowed with the norm $\|\mathbf{x}\| = \sup |x_n|$; give an example of a continuous map $t \mapsto \mathbf{f}(t) = (f_n(t))_{n\in\mathrm{N}}$ of **R** into $\mathcal{B}(\mathbf{N})$ such that each of the functions f_n is differentiable for $t = 0$, but **f** is not differentiable at this point.

§ 2.

1) Let f be a real function defined and left continuous on an open interval $\mathrm{I} = \,]a, b[$ in **R**; suppose that at all the points of the complement B with respect to I of a countable subset of I the function f is *increasing to the right,* that is, at every point $x \in \mathrm{B}$ there exists a y such that $x < y \leqslant b$ and such that for all z such that $x \leqslant z < y$ one has $f(x) \leqslant f(z)$. Show that f is increasing on I (argue as in prop. 2).

2) In the field $\mathbf{Q}_p$ of p-adic numbers (*Gen. Top.*, III, p. 322, exerc. 23) every p-adic integer $x \in \mathbf{Z}_p$ has one and only one expansion in the form $x = a_0 + a_1 p + \cdots + a_n p^n + \cdots$, where the a_j are rational integers such that $0 \leqslant a_j \leqslant p - 1$ for each j. For each $z \in \mathbf{Z}_p$ put

$$f(x) = a_0 + a_1 p^2 + \cdots + a_n p^{2n} + \cdots;$$

show that, on $\mathbf{Z}_p$, f is a continuous function which is not constant on a neighbourhood of any point yet has a *zero derivative* at every point.

3) *a*) Let K be the triadic Cantor set (*Gen. Top.*, IV, p. 338), let $\mathrm{I}_{n,p}$ be the 2^n contiguous intervals of K with length $1/3^{n+1}$ $(1 \leqslant p \leqslant 2^n)$, and $\mathrm{K}_{n,p}$ the 2^{n+1} closed intervals of length $1/3^{n+1}$ whose union is the complement of the union of the $\mathrm{I}_{m,p}$ for $m \leqslant n$. Let α be a number such that $1 < \alpha < 3/2$; for each n we denote by f_n the continuous increasing function on $[0,\ 1]$ which is equal to 0 for $x = 0$, constant on each of the intervals $\mathrm{I}_{m,p}$ for $m \leqslant n$, is affine linear on each of the intervals $\mathrm{K}_{n,p}$ $(1 \leqslant p \leqslant 2^{n+1})$ and such that $f_d'(x) = \alpha^{n+1}$ on each of the interiors of these last intervals. Show that the series with general term f_n is uniformly convergent on $[0,\ 1]$, that its sum is a function f which admits a right derivative (finite or not) everywhere in $[0, 1[$, and that one has $f_r'(x) = +\infty$ at every point of K distinct from the left-hand endpoints of the contiguous intervals $\mathrm{I}_{n,p}$.

b) Let g be a continuous increasing map of $[0,\ 1]$ onto itself, constant on each of the intervals $\mathrm{I}_{n,p}$ (*Gen. Top.*, IV, p. 403, exerc. 9). If $h = f + g$, show that h admits a right derivative equal to $f_d'(x)$ at every point x of $[0, 1[$.

4) Let f be a finite real function, continuous on a compact interval $[a,\ b]$ in **R**, and having a right derivative at every point of the open interval $]a,\ b[$. Let m and M be the greatest lower bound and least upper bound (finite or not) of f_d' over $]a,\ b[$.

a) Show that when x and y run through $]a,\ b[$ keeping $x \neq y$, the set of values of $(f(x) - f(y))/(x - y)$ contains $]m, \mathrm{M}[$ and is contained in $[m, M]$. (Reduce to proving that if f_d' takes two values of opposite sign at the two points c, d of $]a,\ b[$ (with $c < d$), then there exist two distinct points of the interval $]c, d[$ where f takes the same value).

b) If, further, f has a left derivative at every point of $]a,\ b[$ then the infima (resp. suprema) of f_d' and f_g' over $]a,\ b[$ are equal.

c) Deduce that if f is differentiable on $]a, b[$ then the image under f' of every interval contained in $]a, b[$ is itself an interval, and consequently *connected* (use *a*)).

5) Let $\mathbf{f}$ be the vector mapping of $\mathrm{I} = [0, 1]$ into $\mathbf{R}^3$ defined as follows: for $0 \leqslant t \leqslant \frac{1}{4}$, $\mathbf{f}(t) = (-4t, 0, 0)$; for $\frac{1}{4} \leqslant t \leqslant \frac{1}{2}$ let $\mathbf{f}(t) = (-1, 4t-1, 0)$; for $\frac{1}{2} \leqslant t \leqslant \frac{3}{4}$ let $\mathbf{f}(t) = (-1, 1, 4t-2)$; finally, for $\frac{3}{4} \leqslant t \leqslant 1$ take $\mathbf{f}(t) = (4t-4, 1, 1)$. Show that the convex set generated by the set $\mathbf{f}'_d(\mathrm{I})$ is not identical to the closure of the set of values of $\dfrac{\mathbf{f}(y) - \mathbf{f}(x)}{y - x}$ as (x, y) runs through the set of pairs of distinct points of I (*cf.* exerc. 4 *a*)).

6) On the interval $\mathrm{I} = [-1, +1]$ consider the vector function $\mathbf{f}$, with values in $\mathbf{R}^2$, defined as follows: $\mathbf{f}(t) = (0, 0)$ for $-1 \leqslant t \leqslant 0$;

$$\mathbf{f}(t) = \left(t^2 \sin \frac{1}{t},\ t^2 \cos \frac{1}{t}\right)$$

for $0 \leqslant t \leqslant 1$. Show that $\mathbf{f}$ is differentiable on $]-1, +1[$ but that the image of this interval under $\mathbf{f}'$ is not a connected set in $\mathbf{R}^2$ (*cf.* exerc. 4 *c*)).

7) Let $\mathbf{f}$ be a continuous vector function defined on an open interval $\mathrm{I} \subset \mathbf{R}$, with values in a normed space E over $\mathbf{R}$, and admitting a right derivative at every point of I. Show that the set of points of I where $\mathbf{f}$ admits a derivative is the complement of a meagre subset of I (use exerc. 5 *b*) of I, p. 36, and prop. 6 of I, p. 18).

8) Consider, on the interval $[0, 1]$, a family $(\mathrm{I}_{n,p})$ of pairwise disjoint open intervals, defined inductively as follows: the integer n takes all values $\geqslant 0$; for each value of n the integer p takes the values $1, 2, \ldots, 2^n$; one has $\mathrm{I}_{0,1} =]\frac{1}{3}, \frac{2}{3}[$; if J_n is the union of the intervals $\mathrm{I}_{m,p}$ corresponding to the numbers $m \leqslant n$, the complement of J_n is the union of 2^{n+1} pairwise disjoint closed intervals $\mathrm{K}_{n,p}$ $(1 \leqslant p \leqslant 2^{n+1})$. If $\mathrm{K}_{n,p}$ is an interval $[a, b]$ one then takes for $\mathrm{I}_{n+1,p}$ the open interval with endpoints $b - \dfrac{b-a}{3}\left(1 + \dfrac{1}{2^n}\right)$ and $b - \dfrac{b-a}{3.2^n}$. Let E be the perfect set which is the complement with respect to $[0, 1]$ of the union of the $\mathrm{I}_{n,p}$. Define on $[0, 1]$ a continuous real function f which admits a right derivative at every point of $[0, 1[$, but fails to have a left derivative at the uncountable subset of E of points distinct from the endpoints of intervals contiguous to E (*cf.* exerc. 7). (Take $f(x) = 0$ on E, define f suitably on each of the intervals $\mathrm{I}_{n,p}$ in such a way that for every $x \in \mathrm{E}$ there are points $y < x$ not belonging to E, arbitrarily close to x, and such that $\dfrac{f(y) - f(x)}{y - x} = -1$.)

9) Let f and g be two finite real functions, continuous on $[a, b]$, both having a finite derivative on $]a, b[$; show that there exists a c such that $a < c < b$ and that

$$\begin{vmatrix} f(b) - f(a) & g(b) - g(a) \\ f'(c) & g'(c) \end{vmatrix} = 0.$$

¶ 10) Let f and g be two finite real functions, strictly positive, continuous and differentiable on an open interval I. Show that if f' and g' are strictly positive and f'/g' is strictly increasing on I, then either f/g is strictly increasing on I, or else there exists a number

$c \in \mathrm{I}$ such that f/g is strictly decreasing for $x \leqslant c$ and strictly increasing for $x \geqslant c$ (note that if one has $f'(x)/g'(x) < f(x)/g(x)$ then also

$$f'(y)/g'(y) < f(y)/g(y)$$

for all $y < x$).

11) Let f be a complex function, continuous on an open interval I, vanishing nowhere, and admitting a right derivative at every point of I. For $|f|$ to be increasing on I it is necessary and sufficient that $\mathcal{R}(f'_d/f) \geqslant 0$ on I.

¶ 12) Let f be a differentiable real function on an open interval I, g its derivative on I, and $[a, b]$ a compact interval contained in I; suppose that g is differentiable on the open interval $]a, b[$ but not necessarily right (resp. left) continuous at the point a (resp. b); show that there exists c such that $a < c < b$ and that

$$g(b) - g(a) = (b - a)g'(c)$$

(use exerc. 4 c) of I, p. 36).

13) One terms the *symmetric derivative* of a vector function $\mathbf{f}$ at a point x_0 interior to the interval where $\mathbf{f}$ is defined, the limit (when it exists) of $\dfrac{\mathbf{f}(x_0 + h) - \mathbf{f}(x_0 - h)}{2h}$ as h tends to 0 remaining > 0.

a) Generalize to the symmetric derivative the rules of calculus established in § 1 for the derivative.

b) Show that theorems 1 and 2 of § 2 remain valid when one replaces the words "right derivative" by "symmetric derivative".

14) Let $\mathbf{f}$ be a vector function defined and continuous on a compact interval $\mathrm{I} = [a, b]$ in $\mathbf{R}$, with values in a normed space over $\mathbf{R}$. Suppose that $\mathbf{f}$ admits a right derivative at all points of the complement with respect to $[a, b[$ of a countable subset A of this interval. Show that there exists a point $x \in]a, b[\cap \complement\mathrm{A}$ such that

$$\|\mathbf{f}(b) - \mathbf{f}(a)\| \leqslant \left\|\mathbf{f}'_d(x)\right\| (b - a).$$

(Argue by contradiction, decomposing $[a, b[$ into three intervals $[a, t[$, $[t, t + h[$ and $[t + h, b[$ with $t \notin \mathrm{A}$; if $k = \|\mathbf{f}(b) - \mathbf{f}(a)\| /(b - a)$, note that for h sufficiently small one has $\|\mathbf{f}(t + h) - \mathbf{f}(t)\| < k.h$, and use th. 2 of I, p. 15 for the other intervals.)

§ 3.

1) With the same hypotheses as in prop. 2 of I, p. 20 prove the formula

$$[\mathbf{f}^{(n)}.\mathbf{g}] = \sum_{p=0}^{n}(-1)^p \binom{n}{p} \mathrm{D}^{n-p}[\mathbf{f}.\mathbf{g}^{(p)}].$$

2) With the notation of prop. 2 of I, p. 28 suppose that the relation $[\mathbf{a}.\mathbf{y}] = 0$ for all $\mathbf{y} \in \mathrm{F}$ implies that $\mathbf{a} = 0$ in E. Under these conditions, if $\mathbf{g}_i$ $(0 \leqslant i \leqslant n)$ are $n + 1$ vector

functions with values in E, defined on an interval I of **R** and such that for *every* vector function $\mathbf{f}$ with values in F and n times differentiable on I, one has identically

$$[\mathbf{g}_0.\mathbf{f}] + [\mathbf{g}_1.\mathbf{f}'] + \cdots + [\mathbf{g}_n.\mathbf{f}^{(n)}] = 0$$

then the functions $\mathbf{g}_i$ are identically zero.

3) With the notation of exerc. 2 and the same hypothesis on $[\mathbf{x}.\mathbf{y}]$ suppose that each of the functions $\mathbf{g}_k$ is n times differentiable on I; for each function $\mathbf{f}$ which is n times differentiable on I, with values in F, put

$$[\mathbf{g}_0.\mathbf{f}] - [\mathbf{g}_1.\mathbf{f}]' + [\mathbf{g}_2.\mathbf{f}]'' + \cdots + (-1)^n[\mathbf{g}_n.\mathbf{f}]^{(n)} = [\mathbf{h}_0.\mathbf{f}] + [\mathbf{h}_1.\mathbf{f}'] + \cdots + [\mathbf{h}_n.\mathbf{f}^{(n)}],$$

which defines the functions $\mathbf{h}_i$ $(0 \leqslant i \leqslant n)$ without ambiguity (exerc. 2); show that one has

$$[\mathbf{h}_0.\mathbf{f}] - [\mathbf{h}.\mathbf{f}]' + [\mathbf{h}_2.\mathbf{f}]'' + \cdots + (-1)^n[\mathbf{h}_n.\mathbf{f}]^{(n)} = [\mathbf{g}_0.\mathbf{f}] + [\mathbf{g}_1.\mathbf{f}'] + \cdots + [\mathbf{g}_n.\mathbf{f}^{(n)}]$$

identically.

4) Let $\mathbf{f}$ be a vector function which is n times differentiable on an interval $\mathrm{I} \subset \mathbf{R}$. Show that for $1/x \in \mathrm{I}$ one has identically

$$\frac{1}{x^{n+1}}\mathbf{f}^{(n)}\left(\frac{1}{x}\right) = (-1)^n \mathrm{D}^n\left(x^{n-1}\mathbf{f}\left(\frac{1}{x}\right)\right)$$

(argue inductively on n).

5) Let u and v be two real functions which are n times differentiable on an interval $\mathrm{I} \subset \mathbf{R}$. If one puts $\mathrm{D}^n(u/v) = (-1)^n w_n \,/\, v^{n+1}$ at every point where $v(x) \neq 0$, show that

$$w_n = \begin{vmatrix} u & v & 0 & 0 & \dots & 0 \\ u' & v' & v & 0 & \dots & 0 \\ u'' & v'' & 2v' & v & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ u^{(n-1)} & v^{(n-1)} & \binom{n-1}{1}v^{(n-2)} & \binom{n-1}{2}v^{(n-3)} & \dots & v \\ u^{(n)} & v^{(n)} & \binom{n}{1}v^{(n-1)} & \binom{n}{2}v^{(n-2)} & \dots & \binom{n}{n-1}v' \end{vmatrix}$$

(put $w = u/v$ and differentiate n times the relation $u = wv$).

6) Let $\mathbf{f}$ be a vector function defined on an open interval $\mathrm{I} \subset \mathbf{R}$, taking values in a normed space E.

Put $\Delta\mathbf{f}(x; h_1) = \mathbf{f}(x + h_1) - \mathbf{f}(x)$, and then, inductively, define

$$\Delta^p\mathbf{f}(x;h_1, h_2, \dots, h_{p-1}, h_p) = \Delta^{p-1}\mathbf{f}(x + h_p;h_1, \dots, h_{p-1}) - \Delta^{p-1}\mathbf{f}(x;h_1, \dots, h_{p-1});$$

these functions are defined for each $x \in \mathrm{I}$ when the h_i are small enough.

a) If the function $\mathbf{f}$ is n times differentiable *at the point* x (and so $n-1$ times differentiable on a neighbourhood of x), one has

$$\lim_{\substack{(h_1,\dots,h_n)\to(0,\dots,0)\\ h_1h_2\dots h_n\neq 0}} \frac{\Delta^n\mathbf{f}(x;h_1,\dots,h_n)}{h_1h_2\dots h_n} = \mathbf{f}^{(n)}(x)$$

(argue by induction on n, using the mean value theorem).

b) If $\mathbf{f}$ is n times differentiable *on the interval* I, one has

$$\begin{aligned}&\left\|\Delta^n\mathbf{f}(x;h_1,\dots,h_n)-\mathbf{f}^{(n)}(x_0)h_1h_2\dots h_n\right\|\\ &\qquad\leqslant |h_1h_2\dots h_n|\ \sup\left\|\mathbf{f}^{(n)}(x+t_1h_1+\dots+t_nh_n)-\mathbf{f}^{(n)}(x_0)\right\|\end{aligned}$$

the supremum being taken over the set of (t_i) such that $0\leqslant t_i\leqslant 1$ for $1\leqslant i\leqslant n$ (same method).

c) If f is a real function which is n times differentiable on I, one has

$$\Delta^n f(x;h_1,h_2,\dots,h_n) = h_1h_2\dots h_n\ f^{(n)}(x+\theta_1h_1+\dots+\theta_nh_n)$$

the numbers θ_i belonging to $[0,\ 1]$ (same method, using I, p. 22, corollary).

7) Let f be a finite real function n times differentiable at the point x_0, and $\mathbf{g}$ a vector function which is n times differentiable at the point $y_0=f(x_0)$. Let

$$\begin{aligned}f(x_0+h) &= a_0+a_1h+\dots+a_nh^n+r_n(h)\\ \mathbf{g}(y_0+k) &= \mathbf{b}_0+\mathbf{b}_1k+\dots+\mathbf{b}_nk^n+\mathbf{s}_n(k)\end{aligned}$$

be the Taylor expansions of order n of f and $\mathbf{g}$ at the points x_0 and y_0 respectively. Show that the sum of the $n+1$ terms of the Taylor expansion of order n of the composite function $\mathbf{g}\circ f$ at the point x_0 is equal to the sum of the terms of degree $\leqslant n$ in the polynomial

$$\mathbf{b}_0+\mathbf{b}_1(a_1h+\dots+a_nh^n)+\mathbf{b}_2(a_1h+\dots+a_nh^n)^2+\dots+\mathbf{b}_n(a_1h+\dots+a_nh^n)^n.$$

Deduce the two following formulae:

a)

$$\mathrm{D}^n(\mathbf{g}(f(x))) = \sum \frac{n!}{m_1!m_2!\dots m_q!}\,\mathbf{g}^{(p)}(f(x))\left(\frac{f'(x)}{1!}\right)^{m_1}\cdots\left(\frac{f^{(q)}(x)}{q!}\right)^{m_q}$$

the sum being taken over all systems of positive integers $(m_i)_{1\leqslant i\leqslant q}$ such that

$$m_1+2m_2+\dots+qm_q=n$$

where p denotes the sum $m_1+m_2+\dots+m_q$.

b)

$$\mathrm{D}^n(\mathbf{g}(f(x))) = \sum_{p=1}^{n}\frac{1}{p!}\,\mathbf{g}^{(p)}(f(x))\left(\sum_{q=1}^{p}\binom{p}{q}(-f(x))^{p-q}\,\mathrm{D}^n((f(x))^q\right).$$

8) Let f be a real function defined and n times differentiable on an interval I, let $x_1,x_2,\dots,x_p$ be distinct points of I, and n_i $(1\leqslant i\leqslant p)$ be p integers >0 such that

$$n_1+n_2+\dots+n_p=n.$$

Suppose that at the point x_i the function f vanishes together with its first $n_i - 1$ derivatives for $1 \leqslant i \leqslant p$: show that there is a point ξ interior to the smallest interval that contains the x_i and such that $f^{(n-1)}(\xi) = 0$.

9) With the same notation as in exerc. 8 suppose that f is n times differentiable on I but otherwise arbitrary. Let g be the polynomial of degree $n-1$ (with real coefficients) such that at the point x_i $(1 \leqslant i \leqslant p)$ both g and its first $n_i - 1$ derivatives are respectively equal to f and its first $n_i - 1$ derivatives. Show that we have

$$f(x) = g(x) + \frac{(x-x_1)^{n_1}(x-x_2)^{n_2}\dots(x-x_p)^{n_p}}{n!} f^{(n)}(\xi)$$

where ξ is interior to the smallest interval containing the points x_i $(1 \leqslant i \leqslant p)$ and x. (Apply exerc. 8 to the function of t

$$f(t) - g(t) - a\frac{(t-x_1)^{n_1}(t-x_2)^{n_2}\dots(t-x_p)^{n_p}}{n!}$$

where a is a suitably chosen constant.)

10) Let g be an *odd* real function defined on a neighbourhood of 0, and 5 times differentiable on this neighbourhood. Show that

$$g(x) = \frac{x}{3}\left(g'(x) + 2g'(0)\right) - \frac{x^5}{180}g^{(5)}(\xi) \qquad (\xi = \theta x, \quad 0 < \theta < 1)$$

(same method as in exerc. 9).

Deduce that if f is a real function defined on $[a, b]$ and 5 times differentiable on this interval, then

$$f(b) - f(a) = \frac{b-a}{6}\left[f'(a) + f'(b) + 4f'\left(\frac{a+b}{2}\right)\right] - \frac{(b-a)^5}{2880}f^{(5)}(\xi)$$

with $a < \xi < b$ ("Simpson's formula").

11) Let $f_1, f_2, \dots, f_n$ and $g_1, g_2, \dots, g_n$ be $2n$ real functions which are $n-1$ times differentiable on an interval I. Let $(x_i)_{1\leqslant i\leqslant n}$ be a strictly increasing sequence of points in I. Show that the ratio of the two determinants

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_n) \\ \dots & \dots & \dots & \dots \\ f_n(x_1) & f_n(x_2) & \cdots & f_n(x_n) \end{vmatrix} : \begin{vmatrix} g_1(x_1) & g_1(x_2) & \cdots & g_1(x_n) \\ g_2(x_1) & g_2(x_2) & \cdots & g_2(x_n) \\ \dots & \dots & \dots & \dots \\ g_n(x_1) & g_n(x_2) & \cdots & g_n(x_n) \end{vmatrix}$$

is equal to the ratio of the two determinants

$$\begin{vmatrix} f_1(\xi_1) & f_1'(\xi_2) & \cdots & f_1^{(n-1)}(\xi_n) \\ f_2(\xi_1) & f_2'(\xi_2) & \cdots & f_2^{(n-1)}(\xi_n) \\ \dots & \dots & \dots & \dots \\ f_n(\xi_1) & f_n'(\xi_2) & \cdots & f_n^{(n-1)}(\xi_n) \end{vmatrix} : \begin{vmatrix} g_1(\xi_1) & g_1'(\xi_2) & \cdots & g_1^{(n-1)}(\xi_n) \\ g_2(\xi_1) & g_2'(\xi_2) & \cdots & g_2^{(n-1)}(\xi_n) \\ \dots & \dots & \dots & \dots \\ g_n(\xi_1) & g_n'(\xi_2) & \cdots & g_n^{(n-1)}(\xi_n) \end{vmatrix}$$

where

$$\xi_1 = x_1, \quad \xi_1 < \xi_2 < x_2, \quad \xi_2 < \xi_3 < x_3, \quad \ldots, \quad \xi_{n-1} < \xi_n < x_n$$

(apply exerc. 9 of I, p. 38).

Particular case where $g_1(x) = 1, g_2(x) = x, \ldots, g_n(x) = x^{n-1}$.

¶ 12) *a*) Let $\mathbf{f}$ be a vector function defined and continuous on the finite interval $\mathrm{I} = [-a, +a]$, taking its values in a normed space E over $\mathbf{R}$ and twice differentiable on I. If one puts $\mathrm{M}_0 = \sup_{x\in\mathrm{I}} \|\mathbf{f}(x)\|$, $\mathrm{M}_2 = \sup_{x\in\mathrm{I}} \|\mathbf{f}''(x)\|$, show that for all $x \in \mathrm{I}$ one has

$$\left\|\mathbf{f}'(x)\right\| \leqslant \frac{\mathrm{M}_0}{a} + \frac{x^2+a^2}{2a}\mathrm{M}_2$$

(express each of the differences $\mathbf{f}(a) - \mathbf{f}(x)$, $\mathbf{f}(-a) - \mathbf{f}(x)$).

b) Deduce from *a*) that if $\mathbf{f}$ is a twice differentiable function on an interval I (bounded or not), and if $\mathrm{M}_0 = \sup_{x\in\mathrm{I}} \|\mathbf{f}(x)\|$ and $\mathrm{M}_2 = \sup_{x\in\mathrm{I}} \|\mathbf{f}''(x)\|$ are finite, then so is $\mathrm{M}_1 = \sup_{x\in\mathrm{I}} \|\mathbf{f}'(x)\|$, and one has:

$$\mathrm{M}_1 \leqslant 2\sqrt{\mathrm{M}_0\mathrm{M}_2} \qquad \text{if I has length } \geqslant 2\sqrt{\frac{M_0}{\mathrm{M}_2}}$$

$$\mathrm{M}_1 \leqslant \sqrt{2}\sqrt{\mathrm{M}_0\mathrm{M}_2} \qquad \text{if } \mathrm{I} = \mathbf{R}.$$

Show that in these two inequalities the numbers 2 and $\sqrt{2}$ respectively cannot be replaced by smaller numbers (consider first the case where one supposes merely that $\mathbf{f}$ admits a second right derivative, and show that in this case the two terms of the preceding inequalities can become equal, taking for $\mathbf{f}$ a real function equal "in pieces" to second degree polynomials).

c) Deduce from *b*) that if $\mathbf{f}$ is p times differentiable on $\mathbf{R}$, and if $\mathrm{M}_p = \sup_{x\in\mathbf{R}} \left\|\mathbf{f}^{(p)}(x)\right\|$ and $\mathrm{M}_0 = \sup_{x\in\mathbf{R}} \|\mathbf{f}(x)\|$ are finite, then each of the numbers $\mathrm{M}_k = \sup_{x\in\mathbf{R}} \left\|\mathbf{f}^{(k)}(x)\right\|$ is finite (for $1 \leqslant k \leqslant p-1$) and

$$\mathrm{M}_k \leqslant 2^{k(p-k)/2}\ \mathrm{M}_0^{1-k/p}\ \mathrm{M}_p^{k/p}.$$

¶ 13) *a*) Let f be a twice differentiable real function on $\mathbf{R}$, such that $(f(x))^2 \leqslant a$ and $(f'(x))^2 + (f''(x))^2 \leqslant b$ on $\mathbf{R}$; show that

$$(f(x))^2 + (f'(x))^2 \leqslant \max(a, b)$$

on $\mathbf{R}$ (argue by contradiction, noting that if the function $f^2+f'^2$ takes a value $c > \max(a, b)$ at a point x_0 then there exist two points x_1, x_2 such that $x_1 < x_0 < x_2$ and that at x_1 and x_2 the function f' takes values small enough that $f^2 + f'^2$ takes values $< c$; then consider a point of $[x_1, x_2]$ where $f^2 + f'^2$ attains its supremum on this interval).

b) Let f be a real function n times differentiable on $\mathbf{R}$, and such that $(f(x))^2 \leqslant a$ and $(f^{(n-1}(x))^2 + (f^{(n)}(x))^2 \leqslant b$ on $\mathbf{R}$; show that then

$$(f^{(k-1}(x))^2 + (f^{(k)}(x))^2 \leqslant \max(a, b)$$

on $\mathbf{R}$ for $1 \leqslant k \leqslant n$. (Argue by induction on n; note that, by exerc. 12 the supremum c of $(f'(x))^2$ on $\mathbf{R}$ is finite; show that one necessarily has $c \leqslant \max(a, b)$ by reducing to a contradiction: assuming that $c > \max(a, b)$ choose the constants λ and μ so that for the function $g = \lambda f + \mu$ one has $|g(x)| \leqslant 1$, $|g'(x)| \leqslant 1$, yet one cannot have $(g(x))^2 + (g'(x))^2 \leqslant 1$ for all x.)

¶ 14) Let $\mathbf{f}$ be a function which is $n-1$ times differentiable on an interval I containing 0, and let $\mathbf{f}_n$ be the vector function defined for $x \neq 0$ on I by the relation

$$\mathbf{f}(x) = \mathbf{f}(0) + \mathbf{f}'(0)\frac{x}{1!} + \mathbf{f}''(0)\frac{x^2}{2!} + \cdots + \mathbf{f}^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!} + \mathbf{f}_n(x)x^n.$$

a) Show that if $\mathbf{f}$ has an $(n+p)^{th}$ derivative at the point 0 then $\mathbf{f}_n$ has a p^{th} derivative at the point 0 and an $(n+p-1)^{th}$ derivative at all points of a neighbourhood of 0 distinct from 0; moreover, one has $\mathbf{f}_n^{(k)}(0) = \dfrac{k!}{(n+k)!}\mathbf{f}^{(n+k)}(0)$ for $0 \leqslant k \leqslant p$, and $\mathbf{f}_n^{(p+k)}(x)x^k$ tends to 0 with x, for $1 \leqslant k \leqslant n-1$ (express the derivatives of $\mathbf{f}_n$ with the help of the Taylor expansions of the successive derivatives of $\mathbf{f}$, and use prop. 6 of I, p. 18).

b) Conversely, let $\mathbf{f}_n$ be a vector function admitting an $(n+p-1)^{th}$ derivative on a neighbourhood of 0 in I, and such that $\mathbf{f}_n^{(p+k)}(x)x^k$ has a limit for $0 \leqslant k \leqslant n-1$. Show that the function $\mathbf{f}_n(x)x^n$ has an $(n+p-1)^{th}$ derivative on I; if, further, $\mathbf{f}_n$ admits a p^{th} derivative at the point 0, then $\mathbf{f}_n(x)x^n$ admits an $(n+p)^{th}$ derivative at the point 0.

c) Suppose that I is symmetric with respect to 0 and that $\mathbf{f}$ is *even* ($\mathbf{f}(-x) = \mathbf{f}(x)$ on I). Show, with the help of *a*), that if $\mathbf{f}$ is $2n$ times differentiable on I, then there exists a function $\mathbf{g}$ defined and n times differentiable on I, such that $\mathbf{f}(x) = \mathbf{g}(x^2)$ on I.

¶ 15) Let I be an open interval in $\mathbf{R}$, and $\mathbf{f}$ a vector function defined and continuous on I; suppose that there are n vector functions $\mathbf{g}_i$ ($1 \leqslant i \leqslant n$) defined on I, and such that the function of x

$$\frac{1}{h^n}\Big(\mathbf{f}(x+h) - \mathbf{f}(x) - \sum_{p=1}^{n} \frac{h^p}{p!}\,\mathbf{g}_p(x)\Big)$$

tends *uniformly* to 0 on every compact interval contained in I as h tends to 0.

a) We put $\mathbf{f}_p(x,h) = \Delta^p\mathbf{f}(x; h, h, \ldots, h)$ (I, p. 40, exerc. 6). Show that, for $1 \leqslant p \leqslant n$, $(1/h^p)\mathbf{f}_p(x,h)$ tends *uniformly* to $\mathbf{g}_p(x)$ on every compact subinterval of I as h tends to 0, and that the $\mathbf{g}_p$ are continuous on I (prove this successively for $p = n$, $p = n-1$, etc.)

b) Deduce from this that $\mathbf{f}$ has a continuous n^{th} derivative and that $\mathbf{f}^{(p)} = \mathbf{g}_p$ for $1 \leqslant p \leqslant n$ (taking account of the relation $\mathbf{f}_{p+1}(x,h) = \mathbf{f}(x+h,h) - \mathbf{f}_p(x,h)$).

¶ 16) Let f be a real function n times differentiable on $\mathrm{I} =]-1, +1[$, and such that $|f(x)| \leqslant 1$ on this interval.

a) Show that if $m_k(\lambda)$ denotes the minimum of $\left|f^{(k)}(x)\right|$ on an interval of length λ contained in I then one has

$$m_k(\lambda) \leqslant \frac{2^{k(k+1)/2}\,k^k}{\lambda^k} \qquad (1 \leqslant k \leqslant n).$$

(Note that if the interval of length λ is decomposed into three intervals of lengths α, β, γ, one has

$$m_k(\lambda) \leqslant \frac{1}{\beta}(m_{k-1}(\alpha) + m_{k-1}(\gamma)).)$$

b) Deduce from *a*) that there exists a number μ_n depending only on the integer n such that if $|f'(0)| \geqslant \mu_n$, then the derivative $f^{(n)}(x)$ vanishes on at least $n-1$ distinct points of I (show by induction on k that $f^{(k)}$ vanishes at least $k-1$ times on I).

17) *a*) Let $\mathbf{f}$ be a vector function having derivatives of all orders on an open interval $\mathrm{I} \subset \mathbf{R}$. Suppose that, on I, one has $\|\mathbf{f}^{(n)}(x)\| \leqslant a\,n!r^n$, where a and r are two numbers > 0 and independent of x and n; show that at each point x_0 the "Taylor series" with general term $(1/n!)\,\mathbf{f}^{(n)}(x_0)\,(x - x_0)^n$ is convergent, and has sum $\mathbf{f}(x)$ on some neighbourhood of x_0.

b) Conversely, if the Taylor series for $\mathbf{f}$ at a point x_0 converges on a neighbourhood of x_0 there exist two numbers a and r (depending on x_0) such that $\|\mathbf{f}^{(n)}(x_0)\| \leqslant a.n!r^n$ for every integer $n > 0$.

c) Deduce from *a*) and exerc. 16 *b*) that if, on an open interval $\mathrm{I} \subset \mathbf{R}$, a real function f is indefinitely differentiable and if there is an integer p independent of n such that, for all n, the function $f^{(n)}$ does not vanish at more than p distinct points of I, then the Taylor series of f on a neighbourhood of each point $x_0 \in \mathrm{I}$ is convergent, and has sum $f(x)$ at every point of a neighbourhood of x_0.

18) Let $(a_n)_{n\geqslant 0}$ be an arbitrary sequence of complex numbers. For each $n \geqslant 0$ put $s_n^{(0)} = a_n$, and, inductively, for $k \geqslant 0$, define

$$s_n^{(k+1)} = s_0^{(k)} + s_1^{(k)} + \cdots + s_{n-1}^{(k)}.$$

a) Prove "Taylor's formula for sequences": for each integer

$$\left| s_{n+h}^{(k)} - s_n^{(k)} - hs_n^{(k-1)} - \binom{h}{2} s_n^{(k-2)} - \cdots - \binom{h}{k-1} s_n^{(1)} \right| \leqslant \binom{h}{k} \sup_{0\leqslant j\leqslant h-1} |a_{n+j}|$$

(proceed by induction on k).

b) Suppose that there is a number C such that $|na_n| \leqslant C$ for all n, and that the sequence $(s^{(2)}/n)$ formed by the arithmetic means $(s_0 + \cdots + s_{n-1})/n$ of the partial sums $s_n = a_0 + \cdots + a_{n-1}$ tends to a limit σ. Show that the series with general term a_n is convergent and has sum σ ("Hardy-Littlewood tauberian theorem"). (Write

$$s_n = \frac{1}{h}\left(s_{n+h}^{(2)} - s^{(2)}\right) + \frac{h-1}{2} r_n$$

where $|r_n|$ is bounded above with the aid of the inequality $|na_n| \leqslant C$, and h is chosen suitably as a function of n.)

§ 4.

1) *a*) Let H be a set of convex functions on a compact interval $[a, b] \subset \mathbf{R}$; suppose that the sets $\mathrm{H}(a)$ and $\mathrm{H}(b)$ are bounded above in $\mathbf{R}$ and that there exists a point c such that $a < c < b$ and that $\mathrm{H}(c)$ is bounded below in $\mathbf{R}$; show that H is an *equicontinuous* set on $]a, b[$ (*Gen. Top.*, X, p. 283).

b) Let H be a set of convex functions on an interval $\mathrm{I} \subset \mathbf{R}$, and let $\mathfrak{F}$ be a filter on H which converges pointwise on I to a function f_0; show that $\mathfrak{F}$ converges uniformly to f_0 on every compact interval contained in I.

2) Show that every convex function f on a compact interval $\mathrm{I} \subset \mathbf{R}$ is the limit of a decreasing uniformly convergent sequence of convex functions on I which admit a second derivative on I (first consider the function $(x - a)^+$, and approximate f by the sum of an affine linear function and a linear combination $\sum_j c_j(x - a_j)^+$ with coefficients $c_j \geqslant 0$).

3) Let f be a convex function on an interval $\mathrm{I} \subset \mathbf{R}$.

a) Show that if f is not constant it cannot attain its least upper bound at an interior point of I.

b) Show that if I is relatively compact in $\mathbf{R}$ then f is bounded below on I.

c) Show that if $\mathrm{I} = \mathbf{R}$ and f is not constant, then f is not bounded above on I.

4) For a function f to be convex on a compact interval $[a, b] \subset \mathbf{R}$ it is necessary and sufficient that it be convex on $]a, b[$ and that one has $f(a) \geqslant f(a+)$ and $f(b) \geqslant f(b-)$.

5) Let f be a convex function on an open interval $]a, +\infty[$; if there exists a point $c > a$ such that f is strictly increasing on $]c, +\infty[$ then $\lim\limits_{x\to+\infty} f(x) = +\infty$.

6) Let f be a convex function on an interval $]a, +\infty[$; show that $f(x)/x$ has a limit (finite or equal to $+\infty$) as x tends to $+\infty$; this limit is also that of $f'_d(x)$ and of $f'_g(x)$; it is > 0 if $f(x)$ tends to $+\infty$ as x tends to $+\infty$.

7) Let f be a convex function on the interval $]a, b[$ where $a \geqslant 0$; show that on this interval the function $x \mapsto f(x) - xf'(x)$ (the "ordinate at the origin" of the right semi-tangent at the point x to the graph of f) is decreasing (strictly decreasing if f is strictly convex).

Deduce that:

a) If f admits a finite right limit at the point a then $(x-a)f'_d(x)$ has a right limit equal to 0 at this point.

b) On $]a, b[$ either $f(x)/x$ is increasing, or $f(x)/x$ is decreasing, or else there exists a $c \in]a, b[$ such that $f(x)/x$ is decreasing on $]a, c[$ and increasing on $]c, b[$.

c) Suppose that $b = +\infty$: show that if

$$\beta = \lim_{x\to+\infty} \Big(f(x) - xf'_d(x)\Big)$$

is finite, then so is $\alpha = \lim\limits_{x\to+\infty} f(x)/x$, and that the line $y = \alpha x + \beta$ is *asymptotic* [5] to the graph of f, and lies *below* this graph (strictly below if f is strictly convex).

8) Let f be a finite real function, upper semi-continuous on an open interval $\mathrm{I} \subset \mathbf{R}$. Then f is convex if and only if $\limsup\limits_{h\to 0, h\neq 0} \dfrac{f(x+h) + f(x-h) - 2f(x)}{h^2} \geqslant 0$ for all $x \in \mathrm{I}$. (First show that, for all $\varepsilon > 0$ the function $f(x) + \varepsilon x^2$ is convex, using prop. 9 of I, p. 31.)

¶ 9) Let f be a finite real function, lower semi-continuous on an interval $\mathrm{I} \subset \mathbf{R}$. For f to be convex on I it suffices that, for every pair of points a, b of I such that $a < b$ there exists *one* point z such that $a < z < b$, and that M_z be below the segment $\mathrm{M}_a\mathrm{M}_b$ (argue by contradiction, noting that the set of points x such that M_x lies strictly above $\mathrm{M}_a\mathrm{M}_b$ is open).

¶ 10) Let f be a finite real function defined on an interval $\mathrm{I} \subset \mathbf{R}$, such that

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2}\Big(f(x) + f(y)\Big)$$

[5] That is to say, $\lim\limits_{x\to+\infty} \Big(f(x) - (\alpha x + \beta)\Big) = 0$.

for all x, y in I. Show that if f is bounded above on *one* open interval $]a, b[$ contained in I, then f is convex on I (show first that f is bounded above on *every* compact interval contained in I, then that f is continuous at every interior point of I).

¶ 11) Let f be a continuous function on an open interval $\mathrm{I} \subset \mathbf{R}$, having a finite right derivative at every point of I. If for every $x \in \mathrm{I}$ and *every* $y \in \mathrm{I}$ such that $y > x$ the point $\mathrm{M}_y = (y, f(y))$ lies above the right semi-tangent to the graph of f at the point $\mathrm{M}_x = (x, f(x))$, show that f is convex on I (using the mean value theorem show that $f'_d(y) \geqslant \dfrac{f(y) - f(x)}{y - x}$ for $x < y$).

Give an example of a function which is not convex, has a finite right derivative everywhere, and such that for every $x \in \mathrm{I}$ there exists a number $h_x > 0$ depending on x such that M_y lies above the right semi-tangent at the point M_x for all y such that $x \leqslant y \leqslant x + h_x$. This last condition is nevertheless sufficient for f to be convex, if one supposes further that f is differentiable on I (use I, p. 12, corollary).

¶ 12) Let f be a continuous real function on an open interval $\mathrm{I} \subset \mathbf{R}$; suppose that for every pair (a, b) of points of I such that $a < b$ the graph of f lies either entirely above *or* entirely below the segment $\mathrm{M}_a\mathrm{M}_b$ on the interval $[a, b]$. Show that f is convex on all of I or concave on all I (if in $]a, b[$ there is a point c such that M_c lies strictly above the segment $\mathrm{M}_a\mathrm{M}_b$ show that for every $x \in \mathrm{I}$ such that $x > a$ the graph of f lies above the segment $\mathrm{M}_a\mathrm{M}_x$ on the interval $[a, x]$).

13) Let f be a differentiable real function on an open interval $\mathrm{I} \subset \mathbf{R}$. Suppose that for every pair (a, b) of points of I such that $a < b$ there exists a *unique* point $c \in]a, b[$ such that $f(b) - f(a) = (b - a)f'(c)$; show that f is strictly convex on I or strictly concave on I (show that f' is strictly monotone on I).

14) Let f be a convex real function and strictly monotone on an open interval $\mathrm{I} \subset \mathbf{R}$; let g be the inverse function of f (defined on the interval $f(\mathrm{I})$). Show that if f is decreasing (resp. increasing) on I, then g is convex (resp. concave) on $f(\mathrm{I})$.

15) Let I be an interval contained in $]0, +\infty[$; show that if $f(1/x)$ is convex on I then so is $xf(x)$, and conversely.

* 16) Let f be a positive convex function on $]0, +\infty[$, and a, b two arbitrary real numbers. Show that the function $x^a f(x^{-b})$ is convex on $]0, +\infty[$ in the following cases:

1° $a = \frac{1}{2}(b + 1),\ \ |b| \geqslant 1$;

2° $x^a f(x^{-b})$ is increasing, $a(b - a) \geqslant 0,\ \ a \geqslant \frac{1}{2}(b + 1)$;

3° $x^a f(x^{-b})$ is decreasing, $a(b - a) \geqslant 0,\ \ a \leqslant \frac{1}{2}(b + 1)$.

Under the same hypotheses on f show that $e^{x/2} f(e^{-x})$ is convex (use exerc. 2 of I, p. 45).*

17) Let f and g be two positive convex functions on an interval $I = [a, b]$; suppose that there exists a number $c \in \mathrm{I}$ such that in each of the intervals $[a, c]$ and $[c, b]$ the functions f and g vary in the same sense. Show that the product fg is convex on I.

18) Let f be a convex function on an interval $\mathrm{I} \subset \mathbf{R}$ and g a convex increasing function on an interval containing $f(\mathrm{I})$; show that $g \circ f$ is convex on I.

¶ 19) Let f and g be two finite real functions, f being defined and continuous on an interval I, and g defined and continuous on $\mathbf{R}$. Suppose that for every pair (λ, μ) of real numbers the function $g(f(x) + \lambda x + \mu)$ is convex on I.
a) Show that g is convex and monotone on $\mathbf{R}$.
b) If g is increasing (resp. decreasing) on $\mathbf{R}$, show that f is convex (resp. concave) on I (use prop. 7).

20) Show that the set $\mathfrak{K}$ of convex functions on an interval $\mathrm{I} \neq \mathbf{R}$ is reticulated for the order "$f(x) \leqslant g(x)$ for every $x \in \mathrm{I}$" (*Set Theory*, III, p. 146). Give an example of two convex functions f, g on I such that their infimum in $\mathfrak{K}$ takes a value different from $\inf(f(x), g(x))$ at certain points. Give an example of an infinite family (f_α) of functions in $\mathfrak{K}$ such that $\inf\limits_{\alpha} f_\alpha(x)$ is finite at every point $x \in \mathrm{I}$ and yet there is no function in $\mathfrak{K}$ less than all the f_α.

21) Let f be a finite real function, upper semi-continuous on an open interval $\mathrm{I} \subset \mathbf{R}$. For f to be strictly convex on I it is necessary and sufficient that there be no line locally above the graph G of f at a point of G.

22) Let $f_1, \ldots, f_n$ be continuous convex functions on a compact interval $\mathrm{I} \subset \mathbf{R}$; suppose that for all $x \in \mathrm{I}$ one has $\sup(f_j(x)) \geqslant 0$. Show that there exist n numbers $\alpha_j \geqslant 0$ such that $\sum\limits_{j=1}^{n} \alpha_j = 1$ and that $\sum\limits_{j=1}^{n} \alpha_j f_j(x) \geqslant 0$ on I. (First treat the case $n = 2$, considering a point x_0 where the upper envelope $\sup(f_1, f_2)$ attains its minimum; when x_0 is interior to I determine α_1 so that the left derivative of $\alpha_1 f_1 + (1 - \alpha_1) f_2$ is zero at x_0. Pass to the general case by induction on n; use the induction hypothesis for the restrictions of $f_1, \ldots, f_{n-1}$ to the compact interval where $f_n(x) \leqslant 0$.)

23) Let f be a continuous real function on a compact interval $\mathrm{I} \subset \mathbf{R}$; among the functions $g \leqslant f$ which are convex on I there exists one, g_0, larger than all the others. Let $\mathrm{F} \subset \mathrm{I}$ be the set of $x \in \mathrm{I}$ where $g_0(x) = f(x)$; show that F is not empty and that on each of the open intervals contiguous to F the function g_0 is equal to an affine linear function (argue by contradiction).

24) Let $\mathrm{P}(x)$ be a polynomial of degree n with real coefficients all of whose roots are real and contained in the interval $[-1, 1]$. Let k be an integer such that $1 \leqslant k \leqslant n$. Show that the rational function

$$f(x) = x + \frac{\mathrm{P}^{(k-1)}(x)}{\mathrm{P}^{(k)}(x)}$$

is increasing on every interval of $\mathbf{R}$ on which it is defined; if $c_1 < c_2 < \cdots < c_r$ are its poles (contained in $[-1, 1]$), then f is convex for $x < c_1$ and concave for $x > c_r$. Deduce that when a runs through $[-1, 1]$ the length of the largest interval containing the zeros of the k^{th} derivative of $(x - a)\mathrm{P}(x)$ attains its largest value when $a = 1$ or $a = -1$.

25) One says that a real function f defined on $[0, +\infty[$ is *superadditive* if one has $f(x + y) \geqslant f(x) + f(y)$ for $x \geqslant 0, y \geqslant 0$, and if $f(0) = 0$.

a) Give examples of discontinuous superadditive functions.

$b)$ Show that every convex function f on $[0, +\infty[$ such that $f(0) = 0$ is superadditive.

$c)$ If f_1 and f_2 are superadditive then so is $\inf(f_1, f_2)$; using this, exhibit examples of nonconvex continuous superadditive functions.

$d)$ If f is continuous and $\geqslant 0$ on an interval $[0, a]$ $(a > 0)$, such that $f(0) = 0$ and $f(x/n) \leqslant f(x)/n$ for each integer $n \geqslant 1$, show that f has a right derivative at the point 0 (argue by contradiction). In particular, every continuous superadditive function which is $\geqslant 0$ admits a right derivative at the point 0.

CHAPTER II
Primitives and Integrals

§ 1. PRIMITIVES AND INTEGRALS

Unless expressly mentioned to the contrary, in this chapter we shall only consider vector functions of a *real* variable which take their values in a *complete* normed space over **R**. When we deal in particular with real-valued functions it will always be understood that these functions are *finite* unless stated to the contrary.

1. DEFINITION OF PRIMITIVES

A vector function **f** defined on an interval $\mathrm{I} \subset \mathbf{R}$ cannot be the *derivative* at *every point* of this interval of a vector function **g** (defined and continuous on I) unless it satisfies quite stringent conditions: for example, if **f** admits a right limit *and* a left limit at a point x_0 interior to I then **f** must be *continuous* at the point x_0, as follows from prop. 6 of I, p. 18; it follows, that if one takes the interval $[-1,\ 1]$ for I, and for f the real function equal to -1 on $[-1,\ 0[$, and to $+1$ on $[0,\ 1]$, then f is not the derivative of any continuous function on I; all the same, the function $|x|$ has $f(x)$ as its derivative at every point $\neq 0$; one is thus led to make the following definition:

DEFINITION 1. *Given a vector function* **f** *defined on an interval* $\mathrm{I} \subset \mathbf{R}$ *we say that a function* **g** *defined on* I *is a primitive of* **f** *if* **g** *is continuous on* I *and has a derivative equal to* $\mathbf{f}(x)$ *at every point* x *of the complement* (with respect to I) *of a countable subset of* I.

If also **g** admits a derivative equal to $\mathbf{f}(x)$ at *every point* x of I, one says that **g** is a *strict* primitive of **f**.

With this definition, one sees that the real function f considered above admits a primitive equal to $|x|$.

It is clear that if **f** admits a primitive on I then every primitive of **f** is also a primitive of every function which is equal to **f** except at the points of a countable subset of I. By an abuse of language one speaks of a primitive on I of a function $\mathbf{f}_0$ defined only on the complement (with respect to I) of a countable subset of I: this will be the primitive of every function **f** defined on I and equal to $\mathbf{f}_0$ at the points where $\mathbf{f}_0$ is defined.

PROPOSITION 1. *Let* $\mathbf{f}$ *be a vector function defined on* I *with values in* E*; if* $\mathbf{f}$ *admits a primitive* $\mathbf{g}$ *on* I *then the set of primitives of* $\mathbf{f}$ *on* I *is identical to the set of functions* $\mathbf{g} + \mathbf{a}$, *where* $\mathbf{a}$ *is a constant function with its values in* E.

Indeed, it is clear that $\mathbf{g} + \mathbf{a}$ is a primitive of $\mathbf{f}$ for any $\mathbf{a} \in \mathrm{E}$; on the other hand, if $\mathbf{g}_1$ is a primitive of $\mathbf{f}$ then $\mathbf{g}_1 - \mathbf{g}$ admits a derivative equal to 0 except at the points of a countable subset of I, and thus is constant (I, p. 17, corollary).

One says that the primitives of a function $\mathbf{f}$ (when they exist) are defined "up to an additive constant". To define a primitive of $\mathbf{f}$ unambiguously it is enough to assign it (arbitrarily) a value at a point $x_0 \in \mathrm{I}$; in particular, there exists one and only one primitive $\mathbf{g}$ of $\mathbf{f}$ such that $\mathbf{g}(x_0) = 0$; for every primitive $\mathbf{h}$ of $\mathbf{f}$ one has $\mathbf{g}(x) = \mathbf{h}(x) - \mathbf{h}(x_0)$.

2. EXISTENCE OF PRIMITIVES

Let $\mathbf{f}$ be a function defined on an arbitrary interval $\mathrm{I} \subset \mathbf{R}$; for a function $\mathbf{g}$ defined on I to be a primitive of $\mathbf{f}$, it is necessary and sufficient that the restriction of $\mathbf{g}$ to every compact interval $\mathrm{J} \subset \mathrm{I}$ be a primitive of the restriction of $\mathbf{f}$ to J.

THEOREM 1. *Let* A *be a set filtered by a filter* $\mathfrak{F}$, *and* $(\mathbf{f}_\alpha)_{\alpha \in \mathrm{A}}$ *a family of vector functions with values in a complete normed space* E *over* $\mathbf{R}$, *defined on an interval* $\mathrm{I} \subset \mathbf{R}$*: for each* $\alpha \in \mathrm{A}$ *let* $\mathbf{g}_\alpha$ *be a primitive of* $\mathbf{f}_\alpha$. *We suppose that:*

1° *with respect to the filter* $\mathfrak{F}$ *the functions* $\mathbf{f}_\alpha$ *converge uniformly on every compact subset of* I *to a function* $\mathbf{f}$;

2° *there is a point* $a \in \mathrm{I}$ *such that, with respect to the filter* $\mathfrak{F}$, *the family* $(\mathbf{g}_\alpha(a))$ *has a limit in* E.

Under these hypotheses the functions $\mathbf{g}_\alpha$ *converge uniformly (with respect to* $\mathfrak{F}$*) on every compact subset of* I *to a primitive* $\mathbf{g}$ *of* $\mathbf{f}$.

By the remark at the beginning of this subsection we can restrict ourselves to the case where I is a *compact* interval.

First let us show that the $\mathbf{g}_\alpha$ converge uniformly on I to a continuous function $\mathbf{g}$. By hypothesis, for every $\varepsilon > 0$ there is a set $\mathrm{M} \in \mathfrak{F}$ such that, for any two indices α, β belonging to M, one has $\left\|\mathbf{f}_\alpha(x) - \mathbf{f}_\beta(x)\right\| \leqslant \varepsilon$ for each $x \in \mathrm{I}$; in consequence one has (I, p. 15, th. 2)

$$\left\|\mathbf{g}_\alpha(x) - \mathbf{g}_\beta(x) - (\mathbf{g}_\alpha(a) - \mathbf{g}_\beta(a))\right\| \leqslant \varepsilon\,|x - a| \leqslant \varepsilon l$$

where l denotes the length of I; since by hypothesis $\mathbf{g}_\alpha(a)$ approaches a limit with respect to $\mathfrak{F}$, it follows from the Cauchy criterion that the $\mathbf{g}_\alpha$ converge uniformly on I. It remains to show that the limit $\mathbf{g}$ of the $\mathbf{g}_\alpha$ is a primitive of $\mathbf{f}$.

For each integer $n > 0$ let α_n be an index such that $\left\|\mathbf{f}(x) - \mathbf{f}_{\alpha_n}(x)\right\| \leqslant 1/n$ on I; it is clear that the sequence $\left(\mathbf{f}_{\alpha_n}\right)$ converges uniformly to $\mathbf{f}$ and that the sequence $\left(\mathbf{g}_{\alpha_n}\right)$ converges uniformly to $\mathbf{g}$ on I. Let H_n be the countable subset of I where $\mathbf{f}_{\alpha_n}$ is not the derivative of $\mathbf{g}_{\alpha_n}$, and let H be the union of the H_n, which is thus a countable subset of I; we shall see that at every point $x \in \mathrm{I}$ not belonging to H the function $\mathbf{g}$ has a derivative equal to $\mathbf{f}(x)$. Indeed, one sees as above that for every $m \geqslant n$ and every $y \in \mathrm{I}$ one has

$$\left\|\mathbf{g}_{\alpha_m}(y) - \mathbf{g}_{\alpha_m}(x) - \left(\mathbf{g}_{\alpha_n}(y) - \mathbf{g}_{\alpha_n}(x)\right)\right\| \leqslant \frac{2}{n}\,|y-x|\,.$$

Letting m increase indefinitely one also has

$$\left\|\mathbf{g}(y) - \mathbf{g}(x) - \left(\mathbf{g}_{\alpha_n}(y) - \mathbf{g}_{\alpha_n}(x)\right)\right\| \leqslant \frac{2}{n}\,|y-x|$$

for every $y \in \mathrm{I}$; now, there exists an $h > 0$ such that, for $|y-x| \leqslant h$ and $y \in \mathrm{I}$, one has $\left\|\mathbf{g}_{\alpha}(y) - \mathbf{g}_{\alpha_n}(x) - \mathbf{f}_{\alpha_n}(x)\,(y-x)\right\| \leqslant |y-x|\,/n$; since, on the other hand, we have $\left\|\mathbf{f}(x) - \mathbf{f}_{\alpha_n}(x)\right\| \leqslant 1/n$, we finally obtain

$$\|\mathbf{g}(y) - \mathbf{g}(x) - \mathbf{f}(x)(y-x)\| \leqslant \frac{4}{n}\,|y-x|$$

for $y \in \mathrm{I}$ and $|y-x| \leqslant h$, which completes the proof.

COROLLARY 1. *The set* $\mathcal{H}$ *of maps from* I *into* E *which admit a primitive on an interval* I *is a closed (and so complete) vector subspace of the complete vector space* $\mathcal{F}_c(\mathrm{I};\mathrm{E})$ *of maps from* I *into* E*, endowed with the topology of uniform convergence on every compact subset of* I (*Gen. Top.*, X, p. 277).

COROLLARY 2. *Let* x_0 *be a point of* I*, and for each function* $\mathbf{f} \in \mathcal{H}$ *let* $\mathrm{P}(\mathbf{f})$ *be the primitive of* $\mathbf{f}$ *which vanishes at the point* x_0; *the map* $\mathbf{f} \mapsto \mathrm{P}(\mathbf{f})$ *of* $\mathcal{H}$ *into* $\mathcal{F}_c(\mathrm{I};\mathrm{E})$ *is a continuous linear mapping.*

Cor. 1 to th. 1 allows us to establish the existence of primitives for certain categories of functions by the following procedure: if one knows that the functions belonging to a subset $\mathcal{A}$ of $\mathcal{F}_c(\mathrm{I};\mathrm{E})$ admit a primitive, so will the functions belonging to the *closure* in $\mathcal{F}_c(\mathrm{I};\mathrm{E})$ of the vector subspace generated by $\mathcal{A}$. We shall apply this method in the next subsection.

3. REGULATED FUNCTIONS

DEFINITION 2. *One says that a map* f *of an interval* $\mathrm{I} \subset \mathbf{R}$ *into a set* E *is a step function if there is a partition of* I *into a finite number of intervals* J_k *such that* f *is constant on each of the* J_k.

Let $(a_i)_{0 \leqslant i \leqslant n}$ be the strictly increasing sequence formed by the distinct endpoints of the J_k; since the J_k are pairwise disjoint each of them either reduces to a singleton a_i or is an interval with two consecutive points a_i, a_{i+1} as endpoints; moreover, since I is the union of the J_k, the point a_0 is the left-hand endpoint, and a_n is the right-hand endpoint of I. Every step function on I can thus be characterised as a function which is constant on each of the open intervals $]a_i, a_{i+1}[$ $(0 \leqslant i \leqslant n-1)$, where $(a_i)_{0 \leqslant i \leqslant n}$ is a strictly increasing sequence of points of I with a_0 being the left-hand endpoint and a_n the right-hand endpoint of I.

PROPOSITION 2. *The set of step functions defined on* I, *with values in a vector space* E *over* **R**, *is a vector subspace* $\mathcal{E}$ *of the vector space* $\mathcal{F}(\mathrm{I};\mathrm{E})$ *of all maps of* I *into* E.

Indeed, let f and g be two step functions, and (A_i) and (B_j) two partitions of I into a finite number of intervals such that f (resp. g) is constant on each of the A_i (resp. B_j); whatever the real numbers λ, μ, it is clear that $\lambda f + \mu g$ is constant on each of the nonempty intervals $\mathrm{A}_i \cap \mathrm{B}_j$, and that these intervals form a partition of I.

COROLLARY. *The vector subspace* $\mathcal{E}$ *is generated by the characteristic functions of intervals.*

Now let us consider the case where E is a normed space over **R**; it is then immediate that the characteristic function of an interval J with endpoints a, b $(a < b)$ admits a primitive, namely the function equal to a for $x \leqslant a$, to x for $a \leqslant x \leqslant b$, and to b for $x \geqslant b$. The cor. to prop. 2 thus shows that *every step function with values in* E *admits a primitive.*

We can now apply the method set out in n° 2.

DEFINITION 3. *One says that a vector function, defined on an interval* I, *with values in a complete normed space* E *over* **R**, *is a regulated function, if it is the uniform limit of step functions on every compact subset of* I.

In other words, the regulated functions are the elements of the closure in $\mathcal{F}_c(\mathrm{I};\mathrm{E})$ of the subspace $\mathcal{E}$ of step functions; $\overline{\mathcal{E}}$ is a vector subspace of $\mathcal{F}_c(\mathrm{I};\mathrm{E})$ and since $\mathcal{F}_c(\mathrm{I};\mathrm{E})$ is complete, so is $\overline{\mathcal{E}}$; in other words, if a function is the uniform limit of regulated functions on every compact subset of I, then it is regulated on I. For $\mathbf{f}$ to be regulated on an interval I it is necessary and sufficient that its restriction to every compact interval contained in I be regulated.

Cor. 1 to II, p. 53 shows:

THEOREM 2. *Every regulated function on an interval* I *admits a primitive on* I.

We shall transform def. 3 of II, p. 4 to another equivalent one:

THEOREM 3. *For a vector function* $\mathbf{f}$ *defined on an interval* I, *with values in a complete normed space* E *over* **R**, *to be regulated, it is necessary and sufficient that it have a right limit and a left limit at every interior point of* I, *and a right limit at the left-hand endpoint of* I *and a left limit at the right-hand endpoint of* I, *when these points belong to* I. *The set of points of discontinuity of* $\mathbf{f}$ *in* I *is thus countable.*

Since every interval is a countable union of compact intervals, one can restrict oneself to proving th. 3 when I is *compact,* say $\mathrm{I} = [a, b]$.

1° The condition is *necessary.* Suppose that $\mathbf{f}$ is regulated and let x be a point of I different from b. By hypothesis, for every $\varepsilon > 0$ there is a step function $\mathbf{g}$ such that $\|\mathbf{f}(z) - \mathbf{g}(z)\| \leqslant \varepsilon$ for every $z \in \mathrm{I}$; since $\mathbf{g}$ has a right limit at the point x there exists a y such that $x < y \leqslant b$ and such that, for every pair of points z, z' in the interval $]x, y]$ one has $\left\|\mathbf{g}(z) - \mathbf{g}(z')\right\| \leqslant \varepsilon$, and in consequence $\left\|\mathbf{f}(z) - \mathbf{f}(z')\right\| \leqslant 3\varepsilon$; this proves (by Cauchy's criterion) that $\mathbf{f}$ has a right limit at the point x. In the same way one proves that $\mathbf{f}$ has a left limit at every point of I different from a.

2° The condition is *sufficient.* Suppose it is satisfied; for each $x \in \mathrm{I}$ there is an open interval $\mathrm{V}_x =]c_x, d_x[$ containing x and such that on the intersection of I with each of the open intervals $]c_x, x[$, $]x, d_x[$ (when the intersection is not empty) the oscillation of $\mathbf{f}$ is $\leqslant \varepsilon$. Since I is compact there is a finite number of points x_i in I such that the V_{x_i} form a cover of I; let $(a_k)_{0 \leqslant k \leqslant n}$ be the sequence obtained by arranging in increasing order the points of the finite set formed by the points a, b and those points x_i, c_{x_i} and d_{x_i} which belong to I; each of the intervals $]a_k, a_{k+1}[$ $(0 \leqslant k \leqslant n-1)$ being contained in an interval $]c_{x_i}, x_i[$ or $]x_i, d_{x_i}[$, the oscillation of $\mathbf{f}$ there is $\leqslant \varepsilon$; let $\mathbf{c}_k$ be one of the values of $\mathbf{f}$ on $]a_k, a_{k+1}[$; on putting $\mathbf{g}(a_k) = \mathbf{f}(a_k)$ for $0 \leqslant k \leqslant n$, and $\mathbf{g}(x) = \mathbf{c}_k$ for all $x \in]a_k, a_{k+1}[$ $(0 \leqslant k \leqslant n-1)$, one defines a step function $\mathbf{g}$ such that $\|\mathbf{f}(z) - \mathbf{g}(z)\| \leqslant \varepsilon$ on I; so $\mathbf{f}$ is regulated on I.

Finally, let us show that if $\mathbf{f}$ is regulated on I then its set of points of discontinuity is countable. For every $n > 0$ there exists a step function $\mathbf{g}_n$ such that $\|\mathbf{f}(x) - \mathbf{g}_n(x)\| \leqslant 1/n$ on I; since the sequence $(\mathbf{g}_n)$ converges uniformly to $\mathbf{f}$ on I, we see that $\mathbf{f}$ is continuous at every point where the $\mathbf{g}_n$ are all continuous (*Gen. Top.*, X, p. 281, cor. 1); but since $\mathbf{g}_n$ is continuous except at the points of a finite set H_n it follows that $\mathbf{f}$ is continuous at the points of the complement of the set $\mathrm{H} = \bigcup_n \mathrm{H}_n$, which is countable.

COROLLARY 1. *Let* $\mathbf{f}$ *be a regulated function on* I*; at every point of* I*, apart from the right-hand endpoint* (resp. *the left-hand endpoint*) *of* I*, every primitive of* $\mathbf{f}$ *has a right derivative equal to* $\mathbf{f}(x+)$ (resp. *a left derivative equal to* $\mathbf{f}(x-)$)*; in particular, at every point* x *where* $\mathbf{f}$ *is continuous,* $\mathbf{f}(x)$ *is the derivative of one of its primitives.*

This is an immediate consequence of th. 3 of II, p. 54 and of prop. 6 of I, p. 18.

COROLLARY 2. *Let* $\mathbf{f}_i$ $(1 \leqslant i \leqslant n)$ *be* n *regulated functions on an interval* I*, each* $\mathbf{f}_i$ *having its values in a complete normed space* E_i *over* $\mathbf{R}$ $(1 \leqslant i \leqslant n)$*. If* $\mathbf{g}$ *is a continuous map of the subspace* $\prod_{i=1}^{n} \overline{\mathbf{f}_i(\mathrm{I})}$ *of* $\prod_{i=1}^{n} \mathrm{E}_i$ *into a complete normed space* F *over* $\mathbf{R}$*, then the composite function* $x \mapsto \mathbf{g}(\mathbf{f}_1(x), \mathbf{f}_2(x), \ldots, \mathbf{f}_n(x))$ *is regulated on* I.

Indeed, it clearly satisfies the conditions of th. 3 of II, p. 54.

Thus one sees that if $\mathbf{f}$ is a regulated vector function on I, then the real function $x \mapsto \|\mathbf{f}(x)\|$ is also regulated. Further, the real regulated functions on I form a *ring;*

moreover, if f and g are two real regulated functions, then $\sup(f, g)$ and $\inf(f, g)$ are regulated.

Remark. 1) If f is a real regulated function on I, and $\mathbf{g}$ is a regulated vector function on an interval containing $f(\mathrm{I})$, then the composite function $\mathbf{g}\circ f$ is not necessarily regulated (*cf.* II, p. 79, exerc. 4).

Two particular cases of th. 3 of II, p. 54 are especially important:

PROPOSITION 3. *Every continuous vector function on an interval* $\mathrm{I} \subset \mathbf{R}$ *taking its values in a complete normed space* E *over* $\mathbf{R}$ *is regulated, and admits a primitive on* I*, of which it is the derivative at every point.*

Remarks. 2) To show that a continuous function admits a primitive, one can use the fact that every *polynomial function* of a real variable (with coefficients in E) admits a primitive; since from the theorem of Weierstrass (*Gen. Top.*, X, p. 313, prop. 3) every continuous function is the uniform limit of polynomials on every compact interval, th. 1 of II, p. 52 shows that every continuous function admits a primitive.

3) The principle of the preceding remark extends without significant modification to vector functions of a *complex* variable taking values in a complete normed space over $\mathbf{C}$. If U is an open set in $\mathbf{C}$, homeomorphic to $\mathbf{C}$, a *primitive* of such a vector function $\mathbf{f}$ defined on U is by definition a continuous function on U, having derivative equal to $\mathbf{f}$ at every point of U. With this definition, th. 1 of II, p. 52 extends without modification (one proves, using the connectedness of U, that $(\mathbf{g}_\alpha)$ is uniformly convergent with respect to $\mathcal{F}$ on a neighbourhood of each point of U, from which it follows that $(\mathbf{g}_\alpha)$ is uniformly convergent with respect to $\mathcal{F}$ on every compact subset of U; the proof is completed using prop. 4 of I, p. 18). Consequently, every function which is a *uniform limit of polynomials* on every compact subset of U, admits a primitive on U; these functions are no other than the functions called *holomorphic* on U, which we shall study further in detail in a later Book.

PROPOSITION 4. *Every monotone real function* f *on an interval* $\mathrm{I} \subset \mathbf{R}$ *is regulated, and every primitive of* f *is convex on* I.

Indeed, f satisfies the criterion of th. 3 of *Gen. Top.*, IV, p. 350, prop. 4; the second part of the proposition follows from cor. 1, from II, p. 55, and from prop. 5 of I, p. 27.

Remark. 4) One must not think that the regulated functions on an interval I are the only functions having a primitive on I (*cf.* II, p. 80, exercises 7 and 8).

4. INTEGRALS

We have obtained (II, p. 54, th. 2) a primitive of a regulated function on an interval I as the uniform limit of primitives of step functions. This procedure can be expressed in a slightly different way: let x_0, x be two arbitrary points of I such that $x_0 < x$; we call a *subdivision* of the interval $[x_0,\ x]$ any sequence of intervals $[x_i,\ x_{i+1}]$ with union $[x_0,\ x]$, where $(x_i)_{0 \leqslant i \leqslant n}$ is a strictly increasing sequence of points of $[x_0,\ x]$ such that $x_n = x$. We shall call a *Riemann sum*, relative to a vector function $\mathbf{f}$ defined on I and to the subdivision formed by the $[x_i,\ x_{i+1}]$, any expression of the form $\sum_{i=0}^{n-1} \mathbf{f}(t_i)(x_{i+1} - x_i)$ where the t_i belong to $[x_i,\ x_{i+1}]$ for $0 \leqslant i \leqslant n-1$. One then has the following proposition:

PROPOSITION 5. *Let* $\mathbf{f}$ *be a regulated function on an interval* I*, let* $\mathbf{g}$ *be a primitive of* $\mathbf{f}$ *on* I*, and* $[x_0,\ x]$ *a compact interval contained in* I*. For every* $\varepsilon > 0$ *there exists a number* $\rho > 0$ *such that for every subdivision of* $[x_0,\ x]$ *into intervals of length* $\leqslant \rho$ *one has*

$$\left\| \mathbf{g}(x) - \mathbf{g}(x_0) - \sum_{i=0}^{n-1} \mathbf{f}(t_i)(x_{i+1} - x_i) \right\| \leqslant \varepsilon \tag{1}$$

for every Riemann sum relative to this subdivision.

Indeed, let $\mathbf{f}_1$ be a step function such that $\|\mathbf{f}(y) - \mathbf{f}_1(y)\| \leqslant \varepsilon$ for every $y \in [x_0, x]$; one has, denoting a primitive of $\mathbf{f}_1$ on I by $\mathbf{g}_1$,

$$\|\mathbf{g}(x) - \mathbf{g}(x_0) - (\mathbf{g}_1(x) - \mathbf{g}_1(x_0))\| \leqslant \varepsilon(x - x_0)$$

by the mean value theorem, and on the other hand

$$\left\| \sum_{i=0}^{n-1} \mathbf{f}(t_i)(x_{i+1} - x_i) - \sum_{i=0}^{n-1} \mathbf{f}_1(t_i)(x_{i+1} - x_i) \right\| \leqslant \varepsilon(x - x_0).$$

It thus suffices to prove the proposition when $\mathbf{f}$ is a *step function*. Let $(y_k)_{1 \leqslant k \leqslant m}$ be the strictly increasing finite sequence of points of discontinuity of $\mathbf{f}$ in $[x_0,\ x]$. For every subdivision of $[x_0,\ x]$ into intervals of length $\leqslant \rho$ each of the points y_k belongs to at most two of these intervals; there can therefore be no more than $2m$ intervals on which $\mathbf{f}$ is not constant; but, on such an interval $[x_i,\ x_{i+1}]$ one has

$$\|\mathbf{g}(x_{i+1}) - \mathbf{g}(x_i) - \mathbf{f}(t_i)(x_{i+1} - x_i)\| \leqslant 2\mathrm{M}\,(x_{i+1} - x_i)$$

on denoting by M the least upper bound of $\|\mathbf{f}\|$ on $[x_0,\ x]$; on the other hand, when $\mathbf{f}$ is constant on $[x_i,\ x_{i+1}]$ one has

$$\mathbf{g}(x_{i+1}) - \mathbf{g}(x_i) - \mathbf{f}(t_i)(x_{i+1} - x_i) = 0.$$

One thus sees that the difference $\left\| \mathbf{g}(x) - \mathbf{g}(x_0) - \sum_{i=0}^{n-1} \mathbf{f}(t_i)(x_{i+1} - x_i) \right\|$ cannot exceed $4\mathrm{M}m\rho$; it therefore suffices to take $\rho \leqslant \varepsilon/4\mathrm{M}m$ to obtain (1).

Remark. 1) When $\mathbf{f}$ is *continuous* prop. 5 can be proved more simply: since $\mathbf{f}$ is uniformly continuous on $[x_0,\ x]$ there exists a $\rho > 0$ such that on every interval of length $\leqslant \rho$ contained in $[x_0,\ x]$ the oscillation of $\mathbf{f}$ is $\leqslant \dfrac{\varepsilon}{x - x_0}$; for every subdivision of $[x_0,\ x]$ into intervals $[x_i,\ x_{i+1}]$ of length $\leqslant \rho$ and every choice of t_i in $[x_i,\ x_{i+1}]$ for $0 \leqslant i \leqslant n-1$ the step function $\mathbf{f}_1$ equal to $\mathbf{f}(t_i)$ on $[x_i,\ x_{i+1}[$ $(0 \leqslant i \leqslant n-1)$, and to $\mathbf{f}(x)$ at the point

x, is such that $\|\mathbf{f}(y) - \mathbf{f}_1(y)\| \leqslant \dfrac{\varepsilon}{x - x_0}$ on $[x_0, x]$; if $\mathbf{g}_1$ is a primitive of $\mathbf{f}_1$ one has $\mathbf{g}_1(x) - \mathbf{g}_1(x_0) = \sum\limits_{i=0}^{n-1} \mathbf{f}(t_i)(x_{i+1} - x_i)$, so the relation (1) follows immediately from the mean value theorem.

In the rest of this chapter we shall confine ourselves to the study of primitives of *regulated* functions on an interval I. For such a function $\mathbf{f}$, with values in E, a primitive $\mathbf{g}$ of $\mathbf{f}$, and for two arbitrary points x_0, x of I, the element $\mathbf{g}(x) - \mathbf{g}(x_0)$ of E (which is clearly the same, no matter which primitive $\mathbf{g}$ of $\mathbf{f}$ one considers) is called the *integral of the function* $\mathbf{f}$ *from* x_0 *to* x (or *over the compact interval* $[x_0, x]$) and is denoted by $\int_{x_0}^{x} \mathbf{f}(t)dt$ or $\int_{x_0}^{x} \mathbf{f}$. This name and notation have their origin in prop. 5 of II, p. 57, which shows that an integral can be approximated arbitrarily closely by a Riemann sum; more particularly, one can write, taking subdivisions of $[x_0, x]$ into equal intervals,

$$\frac{1}{x - x_0} \int_{x_0}^{x} \mathbf{f}(t)\, dt = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{f}\left(x_0 + k\frac{x - x_0}{n}\right). \tag{2}$$

In other words, the element $\dfrac{1}{x - x_0} \int_{x_0}^{x} \mathbf{f}(t)\, dt$ is the limit of the *arithmetic means* of the values of $\mathbf{f}$ at the left-hand endpoints of the intervals of a subdivision of $[x_0, x]$ into equal intervals; one also calls it the *mean* (or *mean value*) of the function $\mathbf{f}$ on the interval $[x_0, x]$.

By definition, the function $x \mapsto \int_{x_0}^{x} \mathbf{f}(t)\, dt$ is none other than the primitive of $\mathbf{f}$ which vanishes at the point $x_0 \in$ I; one also denotes it by $\int_{x_0} \mathbf{f}(t)\, dt$ or $\int_{x_0} \mathbf{f}$.

Remarks. 2) For an arbitrary function $\mathbf{h}$ defined on I, with values in E, the element $\mathbf{h}(x) - \mathbf{h}(x_0)$ is also written as $\mathbf{h}(t)\big|_{x_0}^{x}$; with this notation one sees that if $\mathbf{g}$ is any primitive of a regulated function $\mathbf{f}$ on I, one has

$$\int_{x_0}^{x} \mathbf{f}(t)\, dt = \mathbf{g}(t)\,\Big|_{x_0}^{x}. \tag{3}$$

3) The expressions $\int_{x_0}^{x} \mathbf{f}(t)\, dt$ and $\mathbf{g}(t)\Big|_{x_0}^{x}$ are abbreviating symbols representing assemblies in which the letters $x, x_0, \mathbf{f}, \mathbf{g}$, but *not* the letter t, appear (*cf. Set Theory*, I, p. 15); one says that among these symbols t is a "*dummy* variable"; one can thus replace t by any other variable distinct from $x, x_0, \mathbf{f}$ and $\mathbf{g}$ (and from variables which may possibly enter into the proof where these symbols appear) without changing the sense of the symbol so obtained (the reader may compare these symbols with symbols such as $\sum_{i=1}^{n} x_i$, or $\bigcup\limits_i \mathrm{X}_i$, where i is likewise a dummy variable).

4) The approximation of an integral by Riemann sums is closely connected to one of the historical origins of the concept of an integral, the problem of the *measure* of areas. We shall come back to this point in the Book on Integration which is devoted to generalizations of the concept of integral to which this problem leads; in these generalizations the functions to be "integrated" are not necessarily defined on a subset of $\mathbf{R}$; moreover, even when one

deals with (not necessarily regulated) real functions f of a real variable for which one can define an integral $\int_{x_0}^{x} f(t)\,dt$, the function $x \mapsto \int_{x_0}^{x} f(t)\,dt$ is not always a primitive of f, and there exist functions which have a primitive but are not "integrable" in the sense to which we allude.

5. PROPERTIES OF INTEGRALS

The properties of the integrals of regulated functions are simply a *translation,* into the appropriate notation, of the properties of derivatives demonstrated in chap. I.

In the first place, the formula (3) shows that no matter what the points x, y, z of I, one has

$$\int_x^x \mathbf{f}(t)\,dt = 0 \tag{4}$$

$$\int_x^y \mathbf{f}(t)\,dt + \int_y^x \mathbf{f}(t)\,dt = 0 \tag{5}$$

$$\int_x^y \mathbf{f}(t)\,dt + \int_y^z \mathbf{f}(t)\,dt + \int_z^x \mathbf{f}(t)\,dt = 0 \tag{6}$$

From prop. 1 of II, p. 52 one has

$$\int_{x_0} (\mathbf{f}+\mathbf{g}) = \int_{x_0} \mathbf{f} + \int_{x_0} \mathbf{g} \tag{7}$$

and for every scalar k

$$\int_{x_0} k\mathbf{f} = k \int_{x_0} \mathbf{f}. \tag{8}$$

Let E, F be two complete normed spaces over $\mathbf{R}$, and $\mathbf{u}$ a continuous linear map from E into F. If $\mathbf{f}$ is a regulated function on I with values in E, then $\mathbf{u} \circ \mathbf{f}$ is a regulated function on I with values in F (II, p. 6, cor. 2), and one has for $a, b \in$ I (I, p. 13, prop. 2)

$$\int_a^b \mathbf{u}(\mathbf{f}(t))\,dt = \mathbf{u}\left(\int_a^b \mathbf{f}(t)\,dt\right). \tag{9}$$

Now let E, F, G be three complete normed spaces over $\mathbf{R}$, and $(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}.\mathbf{y}]$ a continuous bilinear map of E $\times$ F into G. Let $\mathbf{f}$ and $\mathbf{g}$ be two vector functions defined and continuous on I, taking their values in E and F respectively; suppose moreover that $\mathbf{f}$ and $\mathbf{g}$ are two primitives of *regulated* functions, which we denote by $\mathbf{f}'$ and $\mathbf{g}'$ by abuse of language (these functions are not actually guaranteed to be equal to the derivatives of $\mathbf{f}$ and $\mathbf{g}$ respectively except on the complement of a countable set). By prop. 3 of I, p. 6, the function $\mathbf{h}(x) = [\mathbf{f}(x).\mathbf{g}(x)]$ has, at every point of the complement of a countable subset of I, a derivative equal to $[\mathbf{f}(x).\mathbf{g}'(x)] + [\mathbf{f}'(x).\mathbf{g}(x)]$. Now, by the continuity of $[\mathbf{x}.\mathbf{y}]$ and cor. 2 of II, p. 55, each of the functions $[\mathbf{f}.\mathbf{g}']$ and $[\mathbf{f}'.\mathbf{g}]$ is a regulated function on I; so one has the formula

$$\int_a^b [\mathbf{f}'(t).\mathbf{g}(t)]\,dt = [\mathbf{f}(t).\mathbf{g}(t)]\Big|_a^b - \int_a^b [\mathbf{f}(t).\mathbf{g}'(t)]\,dt \tag{10}$$

called the *formula for integration by parts,* which allows one to evaluate many primitives.

For example, the formula for integration by parts yields the following formula

$$\int_{x_0}^{x} t\mathbf{f}'(t)\,dt = t\mathbf{f}(t)\Big|_{x_0}^{x} - \int_{x_0}^{x} \mathbf{f}(t)\,dt$$

so reducing the evaluation of primitives of one of the two functions $\mathbf{f}(x)$ and $x\mathbf{f}'(x)$ to the other.

Likewise, if $\mathbf{f}$ and $\mathbf{g}$ are n times differentiable on an interval I, and if $\mathbf{f}^{(n)}$ and $\mathbf{g}^{(n)}$ are regulated functions on I, then formula (5) of I, p. 21 is equivalent to the following:

$$\begin{aligned}\int_a^b [\mathbf{f}^{(n)}(t).\mathbf{g}(t)]\,dt \\ = \left(\sum_{p=0}^{n-1} (-1)^p\,[\mathbf{f}^{(n-p-1)}(t).\mathbf{g}^{(p)}(t)]\right)\Bigg|_a^b + (-1)^n \int_a^b [\mathbf{f}(t).\mathbf{g}^{(n)}(t)]\,dt\end{aligned} \tag{11}$$

which one calls the *formula for integration by parts of order n.*

Let us now translate the formula for differentiation of composite functions (I, p. 9, prop. 5). Let f be a real function defined and continuous on I, which is the primitive of a *regulated* function on I (which we again write as f' by abuse of language); let, moreover, $\mathbf{g}$ be a *continuous* vector function (with values in a complete normed space) on an open interval J containing $f(\mathrm{I})$; if $\mathbf{h}$ denotes an arbitrary primitive of $\mathbf{g}$ on J, then $\mathbf{h}$ admits a derivative equal to $\mathbf{g}$ at each point of J (II, p. 56, prop. 3); thus the composite function $\mathbf{h}\circ f$ admits a derivative equal to $\mathbf{g}(f(x))f'(x)$ at all the points of the complement (with respect to I) of a countable subset of I (I, p. 9, prop. 5); since the function $\mathbf{g}(f(x))f'(x)$ is regulated (II, p. 55, cor. 2), one can write the formula

$$\int_a^b \mathbf{g}(f(t))f'(t)\,dt = \int_{f(a)}^{f(b)} \mathbf{g}(u)\,du \tag{12}$$

called the *formula for change of variables,* which also facilitates the evaluation of primitives.

If, for example, one takes $f(x) = x^2$, one sees that the formula (12) reduces from one to the other the evaluation of the primitives of the functions $\mathbf{g}(x)$ and $x\mathbf{g}(x^2)$.

To translate the mean value theorem (I, p. 14, th. 1) for primitives of real regulated functions, we first remark that a real regulated function f on a compact interval I is bounded on I; let J be the set of points of I where f is *continuous,* and put $m = \inf_{x\in \mathrm{J}} f(x)$, $\mathrm{M} = \sup_{x\in \mathrm{J}} f(x)$; one knows (II, p. 54, th. 3) that $\mathrm{I} \cap \complement \mathrm{J}$ is countable; further, if B is the complement, with respect to I, of any countable subset of I, and $m' = \inf_{x\in \mathrm{B}} f(x)$, $\mathrm{M}' = \sup_{x\in \mathrm{B}} f(x)$, then one has $m' \leqslant m \leqslant M \leqslant M'$: indeed, for every point $x \in \mathrm{J}$, there are points y of B arbitrarily close to x, whence $m' \leqslant f(y) \leqslant \mathrm{M}'$; since f is continuous at the point x one sees, making y approach x (y remaining in B) that $m' \leqslant f(x) \leqslant \mathrm{M}'$, which proves our assertion. This being so, translating the mean value theorem gives the following proposition:

PROPOSITION 6 (theorem of the mean). *Let f be a real regulated function on a compact interval $\mathrm{I} = [a, b]$; if J is the set of points of I where f is continuous, and $m = \inf_{x \in \mathrm{J}} f(x)$, $\mathrm{M} = \sup_{x \in \mathrm{J}} f(x)$, then*

$$m < \frac{1}{b-a} \int_a^b \mathbf{f}(t)\,dt < \mathrm{M} \tag{13}$$

except when f is constant on J, in which case the three members of (13) are equal.

In other words, the *mean* of the regulated function f in I lies between the bounds of f over the subset of I where f is continuous.

COROLLARY 1. *If a real regulated function f on I is such that $f(x) \geqslant 0$ at the points where f is continuous, then $\dfrac{1}{b-a} \displaystyle\int_a^b f(t)\,dt > 0$ unless $f(x) = 0$ at the points where f is continuous.*

COROLLARY 2. *Let f and g be two real regulated functions on I, such that $g(x) \geqslant 0$ at the points where g is continuous; if m and M are the greatest lower bound and least upper bound of f over the set of points where f is continuous, then*

$$\frac{m}{b-a} \int_a^b g(t)\,dt \leqslant \frac{1}{b-a} \int_a^b f(t)g(t)\,dt \leqslant \frac{\mathrm{M}}{b-a} \int_a^b g(t)\,dt. \tag{14}$$

The first two terms (resp. *the two last*) *are unequal unless $g(x)(f(x) - m) = 0$* (resp. $g(x)(f(x) - \mathrm{M}) = 0$) *at every point where f and g are continuous.*

For vector functions the mean value theorem (I, p. 15, th. 2) yields the following proposition:

PROPOSITION 7. *Let $\mathbf{f}$ be a regulated vector function on a compact interval $\mathrm{I} = [a, b]$, with values in a complete normed space E, and let g be a real regulated function on I, such that $g(x) \geqslant 0$ at the points where g is continuous; in these circumstances*

$$\left\| \int_a^b \mathbf{f}(t)g(t)\,dt \right\| \leqslant \int_a^b \|\mathbf{f}(t)\|\, g(t)\,dt. \tag{15}$$

In particular,

$$\left\| \int_a^b \mathbf{f}(t)\,dt \right\| \leqslant \int_a^b \|\mathbf{f}(t)\|\; dt. \tag{16}$$

6. INTEGRAL FORMULA FOR THE REMAINDER IN TAYLOR'S FORMULA; PRIMITIVES OF HIGHER ORDER

The formula for integration by parts of order n (II, p. 60, formula (11)) allows one to express in terms of an integral the *remainder* $\mathbf{r}_n(x)$ in the Taylor expansion of order n of a function which admits a *regulated* $(n+1)^{th}$ derivative on an interval I (I, p. 22); indeed, on replacing, in (12), $\mathbf{f}$ by $\mathbf{f}'$, b by x, and $g(t)$ by the function $(t-x)^n/n!$, it follows that

$$\begin{aligned}\mathbf{f}(x) = \mathbf{f}(a) + \mathbf{f}'(a)\frac{(x-a)}{1!} + \mathbf{f}''(a)\frac{(x-a)^2}{2!} + \cdots \\ + \mathbf{f}^{(n)}(a)\frac{(x-a)^n}{n!} + \int_a^x \mathbf{f}^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt\end{aligned} \tag{17}$$

in other words

$$\mathbf{r}_n(x) = \int_a^x \mathbf{f}^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt, \tag{18}$$

which formula often permits one to obtain simple bounds for the remainder.

Given a regulated function $\mathbf{f}$ on an interval I, an arbitrary primitive $\mathbf{g}$ of $\mathbf{f}$, being continuous in I, admits a primitive in its turn; an arbitrary primitive of an arbitrary primitive of $\mathbf{f}$ is called a *second primitive* of $\mathbf{f}$. More generally, a primitive of a primitive of order $n-1$ of $\mathbf{f}$ is termed a *primitive of order* n of $\mathbf{f}$. One sees immediately, by induction on n, that the difference of two primitives of order n of $\mathbf{f}$ is a *polynomial of degree at most equal to* $n-1$ (with coefficients in E). A primitive of order n of $\mathbf{f}$ is entirely determined if one specifies its value and those of its first $n-1$ derivatives at a point $a \in$ I.

In particular, $\int_a^{(n)} \mathbf{f}$ denotes that primitive of order n of $\mathbf{f}$ which vanishes, together with its first $n-1$ derivatives, at the point a. The Taylor formula of order $n-1$, applied to this primitive, shows that if $\mathbf{g} = \int_a^{(n)} \mathbf{f}$, then

$$\mathbf{g}(x) = \int_a^x \mathbf{f}(t)\frac{(x-t)^{n-1}}{(n-1)!}\,dt \tag{19}$$

so reducing the determination of a primitive of order n to one single integral.

§ 2. INTEGRALS OVER NON-COMPACT INTERVALS

1. DEFINITION OF AN INTEGRAL OVER A NON-COMPACT INTERVAL

Let I be a compact interval $[a, b]$ in the *extended line* $\overline{\mathbf{R}}$ (a and b may be infinite); let $\mathbf{f}$ be a function defined on $]a, b[$, taking its values in a complete normed space E over $\mathbf{R}$. Generalizing def. 1 of II, p. 51, we shall say that a function $\mathbf{g}$, defined on $[a, b]$ with values in E, is a *primitive* of $\mathbf{f}$ if it is continuous on $[a, b]$ (and in particular at the endpoints a and b) and admits a derivative equal to $\mathbf{f}(x)$ at all the points of the complement with respect to $]a, b[$ of a countable subset of this interval.

We shall restrict ourselves to the following case: there exists a finite strictly increasing sequence $(c_i)_{0 \leqslant i \leqslant n}$ of points of $\mathrm{I} = [a,\ b]$, such that $c_0 = a,\ c_n = b$, and such that $\mathbf{f}$ is *regulated* on each of the open intervals $]c_i,\ c_{i+1}[$, though not necessarily regulated on every open interval containing at least one point c_i interior to I; such a function will be called *piecewise regulated* on $]a,\ b[$. We remark that a regulated function on $]a,\ b[$ is piecewise regulated (taking $n = 1$ in the preceding definition).

If $\mathbf{f}$ admits a primitive $\mathbf{g}$ (in the sense made precise above), and if c is a point of the interval $]c_i,\ c_{i+1}[$ $(0 \leqslant i \leqslant n-1)$, one has, by hypothesis, for each x in this interval, that $\mathbf{g}(x) - \mathbf{g}(c) = \int_c^x \mathbf{f}(t)\,dt$; since $\mathbf{g}$ is continuous on I by hypothesis, one sees that $\int_c^x \mathbf{f}(t)\,dt$ tends to a limit in E when x tends to c_i from the right and when x tends to c_{i+1} from the left. Conversely, suppose that these conditions are satisfied for all t, and let $\mathbf{g}_i$ be a primitive of $\mathbf{f}$ on the interval $]c_i,\ c_{i+1}[$ $(0 \leqslant i \leqslant n-1)$; we note immediately that the function $\mathbf{g}$, defined on the complement with respect to I of the set of c_i, by the condition that it be equal to $\mathbf{g}_i(x) + \sum_{k=1}^{i} (\mathbf{g}_{k-1}(c_k-) - \mathbf{g}_k(c_k+))$ on $]c_i,\ c_{i+1}[$ for $0 \leqslant i \leqslant n-1$, is continuous at every point of I distinct from the c_i and admits a limit at each of these points; it can therefore be extended by continuity to each of these points, and the extended function is evidently a primitive of $\mathbf{f}$ on I. It is clear, moreover, that every other primitive of $\mathbf{f}$ is of the form $\mathbf{g} + \mathbf{a}$ ($\mathbf{a}$ an element of E).

DEFINITION 1. *One says that a vector function* $\mathbf{f}$, *piecewise regulated on an interval* $]a,\ b[$ *of* $\overline{\mathbf{R}}$, *admits an integral on this interval if* $\mathbf{f}$ *admits a primitive on* $[a,\ b]$; *if* $\mathbf{g}$ *is any one of the primitives of* $\mathbf{f}$ *on* $[a,\ b]$, *and* x_0 *and* x *are any two points of* $[a,\ b]$, *one calls the element* $\mathbf{g}(x) - \mathbf{g}(x_0)$ *the integral of* $\mathbf{f}$ *from* x_0 *to* x, *and one denotes it by* $\int_{x_0}^x \mathbf{f}(t)\,dt$.

This concept clearly agrees with that defined when the interval $[x_0,\ x]$ contains none of the points c_i.

The remarks which precede def. 1 show that for $\mathbf{f}$ to have an integral on $]a,\ b[$ it is necessary and sufficient that its restriction to each of the intervals $]c_i,\ c_{i+1}[$ should admit an integral over this interval. In other words, one reduces to the case where $\mathbf{f}$ is regulated on a *non-compact* interval $\mathrm{I} \subset \mathbf{R}$, with endpoints $a,\ b$ $(a < b)$, and where: 1° either one of the numbers $a,\ b$ (at least) is infinite; 2° or $\mathbf{f}$ is not regulated on a compact interval containing at least one of the points $a,\ b$ (these two hypotheses not being mutually exclusive). For $\mathbf{f}$ to have an integral over I it is necessary and sufficient that the integral $\int_x^y \mathbf{f}(t)\,dt$ approaches a limit when the point $(x,\ y)$ approaches $(a,\ b) \in \overline{\mathbf{R}}^2$ while remaining in $\mathrm{I} \times \mathrm{I}$: and this limit is no other than $\int_a^b \mathbf{f}(t)\,dt$ according to def. 1. By an abuse of language, instead of saying that $\mathbf{f}$ has an integral over I, one says that the integral $\int_a^b \mathbf{f}(t)\,dt$ is *convergent (or converges)*.

Examples. 1) The integral $\int_1^{+\infty} dt/t^2$ is convergent and equal to 1, for

$$\int_1^x \frac{dt}{t^2} = 1 - \frac{1}{x}.$$

2) The integral $\int_0^1 dt/\sqrt{t}$ is convergent and equal to 2, for

$$\int_x^1 \frac{dt}{\sqrt{t}} = 2(1-\sqrt{x}) \quad \text{for} \quad x > 0.$$

3) Let $(\mathbf{u}_n)_{n\geqslant 1}$ be an infinite sequence of points of E, and let $\mathbf{f}$ be the step function defined on the interval $[1, +\infty[$ by the conditions: $\mathbf{f}(x) = \mathbf{u}_n$ for $n \leqslant x < n+1$. Then for the integral $\int_1^{+\infty} \mathbf{f}(t)\,dt$ to be convergent it is necessary and sufficient that the series with general term $\mathbf{u}_n$ be *convergent* in E; indeed, one has

$$\int_1^n \mathbf{f}(t)\,dt = \sum_{p=1}^{n-1} \mathbf{u}_p,$$

so the condition is necessary; conversely, if the series with general term $\mathbf{u}_n$ converges in E, then $\lim\limits_{n\to\infty} \mathbf{u}_n = 0$; now, if $n \leqslant x \leqslant n+1$, one has $\displaystyle\int_1^n \mathbf{f}(t)\,dt = \sum_{p=1}^{n-1} \mathbf{u}_p + \mathbf{u}_n(x-n)$, so this integral has the limit $\displaystyle\sum_{n=1}^{\infty} \mathbf{u}_n$ when x tends to $+\infty$.

It is immediate that if a piecewise regulated function $\mathbf{f}$ admits an integral over I then the formulae (4) to (9) of II, p. 59 remain valid. Similarly, formula (10) of II, p. 59 extends in the following manner: $\mathbf{f}$ and $\mathbf{g}$ are assumed to be primitives of the regulated functions $\mathbf{f}'$ and $\mathbf{g}'$ on $]a, b[$, and one denotes by $\left[\mathbf{f}.\mathbf{g}\right]\Big|_a^b$ the limit (if it exists) of $\left[\mathbf{f}.\mathbf{g}\right]\Big|_x^y$ as (x, y) tends to (a, b) (with $a < x \leqslant y < b$); then, if two of the (three) expressions $\left[\mathbf{f}.\mathbf{g}\right]\Big|_a^b$, $\int_a^b \left[\mathbf{f}(t).\mathbf{g}'(t)\right] dt$, and $\int_a^b \left[\mathbf{f}'(t).\mathbf{g}(t)\right] dt$ have a meaning, then so has the third, and the formula (10) of II, p. 59 is valid.

Finally, let f be a real function which is defined and continuous on $\mathrm{I} =]a, b[$, and is the primitive of a regulated function f' on $]a, b[$; let on the other hand $\mathbf{g}$ be a continuous vector function on an open interval J containing $f(\mathrm{I})$; if the function $\mathbf{g}(f(x))f'(x)$ admits an integral over I, and if f tends to a limit (finite or not) at the points a and b, then $\mathbf{g}$ admits an integral from $f(a+)$ to $f(b-)$, and one has the formula

$$\int_a^b \mathbf{g}(f(t))f'(t)\,dt = \int_{f(a+)}^{f(b-)} \mathbf{g}(u)\,du. \tag{1}$$

Indeed, if (x, y) tends to (a, b), then $(f(x), f(y))$ tends to $(f(a+), f(b-))$ by hypothesis; it suffices to apply formula (12) of II, p. 60 between x and y, and to pass to the limit to obtain (1).

Given a regulated function $\mathbf{f}$ on a non-compact interval $\mathrm{I} \subset \mathbf{R}$, with endpoints a and b ($a < b$), the condition for $\mathbf{f}$ to have an integral over I can be presented in the following manner. The compact intervals $\mathrm{J} \subset \mathrm{I}$ form a *directed set* $\mathfrak{K}(I)$ with respect to the relation $\subset$ [1], for if $[\alpha,\ \beta]$ and $[\gamma,\ \delta]$ are two compact intervals contained in I, and if one puts $\lambda = \min(\alpha,\ \gamma),\ \mu = \max(\beta,\ \delta)$, then the interval $[\lambda,\ \mu]$ is contained in I and contains the two intervals considered. For each compact interval $J = [\alpha,\ \beta]$ contained in I, let us put

$$\int_J \mathbf{f}(t)\,dt = \int_\alpha^\beta \mathbf{f}(t)\,dt;$$

for $\mathbf{f}$ to admit an integral over I it is necessary and sufficient that the map $J \mapsto \int_J \mathbf{f}(t)\,dt$ have a limit with respect to the directed set $\mathfrak{K}(I)$; this limit is then the integral $\int_a^b \mathbf{f}(t)\,dt$, which we again denote by $\int_I \mathbf{f}(t)\,dt$.

PROPOSITION 1 (Cauchy's criterion for integrals). *Let* $\mathbf{f}$ *be a regulated function on an interval* $I \subset \mathbf{R}$ *having endpoints a and b* ($a < b$). *For the integral* $\int_a^b \mathbf{f}(t)\,dt$ *to exist it is necessary and sufficient that for every* $\varepsilon > 0$ *there exist a compact interval* $J_0 = [\alpha,\ \beta]$ *contained in* I, *such that for any compact interval* $K = [x,\ y]$ *contained in* I *and having no interior points in common with* J_0, *one has* $\left\|\int_K \mathbf{f}(t)\,dt\right\| \leqslant \varepsilon$.

Indeed, since E is complete the Cauchy criterion shows that for the integral $\int_I \mathbf{f}(t)\,dt$ to be convergent it is necessary and sufficient that for any $\varepsilon > 0$ there exists a compact interval $J_0 = [\alpha, \beta]$ with for every compact interval J such that $J_0 \subset J \subset I$ one has $\left\|\int_J \mathbf{f}(t)\,dt - \int_{J_0} \mathbf{f}(t)\,dt\right\| \leqslant \varepsilon$. The proposition will follow from the following lemma:

Lemma. *Let* $J_0 = [\alpha,\ \beta]$ *be a compact interval contained in* I. *In order that* $\left\|\int_J \mathbf{f}(t)\,dt - \int_{J'} \mathbf{f}(t)\,dt\right\| \leqslant \varepsilon$ *for every pair of compact intervals* J, J′ *contained in* I *and containing* J' *it is necessary that* $\left\|\int_K \mathbf{f}(t)\,dt\right\| \leqslant \varepsilon$, *and it suffices that* $\left\|\int_K \mathbf{f}(t)\,dt\right\| \leqslant \varepsilon/2$, *for every compact interval* K *contained in* I *and having no interior point in common with* J_0.

Indeed, if for $J_0 \subset J \subset I$ and $J_0 \subset J' \subset I$, one has

$$\left\|\int_J \mathbf{f}(t)\,dt - \int_{J'} \mathbf{f}(t)\,dt\right\| \leqslant \varepsilon$$

one sees in particular that, for $x \leqslant y \leqslant \alpha$, or for $\beta \leqslant x \leqslant y$ (x and y in I), one has $\left\|\int_x^y \mathbf{f}(t)\,dt\right\| \leqslant \varepsilon$. Conversely, if $\left\|\int_K \mathbf{f}(t)\,dt\right\| \leqslant \varepsilon/2$ for every compact interval $K \subset I$ such that $K \cap J_0 = \emptyset$, and if $J = [x,\ y],\ J' = [z,\ t]$ are two compact intervals contained in I and containing J_0, one has

[1] Recall (*Set Theory*, III, p. 144) that a set $\mathfrak{F}$ of subsets of I is *directed with respect to the relation* $\subset$ if, for any $X \in \mathfrak{F},\ Y \in \mathfrak{F}$, there exists $Z \in \mathfrak{F}$ such that $X \subset Z$ and $Y \subset Z$. If S(X) denotes the subset of $\mathfrak{F}$ formed by the $U \in \mathfrak{F}$ such that $U \supset X$, then the S(X) form a base for a filter on $\mathfrak{F}$, called the *filter of sections* of $\mathfrak{F}$; the limit (if it exists) of a map f of $\mathfrak{F}$ into a topological space, with respect to the filter of sections of $\mathfrak{F}$, is called the *limit of f with respect to the directed set* $\mathfrak{F}$ (*cf. Gen. Top.*, I, p. 70 and *Gen. Top.*, IV, p. 348).

$$\left\| \int_{\mathrm{J}} \mathbf{f}(t)\,dt - \int_{\mathrm{J}'} \mathbf{f}(t)\,dt \right\| = \left\| \int_x^z \mathbf{f}(t)\,dt + \int_t^y \mathbf{f}(t)\,dt \right\| \leqslant \varepsilon,$$

since

$$x \leqslant \alpha \leqslant \beta \leqslant y \qquad \text{and} \qquad z \leqslant \alpha \leqslant \beta \leqslant t.$$

Example. If the interval I is *bounded,* and if $\mathbf{f}$ is *bounded* on I, then the integral $\int_{\mathrm{I}} \mathbf{f}(t)\,dt$ always exists, for, by the mean value theorem, one has, for $y \leqslant \alpha \leqslant \beta \leqslant z$,

$$\left\| \int_y^\alpha \mathbf{f}(t)\,dt \right\| \leqslant (\alpha - a) \sup_{x\in \mathrm{I}} \|\mathbf{f}(x)\|\,, \qquad \left\| \int_\beta^z \mathbf{f}(t)\,dt \right\| \leqslant (b - \beta) \sup_{x\in \mathrm{I}} \|\mathbf{f}(x)\|$$

and it suffices to take $\alpha - a$ and $b - \beta$ small enough for the Cauchy criterion be satisfied.

One may note that in this case a primitive of $\mathbf{f}$ on I does not necessarily have a right (resp. left) derivative at the left-hand endpoint (resp. right-hand endpoint) of I (when this number is finite) contrary to the situation when I is compact and $\mathbf{f}$ is regulated on I (*cf.* II, p. 33, exerc. 1).

2. INTEGRALS OF POSITIVE FUNCTIONS OVER A NON-COMPACT INTERVAL

PROPOSITION 2. *Let f be a real regulated function $\geqslant 0$ on an interval* $\mathrm{I} \subset \mathbf{R}$ *with endpoints a and b ($a < b$). For the integral $\int_a^b f(t)\,dt$ to exist it is necessary and sufficient that the set of numbers $\int_{\mathrm{J}} f(t)\,dt$ be bounded above when* J *runs through the set of compact intervals contained in* I*; the integral $\int_a^b f(t)\,dt$ is then the least upper bound of the set of $\int_{\mathrm{J}} f(t)\,dt$.*

Indeed, since $f \geqslant 0$, the relation $\mathrm{J} \subset \mathrm{J}'$ implies that

$$\int_{\mathrm{J}} f\,dt \leqslant \int_{\mathrm{J}'} f\,dt;$$

the map $\mathrm{J} \mapsto \int_{\mathrm{J}} f\,dt$ is thus increasing, and the proposition follows from the monotone limit theorem (*Gen. Top.*, IV, p. 349, th. 2).

When the map $\mathrm{J} \mapsto \int_{\mathrm{J}} f(t)\,dt$ is not bounded it has limit $+\infty$ with respect to the directed set $\mathfrak{K}(\mathrm{I})$; then one says, by abuse of language, that the integral $\int_a^b f(t)\,dt$ is equal to $+\infty$. The properties of integrals established in n° 1 extend (when dealing with functions $\geqslant 0$) to the case where certain of the integrals concerned are infinite, provided that the relations in which they feature make sense.

PROPOSITION 3 (comparison principle). *Let f and g be two real regulated functions on an interval* $\mathrm{I} \subset \mathbf{R}$, *such that $0 \leqslant f(x) \leqslant g(x)$ at each point where f and g are continuous* (*cf.* II, p. 61, prop. 6). *If the integral of g over* I *is convergent, so also is the integral of f, and one has $\int_{\mathrm{I}} f(t)\,dt \leqslant \int_{\mathrm{I}} g(t)\,dt$. Further, the two integrals cannot be equal unless $f(x) = g(x)$ at every point of* I *where f and g are continuous.*

Now for every compact interval $J \subset I$ one has

$$\int_J f(t)\,dt \leqslant \int_J g(t)\,dt;$$

since $\int_J g(t)\,dt \leqslant \int_I g(t)\,dt$, the integrals $\int_J f\,dt$ are bounded above, so the integral $\int_I f(t)\,dt$ is convergent; further, on passing to the limit, one has $\int_I f(t)\,dt \leqslant \int_I g(t)\,dt$. Suppose further that $f(x) < g(x)$ at a point $x \in I$ at which f and g are continuous; there exists a compact interval $[c, d]$ contained in I, not reducing to a (single) point, and such that $x \in [c, d]$; one has $\int_c^d f(t)\,dt < \int_c^d g(t)\,dt$ (II, p. 61, cor. 1), and since on the other hand $\int_a^c f(t)\,dt \leqslant \int_a^c g(t)\,dt$ and $\int_d^b f(t)\,dt \leqslant \int_d^b g(t)\,dt$ from the above, one sees, on adding term-by-term, that $\int_a^b f(t)\,dt < \int_a^b g(t)\,dt$.

This proposition provides the most frequently used means for deciding if the integral of a function $f \geqslant 0$ is or is not convergent: namely, *comparing* f to a simpler function $g \geqslant 0$ whose integral one already knows to be, or not to be, convergent; we shall see in chap. V how to search for comparator functions, in the most usual cases; and we shall deduce everyday criteria for the convergence of integrals and of series.

3. ABSOLUTELY CONVERGENT INTEGRALS

DEFINITION 2. *One says that the integral of a regulated function* $\mathbf{f}$ *over an interval* $I \subset \mathbf{R}$ *is absolutely convergent if the integral of the positive function* $\|\mathbf{f}(x)\|$ *is convergent.*

PROPOSITION 4. *If the integral of* $\mathbf{f}$ *over* I *is absolutely convergent then it is convergent, and one has*

$$\left\| \int_I \mathbf{f}(t)\,dt \right\| \leqslant \int_I \|\mathbf{f}(t)\|\;dt. \tag{2}$$

Indeed, for every compact interval $J \subset I$ one has (II, p. 61, formula (16))

$$\left\| \int_J \mathbf{f}(t)\,dt \right\| \leqslant \int_J \|\mathbf{f}(t)\|\;dt. \tag{3}$$

If the integral of the positive function $\|\mathbf{f}(x)\|$ is convergent, then for every $\varepsilon > 0$ there exists a compact interval $[\alpha, \beta]$ contained in I, such that, for every compact interval $[x, y]$ contained in I and having no interior point in common with $[\alpha, \beta]$, one has $\int_x^y \|\mathbf{f}(t)\,dt\| \leqslant \varepsilon$ (II, p. 65, prop. 1); one deduces that $\left\|\int_x^y \mathbf{f}(t)\,dt\right\| \leqslant \varepsilon$, which shows convergence of the integral over I (II, p. 16, prop. 1); on passing to the limit in (3) one deduces the inequality (2).

COROLLARY. *Let* E, F, G *be three complete normed spaces over* $\mathbf{R}$*, and* $(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}.\mathbf{y}]$ *a continuous bilinear map from* $E \times F$ *into* G. *Let* $\mathbf{f}$, $\mathbf{g}$ *be two regulated functions on* I*, with values in* E *and* F *respectively. If* $\mathbf{f}$ *is bounded on* I *and if the* *integral of* **g** *is absolutely convergent over* I, *then the integral of* $[\mathbf{f}, \mathbf{g}]$ *is absolutely convergent.*

Indeed, there exists a number $h > 0$ such that one has $\|[\mathbf{x}.\mathbf{y}]\| \leqslant h\,\|\mathbf{x}\|\,.\,\|\mathbf{y}\|$ identically (*Gen. Top.*, IX, p. 173, th. 1); if one puts $k = \sup_{x\in \mathrm{I}} \|\mathbf{f}(x)\|$, then one has $\|[\mathbf{f}(x).\mathbf{g}(x)]\| \leqslant hk\,\|\mathbf{g}(x)\|$ on I; the comparison principle now shows that the integral of $[\mathbf{f}.\mathbf{g}]$ is absolutely convergent, and, from (2),

$$\left\|\int_{\mathrm{I}} [\mathbf{f}(t).\mathbf{g}(t)]\,dt\right\| \leqslant hk \int_{\mathrm{I}} \|\mathbf{g}(t)\|\ dt.$$

Remark. An integral can be convergent without being absolutely convergent; this is what is shown by *Example* 3 of II, p. 64, where the series with general term $\mathbf{u}_n$ is convergent without being absolutely convergent.

§ 3. DERIVATIVES AND INTEGRALS OF FUNCTIONS DEPENDING ON A PARAMETER

1. INTEGRAL OF A LIMIT OF FUNCTIONS ON A COMPACT INTERVAL

Th. 1 of II, p. 52, applied to the particular case of regulated functions on a compact interval, translates as follows into the notation appropriate to integrals:

PROPOSITION 1. *Let* A *be a set filtered by a filter* $\mathfrak{F}$, *and* $(\mathbf{f}_\alpha)_{\alpha\in \mathrm{A}}$ *a family of regulated functions on a compact interval* $\mathrm{I} = [a,\ b]$; *if the functions* $\mathbf{f}_\alpha$ *converge uniformly on* I *to a (regulated) function* **f** *with respect to the filter* $\mathfrak{F}$, *then*

$$\lim_{\mathfrak{F}} \int_a^b \mathbf{f}_\alpha(t)\,dt = \int_a^b \mathbf{f}(t)\,dt. \tag{1}$$

Two corollaries to this proposition are important in applications:

COROLLARY 1. *Let* $(\mathbf{f}_n)$ *be a sequence of regulated functions on a compact interval* $\mathrm{I} = [a,\ b]$. *If the sequence* $(\mathbf{f}_n)$ *converges uniformly on* I *to a (regulated) function* **f**, *one has*

$$\lim_{n\to\infty} \int_a^b \mathbf{f}_n(t)\,dt = \int_a^b \mathbf{f}(t)\,dt. \tag{2}$$

In particular, if a *series* whose general term $\mathbf{u}_n$ is a regulated function on I, *converges uniformly* to **f** on I, then the series with general term $\int_a^b \mathbf{u}_n(t)\,dt$ is convergent and its sum is $\int_a^b \mathbf{f}(t)\,dt$ ("term-by-term integration of a uniformly convergent series").

COROLLARY 2. *Let* A *be a subset of a topological space* F*, and* $\mathbf{f}$ *a map from* $\mathrm{I} \times \mathrm{A}$ *into a complete normed space* E *over* $\mathbf{R}$*, such that, for each* $\alpha \in \mathrm{A}$*, the function* $x \mapsto \mathbf{f}(x, \alpha)$ *is regulated on* I*. If the functions* $x \mapsto \mathbf{f}(x, \alpha)$ *converge uniformly on* I *to a (regulated) function* $x \mapsto \mathbf{g}(x)$*, as* α *tends to a point* $\alpha_0 \in \overline{\mathrm{A}}$ *while remaining in* A*, then one has*

$$\lim_{\alpha \to \alpha_0,\ \alpha \in \mathrm{A}} \int_a^b \mathbf{f}(x, \alpha)\, dx = \int_a^b \mathbf{g}(x)\, dx. \tag{3}$$

In particular:

PROPOSITION 2 ("continuity of an integral with respect to a parameter"). *Let* F *be a compact space, let* $\mathrm{I} = [a,\ b]$ *be a compact interval in* $\mathbf{R}$*, and let* $\mathbf{f}$ *be a continuous map of* $\mathrm{I} \times \mathrm{F}$ *into a complete normed space* E *over* $\mathbf{R}$*; then the function* $\mathbf{h}(\alpha) = \int_a^b \mathbf{f}(x, \alpha)\, dx$ *is continuous on* F.

Indeed, since $\mathbf{f}$ is *uniformly continuous* on the compact space $\mathrm{I} \times \mathrm{F}$, the functions $\mathbf{f}(x, \alpha)$ converge uniformly to $\mathbf{f}(x, \alpha_0)$ on I, when α tends to an arbitrary point $\alpha_0 \in \mathrm{F}$.

Here is an application of this proposition: the function $(x, \alpha) \mapsto x^\alpha$ is continuous on the product $\mathrm{I} \times \mathrm{J}$, where $\mathrm{I} = [a,\ b]$ is a compact interval such that $0 < a < b$, and J is any compact interval in $\mathbf{R}$; one concludes that $\int_a^b x^\alpha\, dx$ is a continuous function of α on $\mathbf{R}$; now, for α *rational* and $\neq -1$, this function is equal to $\dfrac{b^{\alpha+1} - a^{\alpha+1}}{\alpha + 1}$, and the function $\alpha \mapsto \dfrac{b^{\alpha+1} - a^{\alpha+1}}{\alpha + 1}$ is continuous on every interval of $\mathbf{R}$ not containing -1; one thus has (extension of identities) $\displaystyle\int_a^b x^\alpha\, dx = \frac{b^{\alpha+1} - a^{\alpha+1}}{\alpha + 1}$ for *all* real $\alpha \neq -1$; this again means that, for all real α, the derivative of x^α is $\alpha x^{\alpha - 1}$ (*cf.* III, p. 94).

2. INTEGRAL OF A LIMIT OF FUNCTIONS ON A NON-COMPACT INTERVAL

Th. 1 of II, p. 52 applies to functions more general than regulated functions, since there one merely assumes that the functions admit primitives. In particular one sees that prop. 1 of II, p. 68 still applies when, on an interval $\mathrm{I} \subset \mathbf{R}$, the functions $\mathbf{f}_\alpha$ are only assumed to be *piecewise regulated* and to admit an *integral* over I; however this result presupposes that the other two hypotheses of prop. 1 are satisfied, namely: 1° I is a *bounded* interval; 2° the $\mathbf{f}_\alpha$ converge *uniformly on* I to $\mathbf{f}$. Formula (1) of II, p. 68 *may fail* when one of these conditions is no longer satisfied: it can happen that one or the other of these two terms does not exist, or that both exist but have different values.

For example, if f_n is the regulated function on $]0,\ 1]$, defined by $f_n(x) = n$ for $0 < x < 1/n$ and $f_n(x) = 0$ for $1/n \leqslant x \leqslant 1$, then the sequence (f_n) converges to 0 *uniformly on every compact interval* contained in $]0,\ 1]$, but not uniformly on $[0,\ 1]$, and one has $\int_0^1 f_n(t)\,dt = 1$ for each n. One has an example where $\int_0^1 f_n(t)\,dt$ does not tend to any limit on replacing the preceding sequence (f_n) by the sequence $((-1)^n f_n)$ which again converges uniformly to 0 on every compact interval contained in $]0,\ 1]$.

On the other hand, on the *unbounded* interval $\mathrm{I} = [0,\ +\infty[$, let f_n be the regulated function such that $f_n(x) = 1/n$ for $n^2 \leqslant x \leqslant (n+1)^2$ and $f_n(x) = 0$ for every other value of x in I ($n \geqslant 1$); the sequence (f_n) converges uniformly to 0 on I, but the integral $\int_0^{+\infty} f_n(t)\,dt = (2n+1)/n$ tends to 2 as n increases indefinitely.

In other words, when I is not bounded, if one denotes by $\mathcal{I}$ the vector space formed by the regulated functions $\mathbf{f}$ on I, with values in E, and admitting an integral over I, then the map $\mathbf{f} \mapsto \int_{\mathrm{I}} \mathbf{f}(t)\,dt$ *is not continuous* when one endows $\mathcal{I}$ with the topology of uniform convergence on I (*cf.* II, p. 53, cor. 2).

We shall seek *sufficient* conditions to assure the validity of prop. 1, under the following hypotheses:

1° I is an arbitrary interval in $\mathbf{R}$, the function $\mathbf{f}_\alpha$ is regulated on I, and admits an integral over I;

2° the family $(\mathbf{f}_\alpha)$ converges uniformly to $\mathbf{f}$ with respect to the filter $\mathfrak{F}$ on every compact interval contained in I.

Writing $\mathfrak{K}(\mathrm{I})$ for the directed set of compact intervals contained in I (II, p. 64), the left-hand side of formula (1) of II, p. 68 can be written as $\lim\limits_{\mathfrak{F}} \left(\lim\limits_{\mathrm{J}\in\mathfrak{K}(\mathrm{I})} \int_{\mathrm{J}} \mathbf{f}_\alpha(t)\,dt \right)$; on the other hand, taking account of prop. 1 (II, p. 68), and also the fact that the family $(\mathbf{f}_\alpha)$ is uniformly convergent on every compact interval $\mathrm{J} \subset \mathrm{I}$, the right-hand side of (1) (II, p. 68) can be written as $\lim\limits_{\mathrm{J}\in\mathfrak{K}(\mathrm{I})} \left(\lim\limits_{\mathfrak{F}} \int_{\mathrm{J}} \mathbf{f}_\alpha(t)\,dt \right)$. One thus sees that prop. 1 of II, p. 19 extends when one can *interchange the limits* of the map $(\mathrm{J},\ \alpha) \mapsto \int_{\mathrm{J}} \mathbf{f}_\alpha(t)\,dt$ with respect to the filter $\mathfrak{F}$ and with respect to the filter Φ of sections of the directed set $\mathfrak{K}(\mathrm{I})$. Now, we know a *sufficient* condition for this interchange to be justified, namely the existence of the limit of the map $(\mathrm{J},\ \alpha) \mapsto \int_{\mathrm{J}} \mathbf{f}_\alpha(t)\,dt$ with respect to the *product filter* $\Phi \times \mathfrak{F}$ (*Gen. Top.*, I, p. 81, cor. to th. 1). We shall transform this condition into an equivalent, more manageable, condition.

In the first place, since E is complete, in order that $(\mathrm{J},\ \alpha) \mapsto \int_{\mathrm{J}} \mathbf{f}_\alpha(t)\,dt$ should have a limit with respect to $\Phi \times \mathfrak{F}$ it is necessary and sufficient that, for every $\varepsilon > 0$, there should exist a compact interval $\mathrm{J}_0 \subset \mathrm{I}$ and a set $\mathrm{M} \in \mathfrak{F}$ such that, for any elements $\alpha,\ \beta$ of M and compact interval $\mathrm{J} \supset \mathrm{J}_0$ contained in I, one has

$$\left\| \int_{\mathrm{J}_0} \mathbf{f}_\alpha(t)\,dt - \int_{\mathrm{J}} \mathbf{f}_\beta(t)\,dt \right\| \leqslant \varepsilon. \tag{4}$$

We shall show on the other hand that this condition is itself equivalent to the following condition: for every $\varepsilon > 0$ there exists a compact interval $\mathrm{J}_0 \subset \mathrm{I}$ and a set $\mathrm{M} \in \mathfrak{F}$ such that, for any α of M and any compact interval $\mathrm{J} \supset \mathrm{J}_0$ contained in I, one has

$$\left\| \int_{\mathrm{J}_0} \mathbf{f}_\alpha(t)\,dt - \int_{\mathrm{J}} \mathbf{f}_\alpha(t)\,dt \right\| \leqslant \varepsilon. \tag{5}$$

It is indeed clear that this last condition is necessary; conversely, if it is satisfied, there exists (by the uniform convergence of $(\mathbf{f}_\alpha)$ on every compact interval) a set $\mathrm{N} \in \mathfrak{F}$ such that, for any α, β in N one has

$$\left\|\int_{\mathrm{J}_0} \mathbf{f}_\alpha(t)\,dt - \int_{\mathrm{J}_0} \mathbf{f}_\beta(t)\,dt\right\| \leqslant \varepsilon; \tag{6}$$

and therefore $\left\|\int_{\mathrm{J}_0} \mathbf{f}_\alpha(t)\,dt - \int_{\mathrm{J}} \mathbf{f}_\beta(t)\,dt\right\| \leqslant 2\varepsilon$ for any α and β in $\mathrm{M} \cap \mathrm{N} \in \mathfrak{F}$ and for any compact interval $\mathrm{J} \supset \mathrm{J}_0$.

Finally, the lemma of II, p. 65 allows us to put this last condition in the following equivalent form: *for every $\varepsilon > 0$ there exist a compact interval $\mathrm{J}_0 \subset \mathrm{I}$ and a set $\mathrm{M} \in \mathfrak{F}$ (depending on ε) such that, for every compact interval $\mathrm{K} \subset \mathrm{I}$ having no interior point in common with J_0, and every $\alpha \in \mathrm{M}$, one has $\left\|\int_{\mathrm{K}} \mathbf{f}_\alpha(t)\,dt\right\| \leqslant \varepsilon$.*

Most often, one uses a more restrictive condition obtained by supposing that, in the last statement, the set M *does not depend on ε*:

DEFINITION 1. *One says that the integral $\int_{\mathrm{I}} \mathbf{f}_\alpha(t)\,dt$ is uniformly convergent for $\alpha \in \mathrm{A}$ (or uniformly convergent over A) if, for every $\varepsilon > 0$ there exists a compact interval $\mathrm{J}_0 \subset J$ such that, for every compact interval $\mathrm{K} \subset \mathrm{I}$ with no interior point in common with J_0, and every $\alpha \in \mathrm{A}$, one has*

$$\left\|\int_{K} \mathbf{f}_\alpha(t)\,dt\right\| \leqslant \varepsilon. \tag{7}$$

This definition is equivalent to saying that the family of maps $\alpha \mapsto \int_{\mathrm{J}} \mathbf{f}_\alpha(t)\,dt$ is *uniformly convergent on* A (towards the map $\alpha \mapsto \int_{\mathrm{I}} \mathbf{f}_\alpha(t)\,dt$) with respect to the filter of sections Φ of $\mathfrak{K}(\mathrm{I})$; each of the integrals $\int_{\mathrm{I}} \mathbf{f}_\alpha(t)\,dt$ is *a fortiori* convergent (the converse being false); Further, from what we have just seen (or from *Gen. Top.*, X, p. 281, cor. 2):

PROPOSITION 3. *Let $(\mathbf{f}_\alpha)$ be a family of regulated functions on an interval* I *such that:* 1° *with respect to the filter $\mathfrak{F}$ the family $(\mathbf{f}_\alpha)$ converges uniformly to a function* $\mathbf{f}$ (regulated on I) *on every compact interval contained in* I; 2° *the integral $\int_{\mathrm{I}} \mathbf{f}_\alpha(t)\,dt$ is uniformly convergent for every $\alpha \in \mathrm{A}$. Under these hypotheses the integral $\int_{\mathrm{I}} \mathbf{f}(t)\,dt$ is convergent, and one has*

$$\lim_{\mathfrak{F}} \int_{\mathrm{I}} \mathbf{f}_\alpha(t)\,dt = \int_{\mathrm{I}} \mathbf{f}(t)\,dt. \tag{8}$$

The hypotheses of prop. 3 are fulfilled when for example I is a *bounded* interval, the $\mathbf{f}_\alpha$ are *uniformly bounded* on I, and converge uniformly to $\mathbf{f}$ on every compact interval contained in I; indeed, if $\|\mathbf{f}_\alpha(x)\| \leqslant h$ for all $x \in \mathrm{I}$ and all α, and if J_0 is such that the difference between the lengths of I and J_0 is $\leqslant \varepsilon/h$, then condition (7) is satisfied for every interval $\mathrm{K} \subset \mathrm{I}$ with no interior point in common with J_0.

As with prop. 1 of II, p. 68, two corollaries to prop. 3 are important in applications:

COROLLARY 1. *Let* $(\mathbf{f}_n)$ *be a sequence of regulated functions on an arbitrary interval* I, *converging uniformly to a function* $\mathbf{f}$ *on each compact interval contained in* I; *if the integral* $\int_{\mathrm{I}} \mathbf{f}_n(t)\,dt$ *is uniformly convergent, then the integral* $\int_{\mathrm{I}} \mathbf{f}(t)\,dt$ *is convergent, and*

$$\lim_{n\to\infty} \int_{\mathrm{I}} \mathbf{f}_n(t)\,dt = \int_{\mathrm{I}} \mathbf{f}(t)\,dt. \tag{9}$$

Remark. The hypotheses imposed in this corollary are sufficient, but not necessary, for the validity of formula (9); we shall generalize this formula later, at the same time as the concept of integral (see INT, IV), and obtain much less restrictive conditions.

COROLLARY 2. *Let* A *be a subset of a topological space* F, *and* $\mathbf{f}$ *a map of* $\mathrm{I}\times\mathrm{A}$ *into a complete normed space* E *over* $\mathbf{R}$, *such that, for every* $\alpha \in \mathrm{A}$, *the function* $x \mapsto \mathbf{f}(x,\alpha)$ *is regulated on* I. *If, on the one hand, the functions* $x \mapsto \mathbf{f}(x,\alpha)$ *converge uniformly on every compact interval contained in* I *to a function* $x \mapsto \mathbf{f}(x)$ *as* α *tends to* $\alpha_0 \in \overline{\mathrm{A}}$ *while remaining in* A; *if, on the other hand, the integral* $\int_{\mathrm{I}} \mathbf{f}(x,\alpha)\,dx$ *is uniformly convergent on* A, *then the integral* $\int_{\mathrm{I}} \mathbf{f}(x)\,dx$ *is convergent, and one has*

$$\lim_{\alpha\to\alpha_0,\ \alpha\in\mathrm{A}} \int_{\mathrm{I}} \mathbf{f}(x,\alpha)\,dx = \int_{\mathrm{I}} \mathbf{f}(x)\,dx. \tag{10}$$

In particular:

PROPOSITION 4 ("continuity of an improper integral with respect to a parameter"). *Let* F *be a compact space, let* I *be any interval in* $\mathbf{R}$, *and* $\mathbf{f}$ *a continuous map from* $\mathrm{I}\times\mathrm{F}$ *into a complete normed space* E *over* $\mathbf{R}$; *if the integral* $\mathbf{h}(\alpha) = \int_{\mathrm{I}} \mathbf{f}(x,\alpha)\,dx$ *is uniformly convergent on* F, *it is a continuous function of* α *on* F.

In view of prop. 2 of II, p. 69, this proposition also follows from the continuity of a uniform limit of continuous functions (*Gen. Top.*, X, p. 282, th. 2).

3. NORMALLY CONVERGENT INTEGRALS

Let $(\mathbf{f}_\alpha)_{\alpha\in\mathrm{A}}$ be a family of regulated functions on an arbitrary interval $\mathrm{I} \subset \mathbf{R}$, with values in a complete normed space E over $\mathbf{R}$. Suppose that there exists a finite real regulated function g on I such that, for every $x \in \mathrm{I}$ and every $\alpha \in \mathrm{A}$, $\|\mathbf{f}_\alpha(x)\| \leqslant g(x)$ and also the integral $\int_{\mathrm{I}} g(t)\,dt$ is convergent. Under these conditions the integral $\int_{\mathrm{I}} \mathbf{f}_\alpha(t)\,dt$ is *absolutely and uniformly convergent* on A; in fact, for every compact interval K contained in I,

$$\left\|\int_{\mathrm{K}} \mathbf{f}_\alpha(t)\,dt\right\| \leqslant \int_{\mathrm{K}} g(t)\,dt$$

and the convergence of the integral $\int_{\mathrm{I}} g(t)\,dt$ implies that for every $\varepsilon > 0$ there exists a compact interval $\mathrm{J} \subset \mathrm{I}$ such that for every compact interval $\mathrm{K} \subset \mathrm{I}$ disjoint from J one has $\int_{\mathrm{K}} g(t)\,dt \leqslant \varepsilon$. When there exists a real function g having the preceding properties one says that the integral $\int_{\mathrm{I}} \mathbf{f}_\alpha(t)\,dt$ is *normally convergent* on A (*cf. Gen. Top.*, X, p. 296).

An integral can be uniformly convergent on A without being normally convergent. *This happens for the sequence (f_n) of real functions defined by the conditions $f_n(x) = 1/x$ for $n \leqslant x \leqslant n+1$, and $f_n(x) = 0$ for the other values of x in $\mathrm{I} = [0, +\infty[$. It is immediate that the integral $\int_{\mathrm{I}}^{\infty} f_n(t)\,dt$ is uniformly convergent, but not normally convergent, since the relation $g(x) \geqslant f_n(x)$ for each $x \in \mathrm{I}$ and all n entails that $g(x) \geqslant 1/x$, and consequently that the integral of g over I is not convergent.*

In particular, let us consider a *series* whose general term $\mathbf{u}_n$ is a regulated function on an interval I, and suppose that the series with general term $\|\mathbf{u}_n(x)\|$ (which is a regulated function on I) converges uniformly on every compact interval contained in I, and such that the series with general term $\int_{\mathrm{I}} \|\mathbf{u}_n(t)\|\,dt$ is convergent; then (II, p. 66, prop. 2) the (regulated) function $g(x)$, the sum of the series with general term $\|\mathbf{u}_n(x)\|$, is such that the integral $\int_{\mathrm{I}} g(t)\,dt$ is convergent. If one puts $\mathbf{f}_n = \sum_{p=1}^{n} \mathbf{u}_p$, then the integral $\int_{\mathrm{I}} \mathbf{f}_n(t)\,dt$ is *normally convergent,* for one has

$$\|\mathbf{f}_n(x)\| \leqslant \sum_{p=1}^{n} \left\|\mathbf{u}_p(x)\right\| \leqslant g(x)$$

for all $x \in \mathrm{I}$ and all n; in consequence, the sum $\mathbf{f}$ of the series with general term $\mathbf{u}_n$ is a regulated function on I such that the integral $\int_{\mathrm{I}} \mathbf{f}(t)\,dt$ is convergent, and one has

$$\int_{\mathrm{I}} \mathbf{f}(t)\,dt = \sum_{n=1}^{\infty} \int_{\mathrm{I}} \mathbf{u}_n(t)\,dt \tag{11}$$

("term-by-term integration of a series on a non-compact interval").

4. DERIVATIVE WITH RESPECT TO A PARAMETER OF AN INTEGRAL OVER A COMPACT INTERVAL

Let A be a compact neighbourhood of a point α_0 in the field $\mathbf{R}$ (resp. the field $\mathbf{C}$), let $\mathrm{I} = [a,\ b]$ be a *compact* interval in $\mathbf{R}$, and $\mathbf{f}$ a *continuous* map of $\mathrm{I} \times \mathrm{A}$ into a complete normed space E over $\mathbf{R}$ (resp. $\mathbf{C}$). We have seen (II, p. 69, prop. 2) that under these conditions $\mathbf{g}(\alpha) = \int_a^b \mathbf{f}(t, \alpha)\,dt$ is a *continuous* function on A. Let us seek *sufficient* conditions for $\mathbf{g}$ to admit a *derivative* at the point α_0. One has, for $\alpha \neq \alpha_0$,

$$\frac{\mathbf{g}(\alpha) - \mathbf{g}(\alpha_0)}{\alpha - \alpha_0} = \int_a^b \frac{\mathbf{f}(t, \alpha) - \mathbf{f}(t, \alpha_0)}{\alpha - \alpha_0}\,dt$$

so (II, p. 69, cor. 2), if the functions $x \mapsto \dfrac{\mathbf{f}(x,\alpha)-\mathbf{f}(x,\alpha_0)}{\alpha-\alpha_0}$ *converge uniformly on* I to a (necessarily continuous) function $x \mapsto \mathbf{h}(x)$ as α tends to α_0 (while remaining $\neq \alpha_0$), then $\mathbf{g}$ admits a derivative equal to $\int_a^b \mathbf{h}(t)\,dt$ at the point α_0; moreover, for each $x \in \mathrm{I}$, $\dfrac{\mathbf{f}(x,\alpha)-\mathbf{f}(x,\alpha_0)}{\alpha-\alpha_0}$ tends to $\mathbf{h}(x)$, so $\mathbf{h}(x)$ is the derivative at the point α_0 of the map $\alpha \mapsto \mathbf{f}(x,\alpha)$; we denote this derivative (called the *partial derivative of* $\mathbf{f}$ *with respect to* α) by the notation $\mathbf{f}'_\alpha(x,\alpha_0)$; the hypotheses we have made imply that

$$\mathbf{g}'(\alpha_0) = \int_a^b \mathbf{f}'_\alpha(t,\alpha_0)\,dt. \tag{12}$$

The following proposition gives a very simple sufficient condition for the validity of formula (12):

PROPOSITION 5. *Suppose that the partial derivative* $\mathbf{f}'_\alpha(x,\alpha)$ *exists for all* $x \in \mathrm{I}$ *and all* α *in an open neighbourhood* V *of* α_0, *and that, for all* $\alpha \in \mathrm{V}$, *the map* $x \mapsto \mathbf{f}'_\alpha(x,\alpha)$ *is regulated on* I. *Under these conditions, if* $x \mapsto \mathbf{f}'_\alpha(x,\alpha)$ *converges uniformly on* I *to* $x \mapsto \mathbf{f}'_\alpha(x,\alpha_0)$ *as* α *tends to* α_0, *then the function* $\mathbf{g}(\alpha) = \int_a^b \mathbf{f}(t,\alpha)\,dt$ *admits a derivative, given by the formula* (12), *at the point* α_0.

Indeed, for every $\varepsilon > 0$ there exists by hypothesis an $r > 0$ such that $|\alpha-\alpha_0| \leqslant r$ implies $\left\|\mathbf{f}'_\alpha(x,\alpha)-\mathbf{f}'_\alpha(x,\alpha_0)\right\| \leqslant \varepsilon$ *for any* $x \in \mathrm{I}$. By props. 3 and 5 of I, p. 17 one has, for $|\alpha-\alpha_0| \leqslant r$ ($\alpha \neq \alpha_0$) and for all $x \in \mathrm{I}$

$$\left\|\frac{\mathbf{f}(x,\alpha)-\mathbf{f}(x,\alpha_0)}{\alpha-\alpha_0} - \mathbf{f}'_\alpha(x,\alpha_0)\right\| \leqslant \varepsilon$$

which proves the uniform convergence of $\dfrac{\mathbf{f}(x,\alpha)-\mathbf{f}(x,\alpha_0)}{\alpha-\alpha_0}$ to $\mathbf{f}'_\alpha(x,\alpha_0)$ on I as α tends to α_0 (remaining $\neq \alpha_0$), and so establishes formula (12).

COROLLARY. *If the partial derivative* $\mathbf{f}'_\alpha(x,\alpha)$ *exists on* $\mathrm{I}\times\mathrm{V}$ *and is a continuous function of* (x,α) *on this set, then the function* $\mathbf{g}$ *admits a derivative given by the formula* (12) *at the point* α_0.

Indeed, if W is a compact neighbourhood of α_0 contained in V, then the map $(x,\alpha) \mapsto \mathbf{f}'_\alpha(x,\alpha)$ is *uniformly continuous* on the compact set $\mathrm{I}\times\mathrm{W}$, so $\mathbf{f}'_\alpha(x,\alpha)$ tends to $\mathbf{f}'_\alpha(x,\alpha_0)$ uniformly on I as α tends to α_0.

From prop. 5 one deduces a more general proposition which allows one to evaluate the derivative of an integral when, not only the integrand $\mathbf{f}$, but also the limits of integration, depend on a parameter α:

PROPOSITION 6. *Supposing that the hypotheses of prop. 5 are satisfied, let* $a(\alpha)$, $b(\alpha)$ *be two functions defined on* V, *with values in* I*; if the derivatives*

$a'(\alpha_0)$, $b'(\alpha_0)$ exist and are finite then the function $\mathbf{g}(\alpha) = \int_{a(\alpha)}^{b(\alpha)} \mathbf{f}(t,\alpha)\,dt$ admits at α_0 a derivative given by the formula

$$\mathbf{g}'(\alpha_0) = \int_{a(\alpha_0)}^{b(\alpha_0)} \mathbf{f}'_\alpha(t,\alpha_0)\,dt + b'(\alpha_0)\mathbf{f}(b(\alpha_0),\alpha_0) - a'(\alpha_0)\mathbf{f}(a(\alpha_0),\alpha_0). \tag{13}$$

Indeed, for all $\alpha \in \mathrm{V}$ distinct from α_0 one can write

$$\frac{\mathbf{g}(\alpha)-\mathbf{g}(\alpha_0)}{\alpha-\alpha_0} = \int_{a(\alpha_0)}^{b(\alpha_0)} \frac{\mathbf{f}(t,\alpha)-\mathbf{f}(t,\alpha_0)}{\alpha-\alpha_0}\,dt + \frac{1}{\alpha-\alpha_0}\int_{b(\alpha_0)}^{b(\alpha)} \mathbf{f}(t,\alpha)\,dt - \frac{1}{\alpha-\alpha_0}\int_{a(\alpha_0)}^{a(\alpha)} \mathbf{f}(t,\alpha)\,dt.$$

By prop. 5 of II, p. 74, the first integral on the right-hand side tends to $\int_{a(\alpha_0)}^{b(\alpha_0)} \mathbf{f}'_\alpha(t,\alpha_0)\,dt$ as α tends to α_0. In the second integral we replace $\mathbf{f}(t,\alpha)$ by $\mathbf{f}(b(\alpha_0),\alpha_0)$ and show that the difference tends to 0. We put $\mathrm{M} = \mathrm{Max}\big(\|\mathbf{f}(b(\alpha_0),\alpha_0)\|, |b'(\alpha_0)|+1\big)$; the function $b(\alpha)$ being continuous at the point α_0 and the function $\mathbf{f}$ continuous at the point $(b(\alpha_0),\alpha_0)$, for every ε such that $0<\varepsilon<1$ there exists an $r>0$ such that the relation $|\alpha-\alpha_0| \leqslant r$ entails that $\|\mathbf{f}(t,\alpha)-\mathbf{f}(b(\alpha_0),\alpha_0)\| \leqslant \varepsilon$ for all t belonging to the interval with endpoints $b(\alpha_0)$ and $b(\alpha)$; thus one may also suppose that the relation $|\alpha-\alpha_0| \leqslant r$ entails $\left|\dfrac{b(\alpha)-b(\alpha_0)}{\alpha-\alpha_0} - b'(\alpha_0)\right| \leqslant \varepsilon$.

By the mean value formula (II, p. 62, formula (17)) one thus has

$$\left\|\frac{1}{\alpha-\alpha_0}\int_{b(\alpha_0)}^{b(\alpha)} \mathbf{f}(t,\alpha)\,dt - \frac{b(\alpha)-b(\alpha_0)}{\alpha-\alpha_0}\mathbf{f}(b(\alpha_0),\alpha_0)\right\| \leqslant \left|\frac{b(\alpha)-b(\alpha_0)}{\alpha-\alpha_0}\right|\varepsilon$$

and consequently

$$\left\|\frac{1}{\alpha-\alpha_0}\int_{b(\alpha_0)}^{b(\alpha)} \mathbf{f}(t,\alpha)\,dt - b'(\alpha_0)\,\mathbf{f}(b(\alpha_0),\alpha_0)\right\| \leqslant 2\mathrm{M}\varepsilon$$

which shows that $\dfrac{1}{\alpha-\alpha_0}\int_{b(\alpha_0)}^{b(\alpha)} \mathbf{f}(t,\alpha)\,dt$ tends to $b'(\alpha_0)\mathbf{f}(b(\alpha_0),\alpha_0)$. In the same way one shows that $\dfrac{1}{\alpha-\alpha_0}\int_{a(\alpha_0)}^{a(\alpha)} \mathbf{f}(t,\alpha)\,dt$ tends to $a'(\alpha_0)\mathbf{f}(a(\alpha_0),\alpha_0)$.

5. DERIVATIVE WITH RESPECT TO A PARAMETER OF AN INTEGRAL OVER A NON-COMPACT INTERVAL

The set V having the same meaning as in prop. 5 of II, p. 74, suppose now that I is *any* interval in $\mathbf{R}$, and that $\mathbf{f}$ is a *continuous* map from $\mathrm{I}\times\mathrm{V}$ into E; if the integral $\mathbf{g}(\alpha) = \int_\mathrm{I} \mathbf{f}(t,\alpha)\,dt$ exists for all $\alpha \in \mathrm{V}$ and is a continuous function of α, the function $\mathbf{g}$ *need not have a derivative equal to* $\int_\mathrm{I} \mathbf{f}'_\alpha(t,\alpha_0)\,dt$ *at the point* α_0, even if $\mathbf{f}'_\alpha(x,\alpha)$

converges uniformly to $\mathbf{f}'_\alpha(x, \alpha_0)$ on every compact interval contained in I, and if the integral $\int_{\mathrm{I}} \mathbf{f}'_\alpha(t, \alpha)\,dt$ exists for all $\alpha \in \mathrm{V}$ (*cf.* II, p. 87, exerc. 3).

A sufficient condition for formula (12) (II, p. 74) to remain valid is given by the following proposition:

PROPOSITION 7. *Let* I *be an arbitrary interval in* $\mathbf{R}$, *and* $\mathbf{f}$ *a continuous function on* $\mathrm{I} \times \mathrm{V}$. *Suppose that:*

1° *the partial derivative* $\mathbf{f}'_\alpha(x, \alpha)$ *exists for all* $x \in \mathrm{I}$ *and all* $\alpha \in \mathrm{V}$, *and, for all* $\alpha \in \mathrm{V}$, *the map* $x \mapsto \mathbf{f}'_\alpha(x, \alpha)$ *is regulated on* I*;*

2° *for all* $\alpha \in \mathrm{V}$, $\mathbf{f}'_\alpha(x, \beta)$ *converges uniformly to* $\mathbf{f}'_\alpha(x, \alpha)$ *on every compact interval contained in* I, *as* β *tends to* α;

3° *the integral* $\int_{\mathrm{I}} \mathbf{f}'_\alpha(t, \alpha)\,dt$ *is uniformly convergent on* V*;*

4° *the integral* $\int_{\mathrm{I}} \mathbf{f}(t, \alpha_0)\,dt$ *is convergent .*

In these circumstances the integral $\mathbf{g}(\alpha) = \int_{\mathrm{I}} \mathbf{f}(t, \alpha)\,dt$ *is uniformly convergent on* V, *and the function* $\mathbf{g}$ *admits at every point of* V *a derivative given by the formula*

$$\mathbf{g}'(\alpha) = \int_{\mathrm{I}} \mathbf{f}'_\alpha(t, \alpha)\,dt. \tag{14}$$

The uniform convergence of $\int_{\mathrm{I}} \mathbf{f}'_\alpha(t, \alpha)\,dt$ on V means that the function $\alpha \mapsto \int_{\mathrm{J}} \mathbf{f}'_\alpha(t, \alpha)\,dt$ converges uniformly on V with respect to the filter of sections Φ of the directed set $\mathfrak{K}(\mathrm{I})$ of compact intervals J contained in I. Let us put $\mathbf{u}_{\mathrm{J}}(\alpha) = \int_{\mathrm{J}} \mathbf{f}(t, \alpha)\,dt$; the hypotheses show that on the one hand $\mathbf{u}_{\mathrm{J}}(\alpha_0)$ has a limit with respect to Φ, and on the other hand, by virtue of prop. 5 of II, p. 74, that $\mathbf{u}'_{\mathrm{J}}(\alpha) = \int_{\mathrm{J}} \mathbf{f}'_\alpha(t, \alpha)\,dt$ for all $\alpha \in \mathrm{V}$. We can therefore apply th. 1 of II, p. 52 to the functions $\mathbf{u}_{\mathrm{J}}$, the rôle of the set of indices being taken here by $\mathfrak{K}(\mathrm{I})$, and that of the filter on this set by the filter Φ; the proposition follows immediately.

Remarks. 1) Conditions 1° and 2° of prop. 7 are satisfied *a fortiori* when $\mathbf{f}'_\alpha(x, \alpha)$ is a *continuous* function of (x, α) on $\mathrm{I} \times \mathrm{V}$.

2) When, in an integral $\int_{a(\alpha)}^{b(\alpha)} \mathbf{f}(t, \alpha)\,dt$, the endpoints of the interval are *finite* functions of the parameter, the study of this integral as a function of α can be related to that of an integral over $[0,\ 1]$; indeed, by the change of variable $t = a(\alpha)(1-u) + b(\alpha)u$, one has

$$\int_{a(\alpha)}^{b(\alpha)} \mathbf{f}(t, \alpha)\,dt = \int_0^1 \mathbf{f}(a(\alpha)(1-u) + b(\alpha)u,\ \alpha)\,(b(\alpha) - a(\alpha))\,du.$$

6. CHANGE OF ORDER OF INTEGRATION

Let $\mathrm{I} = [a,\ b]$ and $\mathrm{A} = [c,\ d]$ be two *compact* intervals in $\mathbf{R}$; let $\mathbf{f}$ be a *continuous* function on $\mathrm{I} \times \mathrm{A}$ with values in a complete normed space E over $\mathbf{R}$; by prop. 2 of II, p. 69, $\int_a^b \mathbf{f}(x, \alpha)\,dx$ is a continuous function of α on A; its integral $\int_c^d \left(\int_a^b \mathbf{f}(x, \alpha)\,dx\right) d\alpha$ is also denoted, for simplicity, by $\int_c^d d\alpha \int_a^b \mathbf{f}(x, \alpha)\,dx$.

PROPOSITION 8. *If* $\mathbf{f}$ *is continuous on* $\mathrm{I} \times \mathrm{A}$ *one has*

$$\int_c^d d\alpha \int_a^b \mathbf{f}(x, \alpha)\, dx = \int_a^b dx \int_c^d \mathbf{f}(x, \alpha)\, d\alpha \tag{15}$$

("formula for interchanging the order of integration").

We shall show that, for all $y \in \mathrm{A}$, one has

$$\int_c^y d\alpha \int_a^b \mathbf{f}(x, \alpha)\, dx = \int_a^b dx \int_c^y \mathbf{f}(x, \alpha)\, d\alpha. \tag{16}$$

Since the two sides of (16) are functions of y, and equal for $y = c$, it will suffice to prove that they are differentiable on $]c, d[$ and that their derivatives are equal at every point of this interval. If one puts $\mathbf{g}(\alpha) = \int_a^b \mathbf{f}(x, \alpha)\, dx$, and $\mathbf{h}(x, y) = \int_c^y \mathbf{f}(x, \alpha)\, dx$, the relation (16) can be written

$$\int_c^y \mathbf{g}(\alpha)\, d\alpha = \int_a^b \mathbf{h}(x, y)\, dx.$$

Now, the derivative of the first term with respect to y is $\mathbf{g}(y)$, while that of the second is $\int_a^b \mathbf{h}'_y(x, y)\, dx$, by II, p. 74, corollary, since $\mathbf{h}'_y(x, y) = \mathbf{f}(x, y)$ is continuous on $\mathrm{I} \times \mathrm{A}$; the two expressions thus obtained are identical.

Suppose now that $\mathrm{A} = [c, d]$ is a *compact* interval in $\mathbf{R}$, and I an *arbitrary* interval in $\mathbf{R}$; let $\mathbf{f}$ be a continuous function on $\mathrm{I} \times \mathrm{A}$, with values in E, such that the integral $\mathbf{g}(\alpha) = \int_\mathrm{I} \mathbf{f}(t, \alpha)\, dt$ is convergent for all $\alpha \in \mathrm{A}$; even if $\mathbf{g}(\alpha)$ is continuous on A one cannot always interchange the order of integration in the integral $\int_c^d d\alpha \int_\mathrm{I} \mathbf{f}(t, \alpha)\, dt$, for the integral $\int_\mathrm{I} dt \int_c^d \mathbf{f}(t, \alpha)\, d\alpha$ may not exist, or it may be different from the integral $\int_c^d d\alpha \int_\mathrm{I} \mathbf{f}(t, \alpha)\, dt$ (*cf.* II, p. 87, exerc. 7). One has, however, the following result:

PROPOSITION 9. *If the function* $\mathbf{f}$ *is continuous on* $\mathrm{I} \times \mathrm{A}$, *and if the integral* $\int_\mathrm{I} \mathbf{f}(t, \alpha)\, dt$ *is uniformly convergent on* A, *then the integral* $\int_\mathrm{I} dt \int_c^d \mathbf{f}(t, \alpha)\, d\alpha$ *is convergent, and one has*

$$\int_c^d d\alpha \int_\mathrm{I} \mathbf{f}(t, \alpha)\, dt = \int_\mathrm{I} dt \int_c^d \mathbf{f}(t, \alpha)\, d\alpha. \tag{17}$$

For every compact interval J contained in I, put $\mathbf{u}_\mathrm{J}(\alpha) = \int_\mathrm{J} \mathbf{f}(t, \alpha)\, dt$. The hypothesis entails that with respect to the filter of sections Φ of the directed set $\mathfrak{K}(\mathrm{I})$ the continuous function $\mathbf{u}_\mathrm{J}$ converges uniformly on A to $\int_\mathrm{I} \mathbf{f}(t, \alpha)\, dt$; thus (II, p. 68, prop. 1), $\int_c^d d\alpha \int_\mathrm{J} \mathbf{f}(t, \alpha)\, dt$ has limit $\int_c^d d\alpha \int_\mathrm{I} \mathbf{f}(t, \alpha)\, dt$ with respect to Φ; but, by prop. 8 (II, p. 77), one has

$$\int_c^d d\alpha \int_\mathrm{J} \mathbf{f}(t, \alpha)\, dt = \int_\mathrm{J} dt \int_c^d \mathbf{f}(t, \alpha)\, d\alpha. \tag{18}$$

The preceding result thus means that the integral $\int_{\mathrm{I}} dt \int_c^d \mathbf{f}(t, \alpha)\, d\alpha$ is convergent, and on passing to the limit with respect to Φ in the relation (18), one obtains (17).

EXERCISES

§ 1.

1) Let (f_α) be a set of real functions, defined on an interval $\mathrm{I} \subset \mathbf{R}$, each admitting a strict primitive on I, and forming a directed set for the relation $\leqslant$. Let f be the upper envelope of the family (f_α); suppose that f admits a strict primitive on I. Show that if g_α (resp. g) is the primitive of f_α (resp. f) which vanishes at a point $x_0 \in \mathrm{I}$, then g is the upper envelope of the family (g_α) on the intersection $\mathrm{I} \cap [x_0, +\infty[$ and its lower envelope on the intersection $\mathrm{I} \cap]-\infty, x_0]$. (Restricting to the first of these intervals, show that if u is the upper envelope of (g_α) on $\mathrm{I} \cap [x_0, +\infty[$ then one has $u(x+h) - u(x) \leqslant g(x+h) - g(x)$ for $h > 0$; then prove that, for all α, one has $u(x+h) - u(x) \geqslant g_\alpha(x+h) - g_\alpha(x)$, and conclude from this last inequality that $\liminf\limits_{h \to 0} (u(x+h) - u(x))/h \geqslant f(x)$; deduce the proposition from this.)

Give an example of an increasing sequence (f_n) of continuous functions on an interval I which are uniformly bounded, but whose upper envelope does not admit a strict primitive on I.

2) Show that for a function $\mathbf{f}$ to be a step function on an interval I it is necessary and sufficient that it have only a finite number of points of discontinuity, and be constant on every interval where it is continuous.

3) Let $\mathbf{f}$ be a regulated function on an interval $\mathrm{I} \subset \mathbf{R}$, taking its values in a complete normed space E over $\mathbf{R}$; show that for every compact subset H of I the set $\mathbf{f}(\mathrm{H})$ is relatively compact in E; give an example where $\mathbf{f}(\mathrm{H})$ is not closed in E.

4) Give an example of a continuous real function f on a compact interval $\mathrm{I} \subset \mathbf{R}$, such that the composite function $x \mapsto \mathrm{sgn}(f(x))$ is not regulated on I (even though sgn is regulated on $\mathbf{R}$).

5) Let $\mathbf{f}$ be a vector function defined on a compact interval $\mathrm{I} = [a, b] \subset \mathbf{R}$, taking its values in a complete normed space E; one says that $\mathbf{f}$ is *of bounded variation* on I if there exists a number $m > 0$ such that, for every finite strictly increasing sequence $(x_i)_{0 \leqslant i \leqslant n}$ of points of I such that $x_0 = a$ and $x_n = b$, one has $\sum\limits_{i=0}^{n-1} \|\mathbf{f}(x_{i+1}) - \mathbf{f}(x_i)\| \leqslant m$.

a) Show that $\mathbf{f}(\mathrm{I})$ is relatively compact in E (argue by contradiction).

b) Show that $\mathbf{f}$ is regulated on I (prove that when x tends to a point $x_0 \in \mathrm{I}$, remaining $> x_0$, the function $\mathbf{f}$ cannot have two different cluster points, and use *a*)).

6) For a function $\mathbf{f}$, with values in a complete normed space E, and defined on an open interval $\mathrm{I} \subset \mathbf{R}$, to be equal to a regulated function on I at all points of the complement of a countable subset of I, it is necessary and sufficient that it satisfy the following condition: for every $x \in \mathrm{I}$ and every $\varepsilon > 0$ there exist a number $h > 0$ and two elements a, b of E such that one has $\|\mathbf{f}(y) - a\| \leqslant \varepsilon$ for every $y \in [x, x+h]$, except for at most a countably infinite number of points of this interval, and $\|\mathbf{f}(z) - b\| \leqslant \varepsilon$ for all $z \in [x-h, x]$ except for at most a countably infinite number of points of this interval.

* 7) Show that the function equal to $\sin(1/x)$ for $x \neq 0$ and to 0 for $x = 0$, admits a strict primitive on $\mathbf{R}$ (remark that $x^2 \sin(1/x)$ admits a derivative at every point).

Deduce from this that if $g(x, u, v)$ is a polynomial in u, v, its coefficients being continuous functions of x on an interval I containing 0, then the function equal to $g(x, \sin 1/x, \cos 1/x)$ for $x \neq 0$, and to a suitable value α (to be determined) for $x = 0$, admits a strict primitive on I; give an example where $\alpha \neq g(0, 0, 0)$. *

* 8) Show that there exists a continuous function on $[-1, +1]$ admitting a finite derivative at every point of this interval, this derivative being equal to $\sin\left(\dfrac{1}{\sin 1/x}\right)$ at points x different from $1/n\pi$ (n an integer $\neq 0$) and from 0. (On a neighbourhood of $x = 1/n\pi$ make the change of variable $x = \dfrac{1}{n\pi + \operatorname{Arc\,sin} u}$ and use exerc. 7; by means of the same change of variable, show that there exists a constant $a > 0$ independent of n such that

$$\left| \int_{2/(2n+1)\pi}^{2/(2n-1)\pi} \sin\left(\frac{1}{\sin \dfrac{1}{x}}\right) dx \right| \leqslant \frac{a}{n^3};$$

and deduce that if one puts

$$g(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{x} \sin\left(\frac{1}{\sin \dfrac{1}{t}}\right) dt,$$

then g has a derivative equal to 0 at the point $x = 0$.) *

9) Let $\mathbf{f}$ be a regulated function on a compact interval $\mathrm{I} = [a, b]$. Show that for every $\varepsilon > 0$ there exists a continuous function $\mathbf{g}$ on I such that $\int_a^b \|\mathbf{f}(t) - \mathbf{g}(t)\|\, dt \leqslant \varepsilon$ (reduce to the case where $\mathbf{f}$ is a step function). Deduce that there exists a polynomial $\mathbf{h}$ (with coefficients in E) such that $\int_a^b \|\mathbf{f}(t) - \mathbf{h}(t)\|\, dt \leqslant \varepsilon$.

10) Let $\mathbf{f}$ be a regulated function on $[a, b]$, taking values in E, let $\mathbf{g}$ be a regulated function on $[a, c]$ ($c > b$), taking its values in F, and let $(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}.\mathbf{y}]$ be a continuous bilinear map from $\mathrm{E} \times \mathrm{F}$ into G (E, F, G being complete normed spaces). Show that

$$\lim_{h \to 0,\, h > 0} \int_a^b [\mathbf{f}(t).\mathbf{g}(t+h)]\, dt = \int_a^b [\mathbf{f}(t).\mathbf{g}(t)]\, dt$$

(reduce to the case where $\mathbf{f}$ is a step function).

11) With the same hypotheses as in exerc. 10 show that for every $\varepsilon > 0$ there exists a number $\rho > 0$ such that for every subdivision of $[a, b]$ into intervals $[x_i, x_{i+1}]$ of length $\leqslant \rho$ $(0 \leqslant i \leqslant n-1)$ one has

$$\left\| \int_a^b [\mathbf{f}(t).\mathbf{g}(t)]\,dt - \sum_{i=0}^{n-1} [\mathbf{f}(u_i).\mathbf{g}(v_i)]\,(x_{i+1} - x_i) \right\| \leqslant \varepsilon$$

for any choice, for each index i, of points u_i, v_i in $[x_i, x_{i+1}]$ (reduce to the case where $\mathbf{f}$ and $\mathbf{g}$ are step functions).

¶ 12) One says that a sequence (x_n) of real numbers in the interval $[0, 1]$ is *uniformly distributed* in this interval if

$$\lim_{n\to\infty} \frac{\nu_n(\alpha, \beta)}{n} = \beta - \alpha \tag{1}$$

for every pair of numbers α, β such that $0 \leqslant \alpha \leqslant \beta \leqslant 1$, where $\nu_n(\alpha, \beta)$ denotes the number of indices i such that $1 \leqslant i \leqslant n$ and $\alpha \leqslant x_i \leqslant \beta$.

Show that if the sequence (x_n) is uniformly distributed and if $\mathbf{f}$ is a regulated function on $[0, 1]$, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{f}(x_i) = \int_0^1 \mathbf{f}(t)\,dt \tag{2}$$

(reduce to the case where $\mathbf{f}$ is a step function). Converse.

Show that for the sequence (x_n) to be uniformly distributed it is enough that the relation (2) should hold for every real function f belonging to a dense set in the space of real continuous functions on $[0, 1]$ endowed with the topology of uniform convergence.

¶ 13) Let f be a real regulated function on a compact interval $[a, b]$. Put

$$r(n) = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right) - \int_a^b f(t)\,dt.$$

a) Show that if f is increasing on $[a, b]$ one has

$$0 \leqslant r(n) \leqslant \frac{b-a}{n}\Big(f(b) - f(a)\Big).$$

b) If f is continuous and admits a regulated bounded right derivative on $[a, b[$, show that one has $\lim\limits_{n\to\infty} n\,r(n) = \dfrac{b-a}{2}(f(b) - f(a))$ (putting $x_k = a + k\,\dfrac{b-a}{n}$, remark that one has

$$r(n) = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \Big(f(x_{k+1}) - f(t)\Big)\,dt,$$

and apply prop. 5 of II, p. 57).

c) Give an example of a function f, increasing and continuous on $[a, b]$, such that $nr(n)$ does not tend to $\dfrac{b-a}{2}(f(b) - f(a))$ as n grows indefinitely $\Big[$take for f the limit of a decreasing sequence (f_n) of increasing functions whose graphs are broken lines such that

$$f_n\left(a + k\,\frac{b-a}{2^n}\right) = f\left(a + k\,\frac{b-a}{2^n}\right) \qquad \text{for} \quad 0 \leqslant k \leqslant 2^n$$

and

$$(b-a)\sum_{k=1}^{2^n} f_n\left(a + k\,\frac{b-a}{2^n}\right) - 2^n \int_a^b f_n(t)\,dt \geqslant \frac{3}{4}(b-a)(f_n(b) - f_n(a))\Big].$$

14) Let $\mathbf{f}$ be a vector function which is the primitive of a regulated function $\mathbf{f}'$ on $[a, b]$, and such that $\mathbf{f}(a) = \mathbf{f}(b) = 0$. Show that if M is the least upper bound of $\|\mathbf{f}'(x)\|$ over the set of points of $[a, b]$ where $\mathbf{f}'$ is continuous, then

$$\left\|\int_a^b \mathbf{f}(t)\,dt\right\| \leqslant \mathrm{M}\frac{(b-a)^2}{4}.$$

15) Let $\mathbf{f}$ be a continuous real function, strictly increasing on an interval $[0, a]$, and such that $f(0) = 0$; let g be its inverse function, defined and strictly increasing on $[0, f(a)]$; show that

$$xy \leqslant \int_0^x f(t)\,dt + \int_0^y g(u)\,du$$

for $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant f(a)$, equality occurring only if $y = f(x)$ (study the variation of $xy - \int_0^x f(t)\,dt$) as a function of x (y remaining fixed). Deduce that for $x \geqslant 0,\ y \geqslant 0,\ p > 1,\ p' = p/(p-1)$, one has $xy \leqslant ax^p + by^{p'}$ for $a > 0,\ b > 0$ and $(pa)^{p'}(p'b)^p \geqslant 1$.

16) Let $\mathbf{f}$ be a regulated vector function on $\mathrm{I} = [a, b] \subset \mathbf{R}$, let $\mathbf{u}$ be a primitive of $\mathbf{f}$ on I and D a closed convex set containing $\mathbf{u}(\mathrm{I})$. Show that if g is a real *monotone* function on I one has

$$\int_a^b \mathbf{f}(t)\,g(t)\,dt = (\mathbf{u}(b) - \mathbf{c})\,g(b) + (\mathbf{c} - \mathbf{u}(a))\,g(a)$$

where $\mathbf{c}$ belongs to D (reduce to the case where g is a monotone step function). Deduce that if f is a real regulated function on I then there exists a $c \in \mathrm{I}$ such that

$$\int_a^b f(t)g(t)\,dt = g(a)\int_a^c f(t)\,dt + g(b)\int_c^b f(t)\,dt$$

("the second mean value theorem").

17) Let g be a real function admitting a continuous derivative, and $\neq 0$ on $[a, x]$; if f is a real function having a regulated $(n+1)^{th}$ derivative on $[a, x]$, show that the remainder $r_n(x)$ in the Taylor expansion of order n for f at the point a can be written

$$r_n(x) = (g(x) - g(a))\,\frac{(x-\xi)^n}{n!}\,\frac{f^{(n+1)}(\xi)}{g'(\xi)}$$

where $a < \xi < x$ (use the integral form for $r_n(x)$).

18) Let f be a finite real function, continuous on an open interval I. For f to be convex on I it is necessary and sufficient that for every $x \in \mathrm{I}$ one has

$$\limsup_{h\to\infty}\left(\frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt - f(x)\right) \geqslant 0$$

(argue as in exerc. 9 of I, p. 46).

19) Let f be a convex function on an interval I, let h be a number > 0, and I_h the intersection of I and the intervals $\mathrm{I}+h$ and $\mathrm{I}-h$; show that, if I_h is not empty, the function

$$g_h(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt$$

is convex on I_h; if $h < k$ one has $g_h \leqslant g_k$. When h tends to 0 show that g_h tends uniformly to f on every compact interval *contained in the interior* of I.

20) Show that as n increases indefinitely the polynomial

$$f_n(x) = \frac{\int_0^x (1-t^2)^n\,dt}{\int_0^1 (1-t^2)^n\,dt}$$

tends uniformly to -1 on every interval $[-1, -\varepsilon]$, and tends uniformly to $+1$ on every interval $[\varepsilon, +1]$, where $\varepsilon > 0$ (note that $\int_0^1 (1-t^2)^n\,dt \geqslant \int_0^1 (1-t)^n\,dt$). Deduce that the polynomial $g_n(x) = \int_0^x f_n(t)\,dt$ tends uniformly to $|x|$ on $[-1, +1]$, which provides a new proof of the Weierstrass theorem (*Gen. Top.*, X, p. 313, prop. 3).

21) Let f be a real increasing convex continuous function on an interval $[0, a]$, and such that $f(0) = 0$. If $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0$ is a finite decreasing sequence of points in $[0, a]$, show that one has

$$f(a_1) - f(a_2) + \cdots + (-1)^{n-1} f(a_n) \geqslant f(a_1 - a_2 + \cdots + (-1)^{n-1} a_n).$$

(One can restrict oneself to the case where $n = 2m$ is even; remark that for $1 \leqslant j \leqslant m$ one has

$$\int_\alpha^\beta f'(t)\,dt \leqslant \int_{a_{2j}}^{a_{2j-1}} f'(t)\,dt$$

where $\alpha = a_{2j+1} - a_{2j+2} + \cdots + a_{2m-1} - a_{2m}$ and $\beta = \alpha + (a_{2j-1} - a_{2j})$.)

22) Let f be a continuous increasing real function and $\geqslant 0$ on the interval $[0, 1]$.

a) Show that there exists a convex function $g \geqslant 0$ on $[0, 1]$ such that $g \leqslant f$ and $\int_0^1 g(t)\,dt \geqslant \frac{1}{2}\int_0^1 f(t)\,dt$ (*cf.* I, p. 48, exerc. 23).

b) Show that there exists a convex function h on $[0, 1]$ such that $h \geqslant f$ and that

$$\int_0^1 h(t)\,dt \leqslant 2\int_0^1 f(t)\,dt.$$

(For every a such that $0 < a \leqslant 1$, let f_a be the function equal to f for $0 \leqslant t \leqslant a$ and to $f(a)$ for $a \leqslant t \leqslant 1$; let A be the set of $a \in]0, 1]$ for which there exists a convex function h_a on $[0, 1]$ such that $h_a \geqslant f_a$ and $\int_0^1 h_a(t)\,dt \leqslant 2\int_0^1 f_a(t)\,dt$. Show that the least upper bound b of A again belongs to A, using exerc. 1 of I, p. 45. Then prove that

$f_b = f$, arguing by contradiction. For this, reduce to proving the following result: if φ is continuous and increasing on $[0, 1]$, not constant, and such that $\varphi(0) = 0$, then there is a point $c \in]0, 1[$ such that $\varphi(c) > 0$ and such that the linear function ψ such that $\psi(c) = \varphi(c)$ and

$$\psi(2c - 1) = 0$$

satisfies the relation $\psi(t) \geqslant \varphi(t)$ for $\max(0, 2c - 1) \leqslant t \leqslant c$.)

23) Let $\mathbf{f}$ be a vector function, continuously differentiable on an interval $[a, b] \subset \mathbf{R}$, with values in a complete normed space.

a) Show that for $a \leqslant t \leqslant b$ one has

$$\mathbf{f}(t) = \frac{1}{b-a}\int_a^b \mathbf{f}(x)\,dx + \int_a^b \frac{x-a}{b-a}\,\mathbf{f}'(x)\,dx + \int_t^b \frac{x-b}{b-a}\,\mathbf{f}'(x)\,dx.$$

b) Deduce the inequalities

$$\|\mathbf{f}(t)\| \leqslant \frac{1}{b-a}\int_a^b \|\mathbf{f}(x)\|\;dx + \int_a^b \left\|\mathbf{f}'(x)\right\|\;dx$$

and

$$\left\|\mathbf{f}\left(\tfrac{1}{2}(a+b)\right)\right\| \leqslant \frac{1}{b-a}\int_a^b \|\mathbf{f}(x)\|\;dx + \frac{1}{2}\int_a^b \left\|\mathbf{f}'(x)\right\|\;dx.$$

§ 2.

1) Let α and β be two finite real numbers such that $\alpha < \beta$. Show that if γ and δ are two numbers such that $\alpha < \gamma \leqslant \delta < \beta$ then there exists a real function f defined on an interval $[0, a]$, taking only the values α and β, such that, on all the interval $[\varepsilon, a]$ (for $\varepsilon > 0$), f is a step function and that, if one puts $g(x) = \int_0^x f(t)\,dt$, one has

$$\liminf_{x\to 0,\, x>0} \frac{g(x)}{x} = \gamma, \qquad \limsup_{x\to 0,\, x>0} \frac{g(x)}{x} = \delta; \tag{$*$}$$

(take for g a function whose graph is a broken line whose consecutive sides have gradients α and β and whose peaks occur alternately on the lines $y = \gamma x$, $y = \delta x$ for $\gamma \neq \delta$, or on the line $y = \gamma x$ and the parabola $y = \gamma x + x^2$ for $\gamma = \delta$).

By the same method show that, whether γ and δ are finite or not ($\gamma < \delta$), there exists a real function f defined on $]0, a]$, such that on every interval $[\varepsilon, a]$ (for $\varepsilon > 0$), f is a step function, that the integral $g(x) = \int_0^x f(t)\,dt$ exists, and that one again has the relations $(*)$.

2) *a*) Let $\mathbf{f}$ be a regulated function on an interval $]0, a]$ such that the integral $\int_0^a \frac{\mathbf{f}(t)}{t}\,dt$ is convergent. Show that the integral $\mathbf{g}(x) = \int_0^x \mathbf{f}(t)\,dt$ is convergent and that $\mathbf{g}$ admits a right derivative at the point $x = 0$ (integrate suitably by parts).

b) Give an example of a real function f such that the integral $g(x) = \int_0^x f(t)\,dt$ is convergent and admits a right derivative at the point 0, and yet $f(t)/t$ does not have an integral on $]0, a]$ (take for $f(x)/x$ the derivative of a function of the form $\cos\varphi(x)$, where φ tends to $+\infty$ as x tends to 0).

3) Let f be a real function $\geqslant 0$, defined on an interval $]0, a]$ and regulated on this interval, such that the integral $g(x) = \int_0^x f(t)\,dt$ is convergent, but that g does not have a right derivative at the point 0. Show that there exists a function f_1, regulated on $]0, a]$, such that $(f_1(x))^2 = f(x)$ for all x, that the integral $g_1(x) = \int_0^x f_1(t)\,dt$ converges, and that g_1 has a right derivative at 0. (First consider the case where f is identical to a step function on every interval $[\varepsilon, a]$ (for $\varepsilon > 0$). In this case, divide each interval on which f is constant into a large enough number of equal parts, and take f_1 to be constant on each of these intervals, the sign of f_1 differing on two consecutive such intervals. Proceed in the same way in the general case.)

4) Define on the interval $[0, 1]$ two real functions f, g such that f and g admit *strict* primitives, but that fg is at every point of $[0, 1[$ the right derivative of a continuous function which has no left derivative at a set of points having the power of the continuum (use constructions similar to those of exerc. 8 of I, p. 38 and exerc. 3 above).

5) *a*) Let $\mathbf{f}$ be a regulated function on a *bounded* open interval $]a, b[$; suppose that there exists a real function g, decreasing on $]a, b[$, such that $\|\mathbf{f}(x)\| \leqslant g(x)$ on $]a, b[$ and such that the integral $\int_a^b g(t)\,dt$ converges. Show that if (ε_n) is a sequence of numbers > 0, tending to 0, and such that $\inf\limits_{n\geqslant 1} n\,\varepsilon_n > 0$, one has

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n-1} \mathbf{f}\left(a + \varepsilon_n + k\,\frac{b-a}{n}\right) = \int_a^b \mathbf{f}(t)\,dt. \qquad (*)$$

b) Give an example of a real regulated function $f > 0$ on $]0, 1]$ such that the integral $\int_0^1 f(t)\,dt$ converges, and yet the relation $(*)$ does not hold for $\varepsilon_n = 1/n$ (take f so that its value for $x = 2^{-p}$ is 2^{2p}).

c) With the same hypotheses as in *a*) show that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (-1)^k\, \mathbf{f}\left(a + k\,\frac{b-a}{n}\right) = 0.$$

6) Let $\mathbf{f}$ be a regulated function on the interval $]a, +\infty[$; suppose that there exists a real decreasing function g on $]a, +\infty[$ such that $\|\mathbf{f}(x)\| \leqslant g(x)$ on this interval and that the integral $\int_a^{+\infty} g(t)\,dt$ converges. Show that the series $\sum_{n=1}^{\infty} \mathbf{f}(a + nh)$ is absolutely convergent for every $h > 0$, and that

$$\lim_{h\to+\infty} h \sum_{n=1}^{\infty} \mathbf{f}(a + nh) = \int_a^{+\infty} \mathbf{f}(t)\,dt.$$

7) Let f and g be two regulated functions, and > 0 on an open interval $]a, b[$. Show that the integrals of the functions $f/(1 + fg)$ and $\inf(f,\ 1/g)$ over $]a, b[$ are simultaneously either convergent or infinite.

8) Let f be a regulated function and $\geqslant 0$ on an interval $[a, +\infty[$, and let g be a differentiable increasing function defined on $[a, +\infty[$, and such that $g(x) - x \geqslant \lambda > 0$ for all $x \geqslant a$.

Show that if one has $f(g(x))\,g'(x) \leqslant kf(x)$ with $k < 1$ (resp. $f(g(x))\,g'(x) \geqslant kf(x)$ with $k > 1$), then the integral $\int_a^{+\infty} f(t)\,dt$ converges (resp. is equal to $+\infty$) (Ermakoff's criterion: denoting the n^{th} iterate of g by g^n, consider the integral $\int_0^{g^n(a)} f(t)\,dt$ and let n increase indefinitely).

9) Let α be a number > 0 and f a function defined on the interval $]0, +\infty[$, $\geqslant 0$ and decreasing, and such that the integral $\int_0^{+\infty} t^\alpha f(t)\,dt$ converges. Show that for all $x > 0$ one has

$$\int_x^{+\infty} f(t)\,dt \leqslant \left(\frac{\alpha}{(\alpha+1)x}\right)^\alpha \int_0^{+\infty} t^\alpha f(t)\,dt.$$

(First prove this when f is constant on an interval $]0, a]$ and zero for $x > a$, then for a sum of such functions, and pass to the limit for the general case.)

§ 3.

1) Let I be an arbitrary interval in **R**, let A be a set, and g a finite real function defined on $\mathrm{I} \times \mathrm{A}$ such that for every $\alpha \in \mathrm{A}$ the map $t \mapsto g(t, \alpha)$ is decreasing and $\geqslant 0$ on I; suppose further that there is a number M independent of α such that $g(t, \alpha) \leqslant \mathrm{M}$ on $\mathrm{I} \times \mathrm{A}$.

a) Show that if $\mathbf{f}$ is a regulated function on I such that the integral $\int_{\mathrm{I}} \mathbf{f}(t)\,dt$ converges, then the integral $\int_{\mathrm{I}} \mathbf{f}(t) g(t, \alpha)\,dt$ is uniformly convergent for $\alpha \in \mathrm{A}$ (use the second mean value theorem; *cf.* II, p. 82, exerc. 16).

b) Suppose that $\mathrm{I} = [a, +\infty[$ and also that $g(t, \alpha)$ tends uniformly to 0 (for $\alpha \in \mathrm{A}$) when t tends to $+\infty$. Show that if $\mathbf{f}$ is a regulated function on I and if there is a number $k > 0$ such that $\left\| \int_{\mathrm{J}} \mathbf{f}(t)\,dt \right\| \leqslant k$ for every compact interval J contained in I, then the integral $\int_{\mathrm{I}} \mathbf{f}(t)\,g(t, \alpha)\,dt$ is uniformly convergent for $\alpha \in \mathrm{A}$ (same method).

c) Suppose that $\mathrm{I} = [a, +\infty[$ with $a > 0$, $\mathrm{A} = [0, +\infty[$, and $g(t, \alpha) = \varphi(\alpha t)$, where φ is a convex decreasing function and $\geqslant 0$ on A, tending to 0 as x tends to $+\infty$ and such that $\varphi(0) = 1$. Suppose further that $\mathbf{f} = \mathbf{h}''$, where $\mathbf{h}$ is a twice differentiable function on $[a, +\infty[$, and, together with $\mathbf{h}'$, is bounded on this interval. The integral $\int_a^{+\infty} \mathbf{f}(t)\,\varphi(\alpha t)\,dt$ is then convergent for $\alpha > 0$; show that when α tends to 0, it tends to $-\mathbf{h}'(a)$ (integrate by parts and use the second mean value theorem).

* *d*) Take $\varphi(x) = e^{-cx}$, where $c > 0$, take $a = 1$, and for f take the complex function $t \mapsto e^{i \log t}/t$, which is the derivative of a bounded function on I; show that for a suitable choice of c the integral $\displaystyle\int_1^{+\infty} e^{-\alpha c t} \frac{e^{i \log t}}{t}\,dt$, which is absolutely convergent for $\alpha > 0$, does not tend to any limit when α tends to 0 (after integrating by parts, make the change of variables $\alpha t = u$; use Laplace transform theory to see that $\displaystyle\int_0^{+\infty} e^{-cu + i \log u}\,du$ does not vanish for certain $c > 0$.)$_*$

2) Let f and g be two real regulated functions on a compact interval $[a, b]$, such that f is decreasing on $[a, b]$, and $0 \leqslant g(t) \leqslant 1$. If one puts $\lambda = \int_a^b g(t)\,dt$, show that one has

$$\int_{b-\lambda}^{b} f(t)\,dt < \int_a^b f(t)\,g(t)\,dt < \int_a^{a+\lambda} f(t)\,dt$$

except when f is constant, or g is equal to 0 (resp. 1) at all the points where it is continuous (in which case all three terms are equal). (Vary one of the limits of integration in the integral $\int_x^y f(t)g(t)\,dt$.)

3) Let $f(x, \alpha) = 1/\sqrt{1 - 2\alpha x + \alpha^2}$ for $-1 < x < +1$ and $\alpha \in \mathbf{R}$; show that the function $g(\alpha) = \int_{-1}^{+1} dx/\sqrt{1 - 2\alpha x + \alpha^2}$ is continuous on $\mathbf{R}$, but has no derivative at $\alpha = 1$ and $\alpha = -1$; show that $f'_\alpha(x, \alpha)$ exists for all $\alpha \in \mathbf{R}$ and for $x \in \mathrm{I} = \left]-1, +1\right[$, and is continuous on $\mathrm{I} \times \mathbf{R}$, and that the integral $\int_{-1}^{+1} f'_\alpha(x, \alpha)\,dx$ exists for all $\alpha \in \mathbf{R}$, but verify that this integral is not uniformly convergent on a neighbourhood of the point $\alpha = 1$ or of the point $\alpha = -1$.

¶ 4) Let I be an interval of $\mathbf{R}$, A a neighbourhood of a point α_0 in the field $\mathbf{R}$ (resp. the field $\mathbf{C}$), and $\mathbf{f}$ a continuous map of $\mathrm{I} \times \mathrm{A}$ into a complete normed space E over $\mathbf{R}$, such that $\mathbf{f}'_\alpha(x, \alpha)$ exists and is continuous on $\mathrm{I} \times \mathrm{A}$. Let $a(\alpha)$, $b(\alpha)$ be two continuous functions defined on A, with values in I, such that one has identically $\mathbf{f}(a(\alpha), \alpha) = \mathbf{f}(b(\alpha), \alpha) = 0$ on A. Show that the function $\mathbf{g}(\alpha) = \int_{a(\alpha)}^{b(\alpha)} \mathbf{f}(t, \alpha)\,dt$ admits a derivative equal to $\int_{a(\alpha_0)}^{b(\alpha_0)} \mathbf{f}'_\alpha(t, \alpha_0)\,dt$ at the point α_0, even if a and b are not differentiable at the point α_0 (let M be the supremum of $\left\|\mathbf{f}'_\alpha(x, \alpha)\right\|$ on a compact neighbourhood of $(b(\alpha_0), \alpha_0)$; note, applying Bolzano's theorem to $b(\alpha)$, that for every x belonging to the interval with endpoints $b(\alpha_0)$ and $b(\alpha)$, one has, for α sufficiently close to α_0, that $\|\mathbf{f}(x, \alpha)\| \leqslant \mathrm{M}\,|\alpha - \alpha_0|$).

5) Let $\mathbf{f}$ be a continuous vector function on a compact interval $\mathrm{I} = [0, a]$. Show that if, at the point $\alpha_0 \in \mathrm{I}$, there is an $\varepsilon > 0$ such that $(\mathbf{f}(x) - \mathbf{f}(\alpha_0))/\left|x - \alpha_0\right|^{1/2+\varepsilon}$ remains bounded as x tends to α_0, then the function $\mathbf{g}(\alpha) = \int_0^\alpha \mathbf{f}(x)\,dx/\sqrt{\alpha - x}$ admits a derivative equal to

$$\frac{1}{\sqrt{\alpha_0}}\mathbf{f}(\alpha_0) - \frac{1}{2}\int_0^{\alpha_0} \frac{\mathbf{f}(x) - \mathbf{f}(\alpha_0)}{(\alpha_0 - x)^{3/2}}\,dx$$

at the point α_0 for $\alpha_0 > 0$, and to 0 for $\alpha = 0$.

When $\mathbf{f}$ is the real function $\sqrt{\alpha_0 - x}$, show that the function $\mathbf{g}$ has an infinite derivative at the point α_0.

¶ 6) Let $\mathrm{I} = [a,\ b]$, $\mathrm{A} = [c, d]$ be two compact intervals on $\mathbf{R}$; let $\mathbf{f}$ be a function defined on $\mathrm{I} \times \mathrm{A}$, with values in a complete normed space E over $\mathbf{R}$, such that for all $\alpha \in \mathrm{A}$, the map $t \mapsto \mathbf{f}(t, \alpha)$ is regulated on I, that $\mathbf{f}$ is bounded on $\mathrm{I} \times \mathrm{A}$, and that the set D of points of discontinuity of $\mathbf{f}$ in $\mathrm{I} \times \mathrm{A}$ is met in a *finite* number of points by each line $x = x_0$ and each line $\alpha = \alpha_0$ ($x_0 \in \mathrm{I}$, $\alpha_0 \in \mathrm{A}$).

a) Show that the function $\mathbf{g}(\alpha) = \int_a^b \mathbf{f}(t, \alpha)\,dt$ is continuous on A (given $\alpha_0 \in \mathrm{A}$ and $\varepsilon > 0$, show that there is a neighbourhood V of α_0 and a finite number of intervals J_k contained in I with the sum of their lengths $\leqslant \varepsilon$, such that, if J denotes the complement in I of $\bigcup_k J_k$, then $\mathbf{f}$ is continuous on $\mathrm{J} \times \mathrm{V}$).

b) Show that the formula for interchanging the order of integration (II, p. 77, formula (15)) is still valid (same method as in *a*)).

7) Let f be a real function defined and having a continuous derivative on the interval $\left]0, +\infty\right[$, and such that

$$\lim_{x \to 0} f(x) = \lim_{x \to +\infty} f(x) = 0.$$

The integral $\int_0^{+\infty} f'(\alpha t)\,dt$ is defined and continuous on every bounded interval $]0, a]$; show that the integral $\int_0^{+\infty} dt \int_0^a f'(\alpha t)\,d\alpha$ may either not exist or may be different from

$$\int_0^a d\alpha \int_0^{+\infty} f'(\alpha t)\,dt.$$

¶ 8) Let I and J be two arbitrary intervals in $\mathbf{R}$, and $\mathbf{f}$ a function defined and continuous on $\mathrm{I} \times \mathrm{J}$, with values in a complete normed space E over $\mathbf{R}$. Suppose that:

1° the integral $\int_{\mathrm{I}} \mathbf{f}(x, y)\,dx$ is uniformly convergent when y runs through an arbitrary compact interval contained in J;

2° the integral $\int_{\mathrm{J}} \mathbf{f}(x, y)\,dy$ is uniformly convergent when x runs through an arbitrary compact interval contained in I;

3° if, for every compact interval H contained in I, one puts $\mathbf{u}_{\mathrm{H}}(y) = \int_{\mathrm{H}} \mathbf{f}(x, y)\,dx$, the integral $\int_{\mathrm{J}} \mathbf{u}_{\mathrm{H}}(y)\,dy$ is uniformly convergent for $\mathrm{H} \in \mathfrak{K}(\mathrm{I})$ (the right directed ordered set of compact intervals contained in I).

Under these conditions, show that the integrals $\int_{\mathrm{I}} dx \int_{\mathrm{J}} \mathbf{f}(x, y)\,dy$ and $\int_{\mathrm{J}} dy \int_{\mathrm{I}} \mathbf{f}(x, y)\,dx$ exist and are equal.

* 9) Deduce from exerc. 8 that the integrals

$$\int_0^{\infty} dx \int_0^{\infty} e^{-yx^2} \sin y\,dy \qquad \text{and} \qquad \int_0^{\infty} dy \int_0^{\infty} e^{-yx^2} \sin y\,dx$$

exist and are equal.*

10) *a*) If h, u, v' are primitives of real regulated functions on an open interval $]a, b[$ of $\mathbf{R}$, if v is a primitive of v', and if $v(x) > 0$ on this interval, then one has the *Redheffer identity*

$$hu'^2 = -h\mathrm{D}\left(\frac{hv'}{v}\right) + hv^2\left(\mathrm{D}\left(\frac{u}{v}\right)\right)^2 + \mathrm{D}\left(\frac{u^2hv'}{v}\right)$$

at those points of $]a, b[$ where the derivatives are defined.

* *b*) Let v, w be two functions > 0 on $]a, b[$, which are primitives of regulated functions $v' > 0$, $w' \leqslant 0$. Deduce from *a*) that for every function u which is a primitive of a regulated function u' on $]a, b[$ and such that $\liminf\limits_{x \to a,\ x > a} u(x) = 0$, the hypothesis that the integral $\displaystyle\int_a^b \frac{w(x)}{v'(x)} u'^2(x)\,dx$ is convergent implies that the integral $\displaystyle\int_a^b \frac{w'(x)}{v(x)} u^2(x)\,dx$ is convergent and that

$$\limsup_{x \to b,\ x < b} u^2(x)w(x)/v(x)$$

is finite, and also the inequality

$$\int_a^b \frac{w(x)}{v'(x)} u'^2(x)\,dx \geqslant -\int_a^b \frac{w'(x)}{v(x)} u^2(x)\,dx + \limsup_{x \to b,\ x < b} \frac{u^2(x)w(x)}{v(x)}. \qquad (*)$$

(Putting $h = w/v'$, first observe that $\int_c^x dt/h(t) \leqslant v(x)/w(x)$ for $a < c < x < b$ and remark that by the Cauchy-Schwarz inequality

$$(u(x)-u(c))^2 \leqslant \left(\int_c^x h(t)u'^2(t)\,dt\right)\left(\int_c^x \frac{dt}{h(t)}\right), \tag{$**$}$$

to deduce that

$$\liminf_{x\to a,\ x>a} u^2(x)w(x)/v(x) = 0,$$

and then integrate the Redheffer identity.)

c) Deduce from $(*)$ that if u is the primitive of a regulated function on $]0,1]$, if

$$\liminf_{x\to 0,\ x>0} u(x) = 0$$

and if the integral $\int_0^1 u'^2(t)\,dt$ converges, then so does $\int_0^1 (u(t)/t)^2\,dt$ and for all $\alpha > 0$ one has the inequality

$$\int_0^1 u'^2(t)\,dt \geqslant \alpha(1-\alpha)\int_0^1 \left(\frac{u(t)}{t}\right)^2 dt + \alpha u^2(1).$$

d) Let α be a number > 0, let K be a function > 0, differentiable and decreasing on $]0,+\infty[$ and such that $\lim_{x\to+\infty} \mathrm{K}(x) = 0$. If u is the primitive of a regulated function on $]0,+\infty[$, such that $\liminf_{x\to 0,\ x>0} u(x) = 0$, and if the integral $\int_0^{+\infty} x^{1-\alpha}\mathrm{K}(x)u'^2(x)\,dx$ converges, then the function $x^{-\alpha}\mathrm{K}(x)u^2(x)$ tends to 0 as x tends to 0 or to $+\infty$, the integral

$$\int_0^{+\infty} x^{-\alpha}\mathrm{K}'(x)u^2(x)\,dx$$

converges, and one has

$$\int_0^{+\infty} x^{1-\alpha}\mathrm{K}(x)u'^2(x)\,dx \geqslant -\alpha\int_0^{+\infty} x^{-\alpha}\mathrm{K}'(x)u^2(x)\,dx$$

(take $v(x) = x^\alpha$, $h(x) = x^{1-\alpha}\mathrm{K}(x)$ in the Redheffer identity).

In particular, for $\alpha = \frac{1}{2}$ and $\mathrm{K}(x) = x^{-1/2}$, one has

$$4\int_0^{+\infty} u'^2(x)\,dx \geqslant \int_0^{+\infty}\left(\frac{u(x)}{x}\right)^2 dx$$

(*Hardy-Littlewood inequality*).

e) Let $\alpha \geqslant -1$, let K be a function $\geqslant 0$, differentiable and increasing on $[0,+\infty[$. Suppose that the integrals

$$\int_0^{+\infty} x^{-\alpha}\mathrm{K}(x)u'^2(x)\,dx \qquad \text{and} \qquad \int_0^{+\infty} x^{\alpha}\mathrm{K}(x)u^2(x)\,dx$$

converge. Then $\mathrm{K}(x)u^2(x)$ tends to 0 as x tends to $+\infty$, and one has

$$\int_0^{+\infty}\mathrm{K}'(x)u^2(x)\,dx \leqslant 2\left(\int_0^{+\infty} x^{-\alpha}\mathrm{K}(x)u'^2(x)\,dx\right)^{1/2}\left(\int_0^{+\infty} x^{\alpha}\mathrm{K}(x)u^2(x)\,dx\right)^{1/2}$$

(take $v(x) = \exp(-cx^\alpha)$ in the Redheffer identity, with a suitable constant c).

In particular, if a and b are constants such that $b+1 \geqslant a$ and $a+b \geqslant 0$, one has

$$(a+b)\int_0^{+\infty} x^{a+b-1}u^2(x)\,dx \leqslant 2\left(\int_0^{+\infty} x^{2a}u'^2(x)\,dx\right)^{1/2}\left(\int_0^{+\infty} x^{2b}u^2(x)\,dx\right)^{1/2}$$

if the integrals on the right-hand side converge (generalized *H. Weyl inequality*).

f) If $0 < \alpha < 2$ and u is a primitive of a regulated function on $[0, \alpha]$ and $u(0) = 0$, one has

$$\int_0^{\alpha} (\alpha - x)^{\alpha-1}\, x^{1-\alpha}\, u'(x)\left(u'(x) - 2u(x)\right) dx \geqslant 0$$

(generalized *Opial inequality*). (Apply (∗) suitably, replacing $u(x)$ by $e^{-x}\,u(x)$.)

g) If u is a primitive of a regulated function on $[0, b]$ and $u(0) = 0$, then

$$\int_0^b e^{-2x}\,u'^2(x)\,dx \geqslant \int_0^b e^{-2x}\,u(x)\,u'(x)\,dx + \tfrac{1}{2}\,u^2(b)\,e^{-2b}$$

(*Hlawka's inequality*) (same method as in *f*)).

h) If u is the primitive of a regulated function on $\mathbf{R}$ then, for all $t \in \mathbf{R}$,

$$|u(t)| \leqslant \left(\int_{-\infty}^{+\infty} u'^2(x)\,dx\right)^{1/4}\left(\int_{-\infty}^{+\infty} u^2(x)\,dx\right)^{1/4}$$

if the two integrals on the right-hand side converge (consider the two intervals $]-\infty, t]$ and $[t, +\infty[$; take $v(x) = e^{\alpha x}$ on the two intervals, $h(x) = 1/\alpha$ on the first interval and $h(x) = -1/\alpha$ on the second, then choose $\alpha > 0$ suitably).$_*$

CHAPTER III
Elementary Functions

§ 1. DERIVATIVES OF THE EXPONENTIAL AND CIRCULAR FUNCTIONS

1. DERIVATIVES OF THE EXPONENTIAL FUNCTIONS; THE NUMBER *e*

We know that every continuous homomorphism of the additive group $\mathbf{R}$ into the multiplicative group $\mathbf{R}^*$ of real numbers $\neq 0$ is a function of the form $x \mapsto a^x$ (called an *exponential function*) where a is a number > 0 (TG, V, p.11); it is an isomorphism of $\mathbf{R}$ onto the multiplicative group $\mathbf{R}_+^*$ of numbers > 0 if $a \neq 1$, and the inverse isomorphism from $\mathbf{R}_+^*$ onto $\mathbf{R}$ is denoted by $\log_a x$ and is called the *logarithm to the base* a.

We shall see that the function $f(x) = a^x$ has, for every $x \in \mathbf{R}$, a derivative of the form $c.a^x$ (where clearly $c = f'(0)$). This results from the following general theorem:

THEOREM 1. *Let* E *be a complete normed algebra over the field* $\mathbf{R}$*, with a unit element* $\mathbf{e}$*, and let* $\mathbf{f}$ *be a continuous group homomorphism of the additive group* $\mathbf{R}$ *into the multiplicative group* G *of invertible elements of* E*. Then the map* $\mathbf{f}$ *is differentiable at every* $x \in \mathbf{R}$*, and*

$$\mathbf{f}'(x) = \mathbf{f}(x)\mathbf{f}'(0). \tag{1}$$

First we note that, E being a complete algebra, G is *open* in E (*Gen. Top.*, IX, p. 179, prop. 14). Consider the function $\mathbf{g}(x) = \int_0^a \mathbf{f}(x+t)\,dt$, where $a > 0$ is a number which we shall choose later; since $\mathbf{f}(x+t) = \mathbf{f}(x)\,\mathbf{f}(t)$ by hypothesis, we have $\mathbf{g}(x) = \int_0^a \mathbf{f}(x)\,\mathbf{f}(t)\,dt = \mathbf{f}(x)\int_0^a \mathbf{f}(t)\,dt$ (I, p. 6, prop. 3). Let $\alpha > 0$ be such that the ball $\|\mathbf{x} - \mathbf{e}\| \leqslant \alpha$ is contained in G; since $\mathbf{f}(0) = \mathbf{e}$ and $\mathbf{f}$ is continuous by hypothesis, one can assume that a is small enough so that $\|\mathbf{f}(t) - \mathbf{e}\| \leqslant \alpha$ on $[0, a]$; consequently (II, p. 61, formula (16)) one has

$$\left\| \frac{1}{a} \int_0^a \mathbf{f}(t)\,dt - \mathbf{e} \right\| \leqslant \alpha,$$

and $\frac{1}{a}\int_0^a \mathbf{f}(t)\,dt$ belongs to G; in other words, is invertible; so too is $\mathbf{b} = \int_0^a \mathbf{f}(t)\,dt$ and one can write $\mathbf{f}(x) = \mathbf{g}(x)\mathbf{b}^{-1}$; it is therefore enough to show that $\mathbf{g}$ is differentiable; now, by the change of variable $x + t = u$ we have $\mathbf{g}(x) = \int_x^{x+a} \mathbf{f}(u)\,du$; since $\mathbf{f}$ is continuous, $\mathbf{g}$ is differentiable for all $x \in \mathbf{R}$ (II, p. 56, prop. 3), and

$$\mathbf{g}'(x) = \mathbf{f}(x+a) - \mathbf{f}(x) = \mathbf{f}(x)\big(\mathbf{f}(a) - \mathbf{e}\big).$$

Hence $\mathbf{f}'(x) = \mathbf{g}'(x)\mathbf{b}^{-1} = \mathbf{f}(x)\mathbf{c}$, where $\mathbf{c} = \big(\mathbf{f}(a) - \mathbf{e}\big)\mathbf{b}^{-1}$, and clearly $\mathbf{f}'(0) = \mathbf{c}$.

Conversely, one can show, either directly (III, p. 115, exerc. 1), or by means of the theory of linear differential equations (IV, p. 188), that every differentiable map $\mathbf{f}$ of $\mathbf{R}$ into a complete normed algebra E, such that $\mathbf{f}'(x) = \mathbf{f}(x)\mathbf{c}$ and $\mathbf{f}(0) = \mathbf{e}$, is a homomorphism of the additive group $\mathbf{R}$ into the multiplicative group G.

PROPOSITION 1. *For every number $a > 0$ and $\neq 1$ the exponential function a^x admits at every point $x \in \mathbf{R}$ a derivative equal to $\big(\log_e a\big)a^x$ where e is a number* > 1 (independent of a).

Applying th. 1 to the case where E is the field $\mathbf{R}$ itself now shows that a^x has a derivative equal to $\varphi(a).a^x$ at every point, where $\varphi(a)$ is a real number $\neq 0$ depending only on a. Let b be a second number > 0 and $\neq 1$; the function b^x has a derivative equal to $\varphi(b).b^x$ from the above; on the other hand, we have $b^x = a^{x.\log_a b}$ so (I, p. 9, prop. 5) the derivative of b^x is equal to $\log_a b.\varphi(a)b^x$; on comparing these two expressions we obtain

$$\varphi(b) = \varphi(a).\log_a b. \tag{2}$$

One deduces that there is one unique number b such that $\varphi(b) = 1$; by (2) this relation is equivalent to $b = a^{1/\varphi(a)}$. It is conventional to denote the real number so obtained by e; from (2) one has $\varphi(a) = \log_e a$, which completes the proof of prop. 1.

One often writes $\exp x$ instead of e^x.

The definition of the number e shows that

$$\mathrm{D}(e^x) = e^x \tag{3}$$

which proves that e^x is strictly increasing, hence that $e > 1$.

In §2 (III, p. 105) we shall see how to calculate arbitrarily close approximations to e.

DEFINITION 1. *Logarithms to the base e are called Naperian logarithms* (or *natural logarithms*).

We usually omit the base in the notation for the Naperian logarithm. Unless it is stated to the contrary, the notation $\log x$ $(x > 0)$ will denote the *Naperian logarithm* of x. With this notation, prop. 1 can be written as the identity

$$\mathrm{D}\big(a^x\big) = (\log a)\,a^x \tag{4}$$

valid for any $a > 0$ ($\log a = 0$ when $a = 1$).

This relation shows that a^x has derivatives of *every order*, and that

$$\mathrm{D}^n\big(a^x\big) = (\log a)^n\, a^x. \tag{5}$$

In particular, for $a > 0$ and $\neq 1$ one has $\mathrm{D}^2\big(a^x\big) > 0$ for all $x \in \mathbf{R}$, and hence a^x is *strictly convex* on $\mathbf{R}$ (I, p. 31, corollary). From this one deduces the following proposition:

PROPOSITION 2 ("geometric mean inequality"). *For any numbers* $z_i > 0$ $(1 \leqslant i \leqslant n)$ *and numbers* $p_i > 0$ *such that* $\sum_{i=1}^{n} p_i = 1$, *one has*

$$z_1^{p_1}\, z_2^{p_2}\, \ldots\, z_n^{p_n} \leqslant p_1 z_1 + p_2 z_2 + \cdots + p_n z_n. \tag{6}$$

Moreover, the two sides of (6) *are equal only if the* z_i *are equal.*

Let us put $z_i = e^{x_i}$; then the inequality (6) can be written

$$\exp\big(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n\big) \leqslant p_1 e^{x_1} + p_2 e^{x_2} + \cdots + p_n e^{x_n}. \tag{7}$$

The proposition thus follows from prop. 1 of I, p. 26 applied to the function e^x, which is strictly convex on $\mathbf{R}$.

One says that the left- (resp. right-) hand side of (6) is the *weighted geometric mean* (resp. *weighted arithmetic mean*) of the n numbers z_i relative to the *weights* p_i $(1 \leqslant i \leqslant n)$. If $p_i = 1/n$ for $1 \leqslant i \leqslant n$, one calls the corresponding arithmetic and geometric means the *ordinary* arithmetic and geometric means of the z_i. Then the inequality (6) can be written

$$\big(z_1 z_2\, \ldots\, z_n\big)^{1/n} \leqslant \frac{1}{n}\big(z_1 + z_2 + \cdots + z_n\big). \tag{8}$$

2. DERIVATIVE OF $\log_a x$

Since a^x is strictly monotone on $\mathbf{R}$ for $a \neq 1$, applying the rule for differentiating inverse functions (I, p. 17, prop. 6) gives, for all $x > 0$

$$\mathrm{D}\big(\log_a x\big) = \frac{1}{x \log a} \tag{9}$$

and in particular

$$\mathrm{D}\big(\log x\big) = \frac{1}{x}. \tag{10}$$

If u is a real function admitting a derivative at the point x_0 and such that $u(x_0) > 0$, then the function $\log u$ admits a derivative equal to $u'(x_0)/u(x_0)$ at the point x_0. In particular, we have $\mathrm{D}\big(\log|x|\big) = 1/|x| = 1/x$ if $x > 0$, and

$$\mathrm{D}\big(\log|x|\big) = -\frac{1}{|x|} = \frac{1}{x}$$

if $x < 0$; in other words, $\mathrm{D}\big(\log|x|\big) = 1/x$ for any $x \neq 0$. One concludes that if, on an interval I, the real function u is not zero and admits a finite derivative, then $\log|u(x)|$ admits a derivative equal to u'/u on I; this derivative is called the *logarithmic derivative* of u. It is clear that the logarithmic derivative of $|u|^\alpha$ is $\alpha u'/u$, and that the logarithmic derivative of a product is equal to the sum of the logarithmic derivatives of the factors; these rules often provide the fastest way to calculate the derivative of a function. They give again, in particular, the formula

$$\mathrm{D}\big(x^\alpha\big) = \alpha x^{\alpha-1} \qquad (\alpha \text{ an arbitrary real number, } x > 0) \qquad (11)$$

which has already been shown by another method (II, p. 69).

Example. If u is a function $\neq 0$ on an interval I, and v is any real function, then $\log(|u|^v) = v.\log|u|$, so if u and v are differentiable

$$\frac{1}{|u|^v}\mathrm{D}(|u|^v) = v'\log|u| + v\frac{u'}{u}.$$

3. DERIVATIVES OF THE CIRCULAR FUNCTIONS; THE NUMBER π

We have defined, in General Topology (*Gen. Top.*, VIII, p. 106), the continuous homomorphism $x \mapsto \mathbf{e}(x)$ of the additive group $\mathbf{R}$ onto the multiplicative group $\mathbf{U}$ of complex numbers of absolute value 1; this is a periodic function with principal period 1, and $\mathbf{e}\left(\frac{1}{4}\right) = i$. One knows (*loc. cit.*) that every continuous homomorphism of $\mathbf{R}$ onto $\mathbf{U}$ is of the form $x \mapsto \mathbf{e}(x/a)$, and one puts $\cos_a x = \mathcal{R}(\mathbf{e}(x/a))$, $\sin_a x = \mathcal{I}(\mathbf{e}(x/a))$ (*trigonometric functions,* or *circular functions,* to base a); these last functions are continuous maps from $\mathbf{R}$ into $\mathbf{[}-1, +1\mathbf{]}$ having principal period a. We have $\sin_a(x + a/4) = \cos_a x$, $\cos_a(x + a/4) = -\sin_a x$, and the function $\sin_a x$ is increasing on the interval $\mathbf{[}-a/4, a/4\mathbf{]}$.

PROPOSITION 3. *The function* $\mathbf{e}(x)$ *has a derivative equal to* $2\pi i\,\mathbf{e}(x)$ *at every point of* $\mathbf{R}$, *where* π *is a constant* > 0.

Now, th. 1 of III, p. 51, applied to the case where E is the field $\mathbf{C}$ of complex numbers, yields the relation $\mathbf{e}'(x) = \mathbf{e}'(0)\,\mathbf{e}(x)$; moreover, since $\mathbf{e}(x)$ has constant euclidean norm, $\mathbf{e}'(x)$ is orthogonal to $\mathbf{e}(x)$ (I, p. 7, *Example* 3); one thus has $\mathbf{e}'(0) = \alpha i$, with α real. Since $\sin_1 x$ is increasing on $\mathbf{[}-\frac{1}{4}, \frac{1}{4}\mathbf{]}$ its derivative for $x = 0$ is

$\geqslant 0$, so $\alpha \geqslant 0$, and since $\mathbf{e}(x)$ is not constant, $\alpha > 0$; it is conventional to denote the number α so obtained by 2π.

In §2 (III, p. 23) we shall show how one can calculate arbitrarily close approximations to the number π.

We thus have the formula

$$\mathrm{D}\left(\mathbf{e}\left(\frac{x}{a}\right)\right) = \frac{2\pi i}{a}\,\mathbf{e}\left(\frac{x}{a}\right). \tag{12}$$

One sees that this formula simplifies when $a = 2\pi$; this is why one uses the circular functions relative to base 2π exclusively in Analysis; we agree to omit the base in the notation for these functions; unless mentioned expressly to the contrary, the notations $\cos x$, $\sin x$ and $\tan x$ denote $\cos_{2\pi} x$, $\sin_{2\pi} x$ and $\tan_{2\pi} x$ respectively.

With these conventions, and $a = 2\pi$, formula (12) can be written

$$\mathrm{D}(\cos x + i \sin x) = \cos\left(x + \frac{\pi}{2}\right) + i \sin\left(x + \frac{\pi}{2}\right), \tag{13}$$

which is equivalent to

$$\mathrm{D}(\cos x) = -\sin x, \qquad \mathrm{D}(\sin x) = \cos x,$$

from which one deduces

$$\mathrm{D}(\tan x) = 1 + \tan^2 x = \frac{1}{\cos^2 x}. \tag{15}$$

Besides the three circular functions $\cos x$, $\sin x$ and $\tan x$ one also uses, in numerical work, the three auxiliary functions: *cotangent, secant* and *cosecant,* defined by the formulae

$$\cot x = \frac{1}{\tan x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \operatorname{cosec} x = \frac{1}{\sin x}.$$

Recall (*Gen. Top.*, VIII, p. 109) that the angle corresponding to the base 2π is called the *radian*.

4. INVERSE CIRCULAR FUNCTIONS

The restriction of the function $\sin x$ to the interval $\mathbf{[}-\pi/2, +\pi/2\mathbf{]}$ is strictly increasing; one denotes its inverse by $\operatorname{Arc\,sin} x$, which is thus a strictly increasing continuous map of the interval $\mathbf{[}-1, +1\mathbf{]}$ onto $\mathbf{[}-\pi/2, +\pi/2\mathbf{]}$ (fig. 6). The formula for differentiating inverse functions (I, p. 9, prop. 6) gives the derivative of this function

$$\mathrm{D}(\operatorname{Arc\,sin} x) = \frac{1}{\cos(\operatorname{Arc\,sin} x)}.$$

Since $-\pi/2 \leqslant \operatorname{Arc\,sin} x \leqslant \pi/2$ we have $\cos(\operatorname{Arc\,sin} x) \geqslant 0$, and since

$$\sin(\operatorname{Arc}\sin x) = x,$$

we have $\cos(\operatorname{Arc}\sin x) = \sqrt{1-x^2}$, from which

$$\mathrm{D}(\operatorname{Arc}\sin x) = \frac{1}{\sqrt{1-x^2}}. \tag{16}$$

Likewise the restriction of $\cos x$ to the interval $[0, \pi]$ is strictly decreasing; one denotes its inverse function by $\operatorname{Arc}\cos x$, and this a strictly decreasing map of $[-1, +1]$ onto $[0, \pi]$ (fig. 6). Moreover

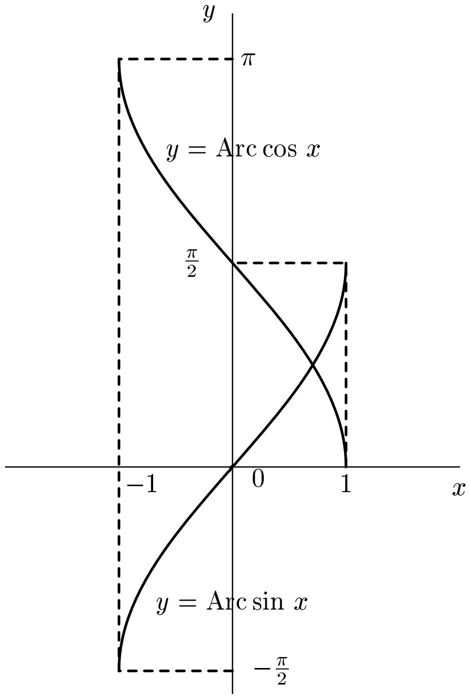

Fig. 6

$$\sin\left(\frac{\pi}{2} - \operatorname{Arc}\cos x\right) = \cos(\operatorname{Arc}\cos x) = x$$

and since $-\pi/2 \leqslant \pi/2 - \operatorname{Arc}\cos x \leqslant \pi/2$, we have

$$\operatorname{Arc}\cos x = \frac{\pi}{2} - \operatorname{Arc}\sin x \tag{17}$$

from which in particular it follows that

$$\mathrm{D}\big(\operatorname{Arc}\cos x\big) = -\frac{1}{\sqrt{1-x^2}}. \tag{18}$$

Finally, the restriction of $\tan x$ to the interval $]-\pi/2, +\pi/2[$ is strictly increasing; one denotes its inverse by $\operatorname{Arc\,tan} x$, and this is a strictly increasing map from $\mathbf{R}$ onto $]-\pi/2, +\pi/2[$ (fig. 7); we have

$$\lim_{x\to-\infty} \operatorname{Arc\,tan} x = -\frac{\pi}{2}, \qquad \lim_{x\to+\infty} \operatorname{Arc\,tan} x = \frac{\pi}{2},$$

and, by applying the formula for differentiating inverse functions and formula (15) of III, p. 95, we have

$$\mathrm{D}\big(\operatorname{Arc\,tan} x\big) = \frac{1}{1+x^2}. \tag{19}$$

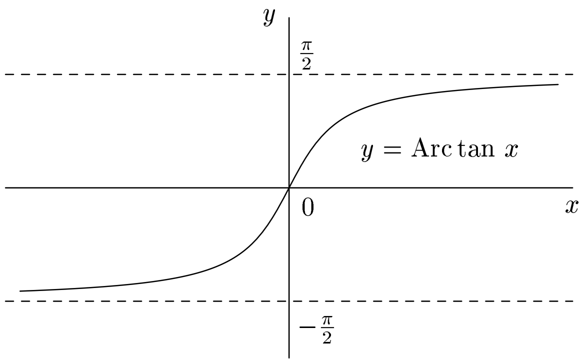

Fig. 7

5. THE COMPLEX EXPONENTIAL

We have determined (*Gen. Top.*, VIII, p. 106) all the continuous homomorphisms of the (additive) topological group $\mathbf{C}$ of complex numbers onto the (multiplicative) topological group $\mathbf{C}^*$ of complex numbers $\neq 0$; these are the maps

$$x + iy \mapsto e^{\alpha x+\beta y}\, \mathbf{e}(\gamma x + \delta y) \tag{20}$$

where $\alpha, \beta, \gamma, \delta$ are four real numbers subject to the single condition $\alpha\delta - \beta\gamma \neq 0$. We now propose to determine which of these homomorphisms $z \mapsto f(z)$ are *differentiable* on $\mathbf{C}$. First we remark that it is enough for f to be differentiable at the point $z = 0$; indeed, for every point $z \in \mathbf{C}$ one has $\dfrac{f(z+h) - f(z)}{h} = f(z)\,\dfrac{f(h)-1}{h}$; if $f'(0)$ exists, then so does $f'(z)$, and $f'(z) = af(z)$, with $a = f'(0)$. On the other hand, if g is a second differentiable homomorphism, such that $g'(z) = bg(z)$, then $g(az/b) = f(z)$, for one notes immediately that the quotient $g(az/b)/f(z)$ has an everywhere zero derivative and is equal to 1 for $z = 0$; all the differentiable homomorphisms are thus of the form $z \mapsto f(\lambda z)$, where f is one of them (assuming they exist) and λ is any (complex) constant.

This being so, if f is differentiable at the point $z = 0$ then each of the maps $x \mapsto f(x)$, $y \mapsto f(iy)$ of $\mathbf{R}$ into $\mathbf{C}$ is necessarily differentiable at the point 0, the first having derivative $f'(0)$, the second $if'(0)$. Now the derivatives of the maps $x \mapsto e^{\alpha x}\mathbf{e}(\gamma x)$, $y \mapsto e^{\beta y}\mathbf{e}(\delta y)$ at the point 0 are respectively equal to $\alpha + 2\pi i\gamma$ and $\beta + 2\pi i\delta$, from which $\beta = -2\pi\gamma$ and $\alpha = 2\pi\delta$; these conditions are, in particular, satisfied by the homomorphism $x + iy \mapsto e^x\mathbf{e}(y/2\pi)$, which we shall denote provisionally by f_0. We shall now show that f_0 is actually differentiable at the point $z = 0$.

It is clear that $x \mapsto f_0(x)$ and $y \mapsto f_0(iy)$ have derivatives of every order; in particular, Taylor's formula of order 1 applied to these functions shows that for every $\varepsilon > 0$ there is an $r > 0$ such that, if one puts

$$f_0(x) = 1 + x + \varphi(x)x, \qquad f_0(iy) = 1 + iy + \psi(y)y,$$

then the conditions $|x| \leqslant r, |y| \leqslant r$ imply that $|\varphi(x)| \leqslant \varepsilon$ and $|\psi(y)| \leqslant \varepsilon$; this being so, we have $f_0(x + iy) = f_0(x)f_0(iy) = 1 + (x + iy) + \theta(x, y)$ with

$$\theta(x, y) = \big(i + \varphi(x)\psi(y)\big)xy + (1 + x)y\,\psi(y) + (1 + iy)x\,\varphi(x);$$

for $|z| \leqslant r$ we have $|x| \leqslant r$ and $|y| \leqslant r$, whence

$$|\theta(x, y)| \leqslant \left(1 + \varepsilon^2\right)|z|^2 + 2\varepsilon\,|z|\left(1 + |z|\right)$$

which proves that the quotient $\dfrac{f_0(z) - 1 - z}{z}$ tends to 0 with z, that is to say, the function f_0 admits a derivative equal to 1 at the point $z = 0$. The above thus proves that, for all $z \in \mathbf{C}$,

$$\mathrm{D}\big(f_0(z)\big) = f_0(z). \tag{21}$$

This property establishes the connection between f_0 and the function e^x, which is moreover the restriction of f_0 to the real axis; for this reason we make the following definition:

DEFINITION 2. *The homomorphism* $x + iy \mapsto e^x\mathbf{e}(y/2\pi)$ *of* $\mathbf{C}$ *onto* $\mathbf{C}^*$ *is called the complex exponential; its value at an arbitrary complex number* z *is denoted by* e^z *or* $\exp z$.

6. PROPERTIES OF THE FUNCTION e^z

The fact that $z \mapsto e^z$ is a homomorphism of $\mathbf{C}$ onto $\mathbf{C}^*$ may be expressed by the identities

$$e^{z+z'} = e^z\,e^{z'}, \qquad e^0 = 1, \qquad e^{-z} = 1/e^z. \tag{22}$$

By definition, one has, for every $z = x + iy$,

$$e^{x+iy} = e^x\,(\cos y + i\sin y) \tag{23}$$

and since $e^x > 0$ one sees that e^z has *absolute value* e^x and *amplitude* y (modulo 2π).

In particular, def. 2 (III, p. 98) gives

$$\mathbf{e}(x) = e^{2\pi i x} \tag{24}$$

which permits us to write the formulae defining $\cos x$ and $\sin x$ in the form

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right), \qquad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \tag{25}$$

(*Euler's formulae*).

Since 2π is the principal period of $\mathbf{e}(y/2\pi)$, $2\pi i$ is the *principal period* of e^z; in other words, the group of periods of e^z is the set of numbers $2n\pi i$, where n runs through $\mathbf{Z}$.

Finally, formula (21) of III, p. 98 can be written

$$\mathrm{D}\left(e^z\right) = e^z \tag{26}$$

whence, for every complex number a

$$\mathrm{D}\left(e^{az}\right) = a\, e^{az}. \tag{27}$$

Remark. If, in formula (27), one restricts the function e^{az} (a complex) to the real axis, one again obtains, for x real,

$$\mathrm{D}(e^{ax}) = a\, e^{ax}. \tag{28}$$

This formula allows us to calculate a primitive for each of the functions $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$ (α and β real); indeed we have $e^{(\alpha+i\beta)x} = e^{\alpha x}\cos\beta x + i e^{\alpha x}\sin\beta x$, so, by (28)

$$\mathrm{D}\left(\mathcal{R}\left(\frac{1}{\alpha+i\beta}\, e^{(\alpha+i\beta)x}\right)\right) = e^{\alpha x}\cos\beta x$$
$$\mathrm{D}\left(\mathcal{I}\left(\frac{1}{\alpha+i\beta}\, e^{(\alpha+i\beta)x}\right)\right) = e^{\alpha x}\sin\beta x.$$

In the same way one reduces the evaluation of a primitive of $x^n e^{\alpha x}\cos\beta x$, or of $x^n e^{\alpha x}\sin\beta x$ (n *an integer* > 0) to that of a primitive of $x^n e^{(\alpha+i\beta)x}$; now, the formula for integration by parts of order $n+1$ (II, p. 60, formula (11)) shows that a primitive of this last function is

$$e^{(\alpha+i\beta)x}\left[\frac{x^n}{\alpha+i\beta} - \frac{nx^{n-1}}{(\alpha+i\beta)^2} + \frac{n(n-1)x^{n-2}}{(\alpha+i\beta)^3} + \cdots + (-1)^n\,\frac{n!}{(\alpha+i\beta)^{n+1}}\right].$$

By Euler's formulae one can on the other hand express every positive integral power of $\cos x$ or of $\sin x$ as a linear combination of exponentials e^{ipx} (p a positive or negative integer). By formula (28) one can thus express a primitive of a function of the form $x^n e^{\alpha x}(\cos\beta x)^r(\sin\gamma x)^s$ as a linear combination of functions of the form $x^p e^{\alpha x}\cos\lambda x$ and $x^p e^{\alpha x}\sin\mu x$ and (n, p, r, s integers, α, β, γ, λ, μ real).

Example. One has

$$\sin^{2n} x = \frac{(-1)^n}{2^{2n}}\left(e^{ix} - e^{-ix}\right)^{2n} = \frac{(-1)^n}{2^{2n}}\left(e^{2nix} - \binom{2n}{1}e^{(2n-2)ix} + \cdots + e^{-2nix}\right)$$

whence

$$\int_0^x \sin^{2n} t\,dt = \frac{(-1)^n}{2^{2n}}\left(\frac{1}{n}\sin 2nx - \binom{2n}{1}\frac{1}{n-1}\sin(2n-2)x + \cdots \right.$$

$$\left. + (-1)^{n-1}\binom{2n}{n-1}\sin 2x + (-1)^n\binom{2n}{n}x\right)$$

and in particular

$$\int_0^{\pi/2} \sin^{2n} t\,dt = \binom{2n}{n}\frac{1}{2^{2n}}\frac{\pi}{2} = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\frac{\pi}{2}. \tag{29}$$

7. THE COMPLEX LOGARITHM

Let B be the “strip” formed by the points $z = x + iy$ such that $-\pi \leqslant y < \pi$; the function e^z takes each of its values once and only once on B; in other words, $z \mapsto e^z$ is a *bijective* continuous map of B onto $\mathbf{C}^*$; the image under this map of the (half-open) segment $x = x_0,\ -\pi \leqslant y < \pi$ is the circle $|z| = e^{x_0}$; the image of the line $y = y_0$ is the (open) half-line defined by $\mathrm{Am}(z) = y_0 \pmod{2\pi}$. The image under $z \mapsto e^z$ of the *interior* $\overset{\circ}{\mathrm{B}}$ of B, that is, of the set of $z \in \mathbf{C}$ such that $|\mathcal{I}(z)| < \pi$, is the complement F of the (closed) negative real half-axis in $\mathbf{C}$; if one agrees to denote by $\mathrm{Am}(z)$ the measure of the amplitude of z which belongs to $[-\pi, \pi[$, then the set F can be defined by the relations $-\pi < \mathrm{Am}(z) < \pi$. Since $z \mapsto e^z$ is a *strict homomorphism* of $\mathbf{C}$ onto $\mathbf{C}^*$ the image under this map of any open subset of $\overset{\circ}{\mathrm{B}}$ (so of $\mathbf{C}$) is an open set in $\mathbf{C}^*$ (so in F); in other words, the restriction of $z \mapsto e^z$ to $\overset{\circ}{\mathrm{B}}$ is a *homeomorphism* of $\overset{\circ}{\mathrm{B}}$ onto F. We denote by $z \mapsto \log z$ the homeomorphism of F onto $\overset{\circ}{\mathrm{B}}$ which is the inverse of the latter; for a complex number $z \in \mathrm{F}$, $\log z$ is called the *principal value of the logarithm of* z. If $z = x + iy$ and $\log z = u + iv$ then $x + iy = e^{u+iv}$, whence $e^u = |z|$, and since $-\pi < v < \pi$, we have $v = \mathrm{Am}(z)$. Moreover, we have $\tan(v + \pi/2) = -x/y$ if $y \neq 0$; thus we can write

$$\begin{cases} u = \log|z| = \dfrac{1}{2}\log(x^2 + y^2) & \\ v = \dfrac{\pi}{2} - \mathrm{Arc}\tan\dfrac{x}{y} & \text{if } y > 0 \\ v = 0 & \text{if } y = 0 \\ v = -\dfrac{\pi}{2} - \mathrm{Arc}\tan\dfrac{x}{y} & \text{if } y < 0. \end{cases} \tag{30}$$

It is clear that $\log z$ is the extension to F of the function $\log x$ defined on the positive open real half-axis $\mathbf{R}_+^*$. If $z,\ z'$ are two points of F such that zz' is not real negative, we have $\log(zz') = \log z + \log z' + 2\varepsilon\pi i$, where $\varepsilon = +1,\ -1$ or 0 depending on the values of $\mathrm{Am}(z)$ and $\mathrm{Am}(z')$.

We note that at the points of the negative real half-axis the function $\log z$ *has no limit;* to be precise, if x tends to $x_0 < 0$ and if y tends to 0 remaining > 0 (resp. < 0), then $\log z$ tends to $\log|x_0| + \pi i$ (resp. $\log|x_0| - \pi i$); when z tends to 0, $|\log z|$ increases indefinitely.

We shall see later how the theory of analytic functions allows us to extend the function $\log z$, and to define the complex logarithm in full generality.

Since $\log z$ is the inverse homeomorphism of e^z, the formula for differentiating inverse functions (I, p. 9, prop. 6) shows that $\log z$ is differentiable at every point $z \in$ F, and that

$$\mathrm{D}(\log z) = \frac{1}{e^{\log z}} = \frac{1}{z} \tag{31}$$

a formula which generalizes formula (10) of III, p. 93.

8. PRIMITIVES OF RATIONAL FUNCTIONS

Formula (31) allows us to evaluate the primitive of an arbitrary rational function $r(x)$ of a *real* variable x, with real or complex coefficients. We know (A.VII.7) that such a function can be written (in unique manner) as a finite sum of terms, which are:

a) either of the form ax^p (p an integer $\geqslant 0$, a a complex number);
b) or of the form $a/(x-b)^m$ (m an integer $\geqslant 0$, a and b complex numbers).

Now it is easy to obtain a primitive of each of these terms:

a) a primitive of ax^p is $a\,\dfrac{x^{p+1}}{p+1}$;

b) if $m > 1$ a primitive of $a/(x-b)^m$ is $\dfrac{a}{(1-m)(x-b)^{m-1}}$;

c) finally, from formulae (10) (III, p. 93) and (31) (III, p. 101), a primitive of $\dfrac{a}{x-b}$ is $a\log|x-b|$ if b is real, $a\log(x-b)$ if b is complex. In the last case, if $b = p + iq$, one has furthermore (III, p. 100, formulae (30))

$$\log(x-b) = \log\sqrt{(x-p)^2 + q^q} + i\,\mathrm{Arc}\tan\frac{x-p}{q} \pm i\frac{\pi}{2}.$$

We postpone the examination of more practical methods for explicitly determining a primitive of a rational function given explicitly to the part of this work dedicated to Numerical Calculus.

One can reduce to the evaluation of a primitive of a rational function:

1° the evaluation of a primitive of a function of the form $r(e^{ax})$, where r is a rational function and a a real number; indeed by the change of variable $u = e^{ax}$ one reduces to finding a primitive for $r(u)/u$;

2° the evaluation of a primitive of a function of the form $f(\sin ax, \cos ax)$, where f is a rational function of two variables and a is a real number; the change of variables $u = \tan(ax/2)$ reduces this to finding a primitive for

$$\frac{2}{1+u^2} f\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right).$$

9. COMPLEX CIRCULAR FUNCTIONS; HYPERBOLIC FUNCTIONS

Euler's formulae (25) (III, p. 99) and the definition of e^z for every complex z allow us to *extend* to $\mathbf{C}$ the functions $\cos x$ and $\sin x$ defined on $\mathbf{R}$, by putting, for all $z \in \mathbf{C}$

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \qquad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \tag{32}$$

(*cf.* III, p. 119, exerc. 19).

These functions are periodic with principal period 2π; that is, one has $\cos(z + \pi/2) = -\sin z$, $\sin(z + \pi/2) = \cos z$; one may also verify the identities

$$\begin{aligned} \cos^2 z + \sin^2 z &= 1 \\ \cos(z + z') &= \cos z \cos z' - \sin z \sin z' \\ \sin(z + z') &= \sin z \cos z' + \cos z \sin z'. \end{aligned}$$

More generally, every algebraic identity between circular functions for *real* variables remains true when one gives these variables arbitrary *complex* values (III, p. 119, exerc. 18).

One puts $\tan z = \sin z/\cos z$ if $z \neq (2k+1)\pi/2$ and $\cot z = \cos z/\sin z$ if $z \neq k\pi$; these are periodic functions with principal period π.

Formula (27) (III, p. 99) shows that $\cos z$ and $\sin z$ are differentiable on $\mathbf{C}$, and that

$$\mathrm{D}\big(\cos z\big) = -\sin z, \qquad \mathrm{D}\big(\sin z\big) = \cos z.$$

For $z = ix$ (x real), the formulae (32) give

$$\cos ix = \frac{1}{2}\left(e^x + e^{-x}\right), \qquad \sin ix = \frac{i}{2}\left(e^x - e^{-x}\right).$$

It is convenient to have a specific notation for the real functions thus introduced; we put

$$\begin{cases} \cosh x = \dfrac{1}{2}\left(e^x + e^{-x}\right) & (\textit{hyperbolic cosine of } x) \\ \sinh x = \dfrac{i}{2}\left(e^x - e^{-x}\right) & (\textit{hyperbolic sine of } x) \\ \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} & (\textit{hyperbolic tangent of } x). \end{cases} \tag{33}$$

One thus has, for every real x

$$\cos ix = \cosh x, \qquad \sin ix = i \sinh x. \tag{34}$$

From every identity between circular functions in a certain number of complex numbers z_k $(1 \leqslant k \leqslant n)$ one can deduce an identity for the hyperbolic functions, by replacing z_k by ix_k (x_k real, $1 \leqslant k \leqslant n$) and using the formulae (34); for instance one has

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1 \\ \cosh(x + x') &= \cosh x \cosh x' + \sinh x \sinh x' \\ \sinh(x + x') &= \sinh x \cosh x' - \cosh x \sinh x'. \end{aligned}$$

The hyperbolic functions allow us to express the real and imaginary parts of $\cos z$ and $\sin z$ for $z = x + iy$, since

$$\begin{aligned} \cos(x + iy) &= \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y \\ \sin(x + iy) &= \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y. \end{aligned}$$

Finally, one has

$$\mathrm{D}\big(\cosh x\big) = \sinh x,\ \mathrm{D}\big(\sinh x\big) = \cosh x,\ \mathrm{D}\big(\tanh x\big) = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}.$$

Since $\cosh x > 0$ for all x one deduces that $\sinh x$ is strictly increasing on $\mathbf{R}$; since $\sinh 0 = 0$, it follows that $\sinh x$ has the same sign as x. In consequence $\cosh x$ is strictly decreasing for $x \leqslant 0$, strictly increasing for $x \geqslant 0$; finally, $\tanh x$ is strictly increasing on $\mathbf{R}$. Moreover

$$\begin{aligned} &\lim_{x \to -\infty} \sinh x = -\infty, \qquad \lim_{x \to +\infty} \sinh x = +\infty \\ &\lim_{x \to -\infty} \cosh x = \lim_{x \to +\infty} \cosh x = +\infty \\ &\lim_{x \to -\infty} \tanh x = -1, \qquad \lim_{x \to +\infty} \tanh x = +1 \qquad \text{(fig. 8 and 9).} \end{aligned}$$

Sometimes we write $\operatorname{Arg}\sinh x$ for the inverse function of $\sinh x$, which is a strictly increasing map from $\mathbf{R}$ onto $\mathbf{R}$; this function can also be expressed in terms of the logarithm, since from the relation $x = \sinh y = \frac{1}{2}(e^y - e^{-y})$ we deduce that $e^{2y} - 2xe^y - 1 = 0$, and since $e^y > 0$, we have $e^y = x + \sqrt{x^2 + 1}$, that is to say

$$\operatorname{Arg}\sinh x = \log\left(x + \sqrt{x^2 + 1}\right).$$

Similarly, we denote by $\operatorname{Arg}\cosh x$ the inverse of the restriction of $\cosh x$ to $[0, +\infty[$; this map is strictly increasing from $[1, +\infty[$ onto $[0, +\infty[$; one shows as above that

$$\operatorname{Arg}\cosh x = \log\left(x + \sqrt{x^2 - 1}\right).$$

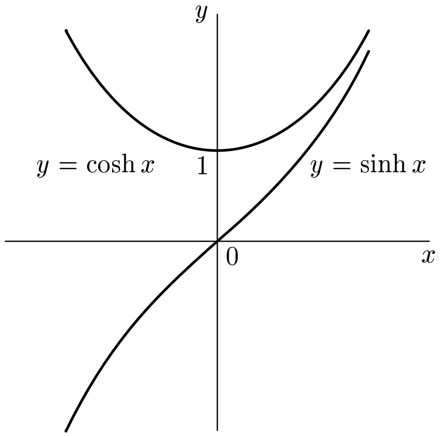

Fig. 8

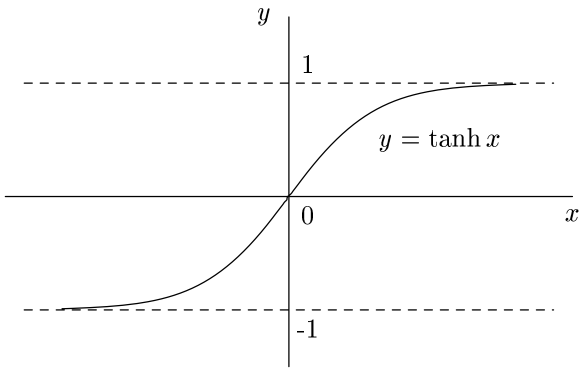

Fig. 9

Finally, we denote by $\operatorname{Arg}\tanh x$ the inverse function of $\tanh x$, which is a strictly increasing map from $\mathbf{]}-1,+1\mathbf{[}$ onto $\mathbf{R}$; one has, moreover,

$$\operatorname{Arg}\tanh x = \frac{1}{2}\log\frac{1+x}{1-x}.$$

Remark. For complex z one sometimes writes

$$\cosh z = \frac{1}{2}(e^z + e^{-z}) = \cos iz$$
$$\sinh z = \frac{1}{2}(e^z - e^{-z}) = -i\sin iz.$$

These functions thus extend to $\mathbf{C}$ the hyperbolic functions defined on $\mathbf{R}$.

§ 2. EXPANSIONS OF THE EXPONENTIAL AND CIRCULAR FUNCTIONS, AND OF THE FUNCTIONS ASSOCIATED WITH THEM

1. EXPANSION OF THE REAL EXPONENTIAL

Since $\mathrm{D}^n\left(e^x\right) = e^x$ the Taylor expansion of order n for e^x is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \int_0^x \frac{(x-t)^n}{n!}\, e^t\, dt. \tag{1}$$

The remainder in this formula is > 0 for $x > 0$ and has the sign of $(-1)^{n+1}$ when $x < 0$; moreover, the inequality of the mean shows that

$$\frac{x^{n+1}}{(n+1)!} < \int_0^x \frac{(x-t)^n}{n!}\, e^t\, dt < \frac{x^{n+1}e^x}{(n+1)!} \qquad \text{for } x > 0 \tag{2}$$

$$\frac{\left|x^{n+1}\right| e^x}{(n+1)!} < \left|\int_0^x \frac{(x-t)^n}{n!}\, e^t\, dt\right| < \frac{\left|x^{n+1}\right|}{(n+1)!} \qquad \text{for } x < 0 \tag{3}$$

Now one knows that the sequence $\left(x^n/n!\right)$ has limit 0 when n increases indefinitely, for all $x \geqslant 0$ (*Gen. Top.*, IV, p. 365); thus, keeping x fixed and letting n grow indefinitely in (1) it follows, from (2) and (3), that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{4}$$

and the series on the right-hand side is *absolutely and uniformly convergent* on every compact interval in $\mathbf{R}$. In particular, one has the formula

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots \tag{5}$$

This formula allows us to calculate rational approximations as close as we desire to the number e; one obtains

$$e = 2.718\,281\,828\ldots$$

to within $1/10^9$. Formula (5) proves, moreover, that e is an *irrational* number [2] (*Gen. Top.*, IV, p. 375).

Remark. Since the remainder in formula (1) is > 0 for $x > 0$ one has, for $x > 0$

$$e^x > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n+1}}{(n+1)!}$$

and *a fortiori*

$$e^x > \frac{x^{n+1}}{(n+1)!}$$

for every integer n : one deduces from this that e^x/x^n *tends to* $+\infty$ with x, for every integer n : we shall find this result again in chap. V by another method (V, p. 231).

[2] Ch. Hermite proved in 1873 that e is a *transcendental* number over the field $\mathbf{Q}$ of rational numbers (in other words, it is not the root of any polynomial with rational coefficients) (*Œuvres*, t. III, p. 150, Paris (Gauthier-Villars), 1912).

2. EXPANSIONS OF THE COMPLEX EXPONENTIAL, OF $\cos x$ AND $\sin x$.

Let z be an arbitrary complex number and consider the function $\varphi(t) = e^{zt}$ of the real variable t; we have $\mathrm{D}^n\varphi(t) = z^n e^{zt}$ and $e^z = \varphi(1)$; expressing $\varphi(1)$ by means of its Taylor series of order n about the point $t = 0$ (II, p. 62) thus gives

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + z^{n+1} \int_0^1 \frac{(1-t)^n}{n!} e^{zt}\, dt \tag{6}$$

a formula which is equivalent to (1) when z is real. The remainder

$$r_n(z) = z^{n+1} \int_0^1 \frac{(1-t)^n}{n!} e^{zt}\, dt$$

in this formula can be bounded above, in absolute value, by using the inequality of the mean; if $z = x + iy$ we have $\left|e^{zt}\right| = e^{xt}$, so $\left|e^{zt}\right| \leqslant 1$ if $x \leqslant 0$, $\left|e^{zt}\right| \leqslant e^x$ if $x > 0$; hence

$$|r_n(z)| \leqslant \frac{|z|^{n+1}}{(n+1)!} \text{ if } x \leqslant 0 \tag{7}$$

$$|r_n(z)| \leqslant \frac{|z|^{n+1}\, e^x}{(n+1)!} \text{ if } x > 0. \tag{8}$$

As above we conclude that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{9}$$

the series being *absolutely and uniformly convergent* on every compact subset of $\mathbf{C}$.

From (6) one deduces in particular that

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \cdots + \frac{i^n x^n}{n!} + i^{n+1} \int_0^x \frac{(x-t)^n}{n!} e^{it}\, dt \tag{10}$$

from which we deduce the Taylor expansions of $\cos x$ and $\sin x$; on taking the real part of (10) for order $2n+1$ we have

$$\cos x = 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1} \int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!} \cos t\, dt \tag{11}$$

with remainder bounded by

$$\left| \int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!} \cos t\, dt \right| \leqslant \frac{|x|^{2n+2}}{(2n+2)!}. \tag{12}$$

Similarly, taking the imaginary part of (10) for order $2n$, we obtain

$$\begin{aligned} \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \\ + (-1)^n \int_0^x \frac{(x-t)^{2n}}{(2n)!} \cos t\, dt \end{aligned} \tag{13}$$

with remainder bounded by

$$\left| \int_0^x \frac{(x-t)^{2n}}{(2n)!} \cos t \, dt \right| \leqslant \frac{|x|^{2n+1}}{(2n+1)!}. \tag{14}$$

Moreover, on comparing the remainders in (11) for orders $2n+1$ and $2n+3$, we have

$$\int_0^x \frac{(x-t)^{2n+3}}{(2n+3)!} \cos t \, dt = \frac{x^{2n+2}}{(2n+2)!} - \int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!} \cos t \, dt$$

and taking (12) into account we see that the remainder in (11) has the *same sign* as $(-1)^{n+1}$ no matter what x; in the same way we can show that the remainder in (13) has the *same sign* as $(-1)^n x$. In particular, for $n = 0$ and $n = 1$ in (11), and for $n = 1$ and $n = 2$ in (13) we obtain the inequalities

$$1 - \frac{x^2}{2} \leqslant \cos x \leqslant 1 \qquad \text{for all } x \tag{15}$$

$$x - \frac{x^3}{6} \leqslant \sin x \leqslant x \qquad \text{for all } x \geqslant 0. \tag{16}$$

Finally, on putting $z = ix$ in (9) we have

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{17}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{18}$$

these series being absolutely and uniformly convergent on every compact interval.

It is clear, furthermore, that the formulae (17) and (18) remain valid for every *complex* x, the series on the right-hand side being absolutely and uniformly convergent on every compact subset of $\mathbf{C}$. In particular, for every x (real or complex)

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

3. THE BINOMIAL EXPANSION

Let m be an *arbitrary* real number. For $x > 0$ we have

$$\mathrm{D}^n\left(x^m\right) = m(m-1)\ldots(m-n+1)\,x^{m-n};$$

the Taylor formula of order n about the point $x = 0$ for the function $(1+x)^m$ shows that for every $x > -1$

$$(1+x)^m = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{n}x^n + r_n(x) \tag{19}$$

with

$$r_n(x) = \frac{m(m-1)\dots(m-n)}{n!} \int_0^x \left(\frac{x-t}{1+t}\right)^n (1+t)^{m-1}\,dt$$

where we put $\binom{m}{n} = \frac{m(m-1)\dots(m-n+1)}{n!}\cdot$ The formula (19) reduces to the binomial formula (*Alg.*, I, p. 99) when m is an integer > 0 and $n \geqslant m$; by extension, we again call it the *binomial formula,* and the coefficients $\binom{m}{n}$ are called the *binomial coefficients,* when m is an *arbitrary* real number and n is an arbitrary integer > 0.

The remainder in (19) has the same sign as $\binom{m}{n+1}$ if $x > 0$, and the sign of $(-1)^{n+1}\binom{m}{n+1}$ if $-1 < x < 0$. Since $\left|\frac{x-t}{1+t}\right| \leqslant |x|$ for $t > -1$ in the interval with endpoints 0 and x, we have the following bound for the remainder, for m and n arbitrary and $x > -1$:

$$\begin{aligned}\left|\frac{m(m-1)\dots(m-n)}{n!} \int_0^x \left(\frac{x-t}{1+t}\right)^n (1+t)^{m-1}\,dt\right| \\ \leqslant \left|\binom{m-1}{n} x^n \left((1+x)^m - 1\right)\right|.\end{aligned} \tag{20}$$

If we suppose $x \geqslant 0$, and $n \geqslant m-1$, then $(1+t)^{n-m+1} \geqslant 1$ on the interval of integration, so

$$0 \leqslant \int_0^x \frac{(x-t)^n}{(1+t)^{n-m+1}}\,dt \leqslant \int_0^x (x-t)^n\,dt = \frac{x^{n+1}}{n+1}$$

which gives the estimate

$$|r_n(x)| \leqslant \left|\binom{m}{n+1}\right| x^{n+1} \qquad (x \geqslant 0,\ n \geqslant m-1) \tag{21}$$

for the remainder. On the other hand, suppose that $-1 \leqslant m < 0$; if one makes the change of variable $u = \frac{x-t}{x(1+t)}$ in the integral (19) one obtains

$$r_n(x) = \frac{m(m-1)\dots(m-n)}{n!}(1+x)^m x^{n+1} \int_0^1 \frac{u^n\,du}{(1+ux)^{m+1}}. \tag{22}$$

To estimate the integral for $x > -1$ we remark that, since $m + 1 < 1$, the integral $\displaystyle\int_0^1 \frac{u^n\,du}{(1-u)^{m+1}}$ converges and bounds the right-hand side of (22) since $1 + ux > 1 - u$. Now, for $-1 < x < 0$ the hypothesis on m implies that all the terms $\binom{m}{1}x, \binom{m}{2}x^2, \ldots, \binom{m}{n}x^n$ which appear in the right-hand side of (19) are $\geqslant 0$, and hence $r_n(x) \leqslant (1+x)^m$, from which, on dividing by $(1+x)^m$,

$$\frac{m(m-1)\ldots(m-n)}{n!}\,x^{n+1}\int_0^1 \frac{u^n\,du}{(1+ux)^{m+1}} \leqslant 1.$$

Moreover, for $-1 < x < 0$ the factor in front of the integral is $\geqslant 0$, so, letting x approach -1,

$$\left|\frac{m(m-1)\ldots(m-n)}{n!}\int_0^1 \frac{u^n\,du}{(1-u)^{m+1}}\right| \leqslant 1$$

and consequently for $-1 \leqslant m < 0$ and $x > -1$ we have

$$|r_n(x)| \leqslant (1+x)^m\;|x|^{n+1}. \tag{23}$$

From these inequalities we can, for a start, deduce that for $|x| < 1$ we have

$$(1+x)^m = \sum_{n=0}^{\infty}\binom{m}{n}x^n \tag{24}$$

the right-hand side (called the *binomial series*) being *absolutely and uniformly convergent* on every *compact* subset of $]-1,+1[$. Indeed one can write

$$\binom{m}{n} = (-1)^n\left(1-\frac{m+1}{1}\right)\left(1-\frac{m+1}{2}\right)\ldots\left(1-\frac{m+1}{n}\right) \tag{25}$$

whence

$$\left|\binom{m}{n}\right| \leqslant \left(1+\frac{|m+1|}{1}\right)\left(1+\frac{|m+1|}{2}\right)\ldots\left(1+\frac{|m+1|}{n}\right).$$

If $|x| \leqslant r < 1$ there is an n_0 such that $1 + \dfrac{|m|}{n_0} < \dfrac{1}{r'}$, where $r < r' < 1$; whence, putting

$$k = \left(1+\frac{|m|}{1}\right)\left(1+\frac{|m|}{2}\right)\ldots\left(1+\frac{|m|}{n_0}\right)$$

we have

$$\left|\binom{m-1}{n}x^n\right| \leqslant k\,|x|^{n_0}\left(\frac{r}{r'}\right)^{n-n_0},$$

which proves the proposition. On the other hand, for $x > 1$, the absolute value of the general term of the series (24) increases indefinitely with n if m is not an integer $\geqslant 0$; indeed, from (25), we have for $n > n_1 \geqslant |m+1|$

$$\left|\binom{m}{n}\right| \geqslant \left|\left(1-\frac{m+1}{1}\right)\left(1-\frac{m+1}{2}\right)\cdots\left(1-\frac{m+1}{n_1}\right)\right| \left(1-\frac{|m+1|}{n_1+1}\right)\cdots\left(1-\frac{|m+1|}{n}\right).$$

Let $n_0 \geqslant n_1$ be such that for $n \geqslant n_0$ we have $1-\dfrac{|m+1|}{n} > \dfrac{1}{x'}$, where $1 < x' < x$. If we put

$$k' = \left|\left(1-\frac{m+1}{1}\right)\cdots\left(1-\frac{m+1}{n_1}\right)\right|\left(1-\frac{|m+1|}{n_1+1}\right)\cdots\left(1-\frac{|m+1|}{n_0}\right)$$

then, for $n > n_0$,

$$\left|\binom{m}{n}x^n\right| \geqslant k'\ |x|^{n_0}\ \left(\frac{x}{x'}\right)^{n-n_0}$$

from which the proposition follows.

We remark that for $m=-1$ the algebraic identity

$$\frac{1}{1+x} = 1-x+x^2-\cdots+(-1)^{n-1}x^{n-1}+(-1)^n\frac{x^n}{1+x} \tag{26}$$

gives the expression for the remainder in the general formula (19) without having to integrate; the formula (23) reduces in this case to the expression for the sum of the *geometric series* (or *progression*) (*Gen. Top.*, IV, p. 364).

In the second place let us study the convergence of the binomial series for $x = 1$ or $x = -1$ (excluding the trivial case $m = 0$) :

a) $m \leqslant -1$. The product with general term $1-\dfrac{m+1}{n}$ converges to $+\infty$ if $m < -1$, to 1 if $m = -1$, so it follows from (25) that for $x = \pm 1$ the general term of the binomial series does not tend to 0.

b) $-1 < m < 0$. This time the product with general term $1-\dfrac{m+1}{n}$ converges to 0, so the inequality (21) shows that $r_n(1)$ tends to 0. Thus the binomial series converges for $x = 1$ and has sum 2^m; moreover, the binomial series is uniformly convergent on every interval $]x_0, 1]$ with $-1 < x_0 \leqslant 1$, by virtue of what we saw above and of (21). On the other hand, for $x = -1$ all the terms on the right-hand side of (24) are $\geqslant 0$; if this series were convergent one could deduce that the binomial series would be normally convergent on $[-1, 1]$ and so would have for its sum a continuous function on this interval, which is absurd because $(1+x)^m$ is not bounded on $]-1, 1]$ for $m < 0$. We conclude that also for $x = 1$ the binomial series is not absolutely convergent.

c) $m > 0$. The definition of $r_n(x)$ shows that $r_n(x)$ tends to the limit $r_n(-1)$ when x tends to -1; on passing to the limit in (20) one concludes that $|r_n(-1)| \leqslant \left|\dbinom{m-1}{n}\right|$, and since $m-1 > -1$ we see that for $x = -1$ the binomial series is convergent. Furthermore, for $n > m+1$ all the terms of this series have the same sign; thus the binomial series is *normally convergent* on the interval $[-1, 1]$ and has sum $(1+x)^m$ on this interval.

4. EXPANSIONS OF log(1 + x), OF Arc tan x AND OF Arc sin x

Let us integrate the two sides of (26) between 0 and x; we obtain the Taylor expansion of order n of $\log(1+x)$, valid for $x > -1$

$$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n \int_0^x \frac{t^n\,dt}{1+t}. \tag{27}$$

The remainder has the same sign as $(-1)^n$ if $x > 0$, and is < 0 if $-1 < x < 0$; further, when $x > 0$, we have $1 + t \geqslant 1$ for $0 \leqslant t \leqslant x$, and, when $-1 < x < 0$, we have $1 + t \geqslant 1 - |x|$ for $x \leqslant 0$; whence the estimates for the remainder

$$\left|\int_0^x \frac{t^n\,dt}{1+t}\right| \leqslant \frac{|x|^{n+1}}{n+1} \qquad \text{for } x \geqslant 0 \tag{28}$$

$$\left|\int_0^x \frac{t^n\,dt}{1+t}\right| \leqslant \frac{|x|^{n+1}}{(n+1)(1-|x|)} \qquad \text{for } -1 < x \leqslant 0. \tag{29}$$

From these last two formulae one deduces immediately that for $-1 < x \leqslant 1$ one has

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} \tag{30}$$

the series being *uniformly convergent* on every compact interval contained in $]-1, +1]$, and *absolutely convergent* for $|x| < 1$.

On the other hand, for $|x| > 1$ the general term in the series on the right-hand side of (30) increases indefinitely in size with n (III, p. 106). For $x = -1$ the series reduces to the harmonic series, which has sum $+\infty$ (*Gen. Top.*, IV, p. 365).

Similarly, let us replace x by x^2 in (26) and integrate both sides between 0 and x; we obtain the Taylor expansion of order $2n - 1$ for $\operatorname{Arc\,tan} x$, valid for all real x

$$\operatorname{Arc\,tan} x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + (-1)^n \int_0^x \frac{t^{2n}\,dt}{1+t^2}. \tag{31}$$

The remainder has the sign of $(-1)^n x$, and since $1 + t^2 \geqslant 1$ for all t we have the estimate

$$\left|\int_0^x \frac{t^{2n}\,dt}{1+t^2}\right| \leqslant \frac{|x|^{2n+1}}{2n+1} \tag{32}$$

from which one deduces that, for $|x| \leqslant 1$,

$$\operatorname{Arc\,tan} x = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^{2n-1}}{2n-1} \tag{33}$$

the series being *uniformly convergent* on $[-1,+1]$, and *absolutely convergent* for $|x| < 1$.

In particular, for $x = 1$ one obtains the formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots + (-1)^n \frac{1}{2n+1} + \cdots \ . \tag{34}$$

For $|x| > 1$ the general term of the right-hand side of (33) increases indefinitely in size with n.

Finally, for the Taylor expansion of $\operatorname{Arc\,sin} x$ we start from the expansion of its derivative $(1-x^2)^{-1/2}$; this last expansion is obtained by replacing x by $-x^2$ in the expansion of $(1+x)^{-1/2}$ as a binomial series; for $|x| < 1$ this gives

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \cdots + \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}x^{2n} + r_n(x)$$

with, by (23), the bound

$$0 \leqslant r_n(x) \leqslant \frac{x^{2n+2}}{\sqrt{1-x^2}}$$

for the remainder.

On taking the primitive of the preceding expansion we obtain

$$\operatorname{Arc\,sin} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots + \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\frac{x^{2n+1}}{2n+1} + \mathrm{R}_n(x) \tag{35}$$

where $\mathrm{R}_n(x)$ has the sign of x and satisfies the inequality

$$|\mathrm{R}_n(x)| \leqslant \int_0^x \frac{t^{2n+2}\,dt}{\sqrt{1-t^2}}. \tag{36}$$

Further, the relation (35) shows that $\mathrm{R}_n(x)$ tends to a limit when x approaches 1 or -1, so one has

$$|\mathrm{R}_n(1)| \leqslant \int_0^1 \frac{t^{2n+2}\,dt}{\sqrt{1-t^2}}. \tag{37}$$

But the right-hand side of (37) tends to 0 when n tends to $+\infty$: for, since the integral $\int_0^1 dt/\sqrt{1-t^2}$ is convergent, for every $\varepsilon > 0$ there is an a such that $0 < a < 1$ and $\int_a^1 dt/\sqrt{1-t^2} \leqslant \varepsilon$; on the other hand we have

$$\int_0^a \frac{t^{2n+2}\,dt}{\sqrt{1-t^2}} \leqslant \frac{1}{\sqrt{1-a^2}} \int_0^a t^{2n+2}\,dt = \frac{a^{2n+3}}{(2n+3)\sqrt{1-a^2}}$$

and so there is an n_0 such that for $n \geqslant n_0$ one has $\dfrac{a^{2n+3}}{(2n+3)\sqrt{1-a^2}} \leqslant \varepsilon$, whence, finally, $|\mathrm{R}_n(x)| \leqslant 2\varepsilon$ for $|x| \leqslant 1$ and $n \geqslant n_0$. Thus one has

$$\operatorname{Arc} \sin x = \sum_{n=0}^{\infty} \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} \frac{x^{2n+1}}{2n+1} \tag{38}$$

the right-hand side being *normally convergent* on the compact interval $[-1, 1]$.

In the opposite case one can show, as for the binomial series, that the general term in the series on the right-hand side of (38) increases indefinitely in absolute value for $|x| > 1$.

On putting $x = \frac{1}{2}$, for example, in (38) we obtain a new expression for the number π;

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} \frac{1}{(2n+1)2^{2n+1}}$$

which is much better suited than formula (34) to calculating approximations to π (see *Calcul numérique*); one thus obtains

$$\pi = 3.141\,592\,653 \ldots$$

accurate to within $1/10^9$. [3]

[3] The number π is not only *irrational* (*cf.* III, p. 126, exerc. 5) but even *transcendental* over the field $\mathbf{Q}$ of rational numbers, as was shown for the first time in 1882 by LINDEMANN (see for example D. HILBERT, *Gesammelte Abhandlungen,* v. 1, p. 1, Berlin (Springer), 1932).

EXERCISES

§ 1.

1) Let $\mathbf{f}$ be a continuous and differentiable map of $\mathbf{R}$ into a complete normed algebra E over $\mathbf{R}$ having a unit element $\mathbf{e}$; suppose that $\mathbf{f}(0) = \mathbf{e}$ and that, identically, $\mathbf{f}'(x) = \mathbf{f}(x)\mathbf{c}$ where $\mathbf{c}$ is an invertible element of E. Show that $\mathbf{f}$ is a homomorphism of the additive group $\mathbf{R}$ into the multiplicative group of invertible elements of E. (Consider the largest open interval I containing 0 such that $\mathbf{f}(z)$ is invertible for all $z \in \mathrm{I}$, and prove that for x, y and $x + y$ in I one has $\mathbf{f}(x + y)\,(\mathbf{f}(x)\,\mathbf{f}(y))^{-1} = \mathbf{e}$ keeping x fixed and letting y vary; then deduce that $\mathrm{I} = \mathbf{R}$.)

2) *a*) In order that the function $(1 + 1/x)^{x+p}$ should be decreasing (resp. increasing) for $x > 0$ it is necessary and sufficient that $p \geqslant \frac{1}{2}$ (resp. $p \leqslant 0$); for $0 < p < \frac{1}{2}$ the function is decreasing on an interval $]0, x_0[$ and increasing on $]x_0, +\infty[$. In every case the function tends to e as x tends to $+\infty$.

b) Study in the same way the functions $(1-1/x)^{x-p}$, $(1+1/x)^x(1+p/x)$ and $(1+p/x)^{x+1}$ for $x > 0$.

3) *a*) Show that $(1 + x)^\alpha \geqslant 1 + \alpha x$ for $\alpha \geqslant 1$ and $x \geqslant 0$, and that $(1 + x)^\alpha \leqslant 1 + \alpha x$ for $0 \leqslant \alpha \leqslant 1$ and $x \geqslant 0$.

b) Show that $(1 - x)^\alpha \leqslant \dfrac{1}{1 + \alpha x}$ for $0 \leqslant x \leqslant 1$ and $\alpha \geqslant 0$.

c) For $0 \leqslant \alpha \leqslant 1$ show that for $x \geqslant 0$ and $y \geqslant 0$ one has $x^\alpha y^{1-\alpha} \leqslant ax + by$ for every pair of numbers > 0 for which $a^\alpha b^{1-\alpha} = \alpha(1 - \alpha)^{1-\alpha}$.

d) Deduce from *c*) that if f, g are two regulated $\geqslant 0$ functions on an interval I such that the integrals $\int_{\mathrm{I}} f(t)\,dt$ and $\int_{\mathrm{I}} g(t)\,dt$ converge, then the integral $\int_{\mathrm{I}} (f(t))^\alpha\,(g(t))^{1-\alpha}\,dt$ converges, and

$$\int_{\mathrm{I}} \bigl(f(t)\bigr)^\alpha \bigl(g(t)\bigr)^{1-\alpha} dt \leqslant a \int_{\mathrm{I}} f(t)\,dt + b \int_{\mathrm{I}} g(t)\,dt.$$

From this deduce the *Hölder inequality*

$$\int_{\mathrm{I}} \bigl(f(t)\bigr)^\alpha \bigl(g(t)\bigr)^{1-\alpha} dt \leqslant \left(\int_{\mathrm{I}} f(t)\,dt\right)^\alpha \left(\int_{\mathrm{I}} g(t)\,dt\right)^{1-\alpha}$$

(which is called the *"Cauchy-Buniakowski-Schwarz inequality"* when $\alpha = \frac{1}{2}$).

4) From the Cauchy-Schwarz inequality deduce that for every function u which is a primitive of a regulated function on $[a, b]$, and every continuous function $h > 0$ on $[a, b]$, one has

$$\left|u(b)^2 - u(a)^2\right| \leqslant 2\left(\int_a^b u'^2(t)\,h(t)dt\right)^{1/2}\left(\int_a^b \frac{u^2(t)}{h(t)}\,dt\right)^{1/2}$$

if the two integrals on the right-hand side converge.

By taking $a = 0,\ b = \pi/2,\ h(t) = 1$ and

$$u(t) = c_1 \cos t + c_2 \cos 3t + \cdots + c_n \cos(2n-1)t,$$

where $c_j \geqslant 0$ for every j, deduce the *Carlson inequality*

$$\sum_j c_j \leqslant \sqrt{\pi}\left(\sum_j c_j^2\right)^{1/4}\left(\sum_j \left(j - \tfrac{1}{2}\right)^2 c_j^2\right)^{1/4}.$$

5) For $x > 0$ and any real y show that

$$xy \leqslant \log x + e^{y-1}$$

(*cf.* II, p. 82, exerc. 15); in what circumstances are the two sides equal?

6) Denote by A_n and G_n the usual arithmetic and geometric means of the first n numbers of a sequence (a_k) $(k \geqslant 1)$ of positive numbers. Show that

$$n\Big(\mathrm{A}_n - \mathrm{G}_n\Big) \leqslant (n+1)\Big(\mathrm{A}_{n+1} - \mathrm{G}_{n+1}\Big)$$

with equality only if $a_{n+1} = \mathrm{G}_n$ (put $a_{n+1} = x^{n+1},\ \mathrm{G}_n = y^{n+1}$).

7) With the notation of exerc. 6 show that if one puts $x_i = a_i/\mathrm{A}_n$ the relation $\mathrm{G}_n \geqslant (1-\alpha)\mathrm{A}_n$ for $0 \leqslant \alpha < 1$ implies, for $1 \leqslant i \leqslant n$,

$$x_i\left(1 - \frac{x_i - 1}{n-1}\right)^{n-1} \geqslant (1-\alpha)^n.$$

Deduce that for each index i one has $1 + x' \leqslant x_i \leqslant 1 + x''$, where x' is the root $\leqslant 0$, and x'' the root $\geqslant 0$, of the equation $(1+x)e^{-x} = (1-\alpha)^n$ (*cf.* III, p. 115, exerc. 2).

8) Consider a sequence of sets D_k $(k \geqslant 1)$ and for each k consider two functions $(x_1, \ldots, x_k) \mapsto f_k(x_1, \ldots, x_k)$, $(x_1, \ldots, x_k) \mapsto g_k(x_1, \ldots, x_k)$ with real values, where $(x_1, \ldots, x_k) \in \mathrm{D}_1 \times \mathrm{D}_2 \times \cdots \times \mathrm{D}_k$. Suppose that for every k there is a real function F_k defined on $\mathbf{R}$ such that for every $\mu \in \mathbf{R}$ and arbitrary $a_1 \in \mathrm{D}_1, \ldots, a_{k-1} \in \mathrm{D}_{k-1}$ one has

$$\sup_{x_k \in \mathrm{D}_k}\Big(\mu f_k(a_1, \ldots, a_{k-1}, x_k) - g_k(a_1, \ldots, a_{k-1}, x_k)\Big) = \mathrm{F}_k(\mu)\, f_{k-1}(a_1, \ldots, a_{k-1}).$$

(By convention one takes $f_0 = 1$.) Show if the sequences of real numbers (μ_k) and (δ_k) satisfy the conditions $\mathrm{F}_1(\mu_1) = 0$ and $\mathrm{F}_k(\mu_k) = \delta_k$ then one has the inequalities

$$(\mu_1 - \delta_2)f_1 + (\mu_2 - \delta_3)f_2 + \cdots + (\mu_{n-1} - \delta_n)f_{n-1} + \mu_n f_n \leqslant g_1 + g_2 + \cdots + g_n$$

for all values of n and of $x_k \in \mathrm{D}_k$.

9) With the notation of exerc. 6 show that

$$\mu_1 G_1 + \mu_2 G_2 + \cdots + \mu_n G_n \leqslant \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n$$

provided that $\lambda_k \geqslant 0$ for each k and

$$\mu_k = k\Big((\lambda_k \beta_k)^{1/k} - \beta_{k+1}^{1/k}\Big) \qquad \text{for} \quad 1 \leqslant k \leqslant n$$

with $\beta_k \geqslant 0$ for all k, $\beta_1 \leqslant 1$ and $\beta_{n+1} = 0$ (method of exerc. 8).

In particular, for suitable choices of λ_k and β_k one has

$$G_1 + G_2 + \cdots + G_n + nG_n \leqslant 2a_1 + \left(\frac{3}{2}\right)^2 a_2 + \cdots + \left(\frac{n+1}{n}\right)^n a_n$$

and in particular (*Carleman's inequality*)

$$G_1 + G_2 + \cdots + G_n < e(a_1 + a_2 + \cdots + a_n);$$
$$eA_n \geqslant \Gamma_n e^{G_n/\Gamma_n}$$

where $\Gamma_n = (G_1 + G_2 + \cdots + G_n)/n$ (strengthening of the Carleman inequality);

$$\frac{A_n}{G_n} \geqslant \frac{1}{2}\left(\frac{\Gamma_n}{G_n} + \frac{G_n}{\Gamma_n}\right)$$

if the sequence (a_k) is increasing;

$$nG_n - mG_m \leqslant a_{m+1} + a_{m+2} + \cdots + a_n.$$

10) The sequence $(a_k)_{k\geqslant 1}$ being formed of numbers > 0, we put $s_n = nA_n = a_1 + \cdots + a_n$. For $p < 1$ and $\lambda_k > 0$ one has

$$\mu_1 s_1^{1/p} + \mu_2 s_2^{1/p} + \cdots + \mu_n s_n^{1/p} \leqslant \lambda_1 a_1^{1/p} + \lambda_2 a_2^{1/p} + \cdots + \lambda_n a_n^{1/p}$$

provided that $\mu_1 \geqslant \lambda_1$ and, for $k \geqslant 2$, $\mu_k = \left(\lambda_k^q + \delta_k^q\right)^{1/\delta} - \delta_{k+1}$, with $q = p/(1-p)$, $\delta_k \geqslant 0$ for every k, and $\delta_{n+1} = 0$ (method of exerc. 8). In particular, one has the inequalities

$$\sum_{k=1}^{n} \Big((k-p)^{1/q} - k^{1/q}\Big) s_k^{1/q} + n^{1/q} s_n^{1/q} \leqslant (1-p)^{1/q} \sum_{k=1}^{n} a_k^{1/q}$$

which imply (with $A_n = s_n/n$)

$$\sum_{k=1}^{n} A_k^{1/p} + \frac{n}{1-p} A_n^{1/p} \leqslant (1-p)^{-1/p} \sum_{k=1}^{n} a_k^{1/p}$$

(strengthened *Hardy inequality*);

$$\frac{1}{q} \sum_{k=1}^{n} (b_k - 1) A_k^{1/p} + nA_n^{1/p} \leqslant \sum_{k=1}^{n} a_k^{1/p} b_k^{1/p}$$

provided that $b_k \geqslant 1$ for all k;

$$\alpha(a_1^2 + a_2^2 + \cdots + a_n^2) \leqslant a_1^2 + (a_2 - a_1)^2 + \cdots + (a_n - a_{n-1})^2 + \beta a_n^2$$

if $\alpha = 2(1 - \cos\theta)$ and $\beta = 1 - \dfrac{\sin(n+1)\theta}{\sin n\theta}$ for some θ such that $0 \leqslant \theta < \dfrac{\pi}{n}$.

Case when $\theta = \dfrac{\pi}{n+1}$; case when $\theta = \dfrac{\pi}{2n+1}$.

11) Let a_{ij} $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n)$ be numbers $\geqslant 0$ such that for each index i one has $\sum_{j=1}^{n} a_{ij} = 1$ and, for each index j one has $\sum_{i=1}^{n} a_{ij} = 1$. Let x_i $(1 \leqslant i \leqslant n)$ be n numbers > 0; put

$$y_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \qquad (1 \leqslant i \leqslant n).$$

Show that $y_1 y_2 \ldots y_n \geqslant x_1 x_2 \ldots x_n$ (bound $\log y_i$ below for each index i).

12) *a*) Let $\sum_{i,j} c_{ij}\, x_i \bar{x}_j$ where $c_{ji} = \bar{c}_{ij}$ be a nondegenerate positive hermitian form, and Δ its determinant; show that $\Delta \leqslant c_{11} c_{22} \ldots c_{nn}$ (express Δ and the c_{ii} in terms of the eigenvalues of the hermitian form, and use prop. 2).

b) Deduce that if (a_{ij}) is a square matrix of order n with arbitrary complex elements and Δ is its determinant, one has ("Hadamard's inequality")

$$|\Delta|^2 \leqslant \left(\sum_{j=1}^{n} |a_{1j}|^2\right)\left(\sum_{j=1}^{n} |a_{2j}|^2\right)\cdots\left(\sum_{j=1}^{n} |a_{nj}|^2\right)$$

with equality only if one of the terms on the right-hand side is zero or if, for all distinct indices h, k

$$a_{h1}\bar{a}_{k1} + a_{h2}\bar{a}_{k2} + \cdots + a_{hn}\bar{a}_{kn} = 0$$

(multiply the matrix (a_{ij}) by the conjugate of its transpose).

13) If x, y, a, b are > 0, show that

$$x \log\frac{x}{a} + y \log\frac{y}{b} \geqslant (x+y)\log\frac{x+y}{a+b}$$

with equality only if $x/a = y/b$.

14) Let a be a real number such that $0 < a < \pi/2$. Show that the function

$$\frac{\dfrac{\tan x}{x} - \dfrac{\tan a}{a}}{x\tan x - a\tan a}$$

is strictly increasing on the interval $]a, \pi/2[$ (*cf.* I, p. 39, exerc. 11).

15) Let u and v be two polynomials in x, with real coefficients, such that $\sqrt{1-u^2} = v\sqrt{1-x^2}$ identically; show that, if n is the degree of u, then $u' = nv$; deduce that $u(x) = \cos(n \operatorname{Arc\,cos} x)$.

16) Show (by induction on n) that

$$\mathrm{D}^n\Big(\operatorname{Arc\,tan} x\Big) = (-1)^{n-1}\frac{(n-1)!}{(1+x^2)^{n/2}}\sin\left(n \operatorname{Arc\,tan}\frac{1}{x}\right).$$

¶ 17) Let f be a real function defined on an open interval $\mathrm{I} \subset \mathbf{R}$, and such that for every system of three points x_1, x_2, x_3 of I satisfying $x_1 < x_2 < x_3 < x_1 + \pi$ one has

$$f(x_1)\sin(x_3 - x_2) + f(x_2)\sin(x_1 - x_3) + f(x_3)\sin(x_2 - x_1) \geqslant 0. \tag{1}$$

Show that:

a) f is continuous at every point of I and has a finite right and a finite left derivative at every point of I; further,

$$f(x)\cos(x - y) - f'_d(x)\sin(x - y) \leqslant f(y) \tag{2}$$

for every pair of points x, y of I such that $|x - y| \leqslant \pi$; there is also an analogue of (2) where one replaces f'_d by f'_g; finally, one has $f'_g(x) \leqslant f'_d(x)$ for every $x \in \mathrm{I}$. (To show (2) let x_2 approach x_1 in (1), keeping x_3 fixed, and so obtain an upper bound for $\limsup\limits_{x_2 \to x_1} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$; then let x_3 approach x_1 in the inequality so obtained; deduce from this the existence of $f'_d(x)$ and the inequality (2) for $y > x$; proceed similarly for the other parts.)

b) Conversely, if (2) holds for every pair of points x, y of I such that $|x - y| \leqslant \pi$, then f satisfies (1) on I (consider the difference $\dfrac{f(x_2)}{\sin(x_3 - x_2)} - \dfrac{f(x_1)}{\sin(x_3 - x_1)}$).

c) If f admits a second derivative on I, then (1) is equivalent to the condition

$$f(x) + f''(x) \geqslant 0 \tag{3}$$

for every $x \in \mathrm{I}$.

*Interpret these results, considering the plane curve defined by $x = \dfrac{1}{f(t)}\cos t$, $y = \dfrac{1}{f(t)}\sin t$ ("convexity with respect to the origin").$_*$

18) *a*) Show that in the vector space of maps from $\mathbf{R}$ into $\mathbf{C}$, the distinct functions of the form $x^n e^{\alpha x}$ (n an integer, α arbitrary complex) form a free system (argue by contradiction, considering a relation among these functions having the least possible number of coefficients $\neq 0$, and differentiating this relation).

b) Let $f(\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_m)$ be a polynomial in m indeterminates, with complex coefficients, such that when one substitutes the function $\cos\left(\sum\limits_{k=1}^{n} p_{jk}x_k\right)$ for X_j for $1 \leqslant j \leqslant r$, and the function $\sin\left(\sum\limits_{k=1}^{n} p_{jk}x_k\right)$ for $r + 1 \leqslant j \leqslant m$, where the p_{jk} are real and the x_k are *real,* one obtains an identically zero function of the x_k; show that the same identity holds when one gives the x_k arbitrary *complex* values (use *a*)).

¶ 19) One knows (A, IX, §10, exerc. 2) that in the *complex plane* $\mathbf{C}^2$ the group A of angles of directed lines is isomorphic to the orthogonal group $\mathbf{O}_2(\mathbf{C})$, the canonical isomorphism associating the angle θ with rotation through the angle θ; one transports the topology of $\mathbf{O}_2(\mathbf{C})$ (considered as a subspace of the space $\mathbf{M}_2(\mathbf{C})$ of matrices of order 2 over $\mathbf{C}$) to the group A by this isomorphism, so making A a locally compact topological group;

further, the map $\theta \mapsto \cos\theta + i\sin\theta$ is an *isomorphism* of the topological group A onto the multiplicative group $\mathbf{C}^*$ of complex numbers $\neq 0$, the inverse isomorphism being defined by the formulae $\cos\theta = \dfrac{1}{2}(z + 1/z)$, $\sin\theta = \dfrac{1}{2i}(z - 1/z)$. Deduce from these relations that every continuous homomorphism $z \mapsto \varphi(z)$ of the additive group $\mathbf{C}$ onto A such that the complex functions $\cos\varphi(z)$, $\sin\varphi(z))$ are differentiable on $\mathbf{C}$, is defined by the relations $\cos\varphi(z) = \cos az$, $\sin\varphi(z) = \sin az$ (a a complex number).

20) Let D be the subset of $\mathbf{C}$ which is the union of the set defined by $-\pi < \mathcal{R}(z) \leqslant \pi$, $\mathcal{I}(z) > 0$, and the segment $\mathcal{I}(z) = 0$, $0 \leqslant \mathcal{R}(z) \leqslant \pi$. Show that the restriction of the function $\cos z$ to D is a bijection of D onto $\mathbf{C}$; the restriction of $\cos z$ to the interior of D is a homeomorphism of this open set onto the complement, in $\mathbf{C}$, of the half-line $y = 0$, $x \leqslant 1$.

21) Let f and g be two relatively prime polynomials (with complex coefficients), the degree of f being strictly less than that of g. Let p be the gcd of g and its derivative g', and let q be the quotient of g by p; show that there are two uniquely determined polynomials u, v whose degrees are, respectively, strictly less than those of p and q, such that

$$\frac{f}{g} = \mathrm{D}\left(\frac{u}{p}\right) + \frac{v}{q}.$$

Deduce that the coefficients of u and v belong to the smallest field (over $\mathbf{Q}$) containing the coefficients of f and g and contained in $\mathbf{C}$.

22) Let f and g be two relatively prime polynomials (with complex coefficients), the degree of f being strictly less than that of g. Let K be a subfield of $\mathbf{C}$ containing the coefficients of f and g, and such that g is irreducible over K. For there to be a primitive of f/g of the form $\sum_i a_i \log u_i$, where the a_i are constants belonging to K and the u_i are irreducible polynomials over K, it is necessary and sufficient that $f = cg'$, where c is a constant in K.

23) If $f(x, y)$ is an arbitrary polynomial in x, y, with complex coefficients, show that the evaluation of a primitive of $f(x, \log x)$ and of $f(x, \operatorname{Arc}\sin x)$ reduces to the evaluation of a primitive of a rational function.

24) Show that one can reduce the evaluation of a primitive of $(ax + b)^p x^q$ (p and q *rational*) to the evaluation of a primitive of a rational function when one of the numbers $p, q, p + q$ is an integer (positive or negative).

* 25) The meromorphic functions on an open disc Δ in $\mathbf{C}$ form a *field* $\mathrm{M}(\Delta)$. A subfield F of $\mathrm{M}(\Delta)$ such that $u \in \mathrm{F}$ implies $\mathrm{D}u \in \mathrm{F}$ is called a *differential* subfield of $\mathrm{M}(\Delta)$.

a) For every polynomial $\mathrm{P(X)} = \mathrm{X}^m + a_1\mathrm{X}^{m-1} + \cdots + a_m$ with coefficients in $\mathrm{M}(\Delta)$ show that there is an open disc $\Delta_1 \subset \Delta$ (not having the same centre as Δ in general) and a function $f \in \mathrm{M}(\Delta_1)$ satisfying

$$\Big(f(z)\Big)^m + a_1(z)\Big(f(z)\Big)^{m-1} + \cdots + a_m(z) = 0$$

at all the points $z \in \Delta_1$ where f and the a_j are holomorphic. If F is a subfield of $\mathrm{M}(\Delta)$ containing the a_j one says that the subfield of $\mathrm{M}(\Delta_1)$ generated by f and the restrictions to Δ_1 of the functions of F is the field $F(f)$ obtained by *adjoining the root f of* P *to* F. This abuse of language does not cause confusion because the restriction map $g \mapsto g|\Delta_1$ of F onto the subfield $M(\Delta_1)$ is injective. If F is differential, so is $F(f)$.

b) For every function $a \in M(\Delta)$ show that there is an open disc $\Delta_2 \subset \Delta$ such that there is a function $g \in M(\Delta_2)$ satisfying the relation $g'(z) = a(z)$ at every point $z \in \Delta_2$ where g and a are holomorphic. If F is a subfield of $M(\Delta)$ containing a one says that the subfield $F(g)$ of $M(\Delta_2)$ generated by g and the restrictions to Δ_2 of the functions of F is the field obtained by *adjoining the primitive* $\int a\,dz$ *to* F. If F is differential, so is $F(g)$.

c) For every function $b \in M(\Delta)$ show that there is an open disc $\Delta_3 \subset \Delta$ such that there is an $h \in M(\Delta_3)$ satisfying the relation $h'(z) = b(z)h(z)$ at every point $z \in \Delta_3$ where h and b are holomorphic and $\neq 0$. If F is a subfield of $M(\Delta)$ containing b, one says that the subfield $F(h)$ of $M(\Delta_3)$ generated by h and the restrictions to Δ_3 of the functions of F is the field obtained by *adjoining the exponential of the primitive* $\exp\left(\int b\,dz\right)$ *to* F. If F is differential, so is $F(h)$.

d) For every subfield F of $M(\Delta)$ and every polynomial $P(X) = X^m + a_1X^{m-1} + \cdots + a_m$ with coefficients in F there is a field K obtained by adjoining to F successively the roots of polynomials such that K is a Galois extension of F and that one has $P(X) = (X-c_1)\ldots(X-c_m)$, where the c_i are meromorphic functions (on a suitable disc) belonging to K. If F is differential one has $(\sigma.g)' = \sigma.g'$ for every $g \in K$ and every element σ of the Galois group of K over F.$_*$

* 26) *a*) Let $F \subset M(\Delta)$ be a differential field and let K be a finite Galois extension of F, a subfield of a $M(\Delta_1)$. Let t be a primitive or an exponential of a primitive of a function of F; show that if t is transcendental over F there is no function $u \in K$ such that $t' = u'$. (Consider separately the two cases where $t' = a \in F$ or $t' = bt$ with $b \in F$; obtain a contradiction by considering the transforms $\sigma.u$ of u by the Galois group of K over F and their derivatives $\sigma.u'$; in the first case show that one would have $t' = c'$ for some $c \in F$, and in the second case, putting $N = [K : F]$, one would have $(t^N/c)' = 0$ for some $c \in F$.)

b) Suppose that t is transcendental over F and that $t' = bt$ with $b \in F$; show that there does not exist any $c \neq 0$ in K such that $(ct^m)' = 0$ (same method).$_*$

* 27) Let $F \subset M(\Delta)$ be a differential field containing **C**, and t a primitive or an exponential of a primitive of a function in F, and suppose that t is transcendental over F. Further, let $c_1, \ldots, c_n$ be elements of F which are linearly independent over the field of rational numbers **Q**; so if $u_1, \ldots, u_n$ and v are elements of the field $F(t)$ and if

$$\sum_{j=1}^{n} c_j \frac{u_j'}{u_j} + v'$$

belongs to the *ring* $F[t]$, then necessarily $v \in F[t]$. Further, if $t \in F$, one necessarily has $u_j \in F$ for $1 \leqslant j \leqslant n$; if $t'/t \in F$ there exists for each j an integer $\nu_j \geqslant 0$ such that $u_j/t^{\nu_j} \in F$. (Decompose the u_j'/u_j and v into simple elements in a suitable Galois extension of F and use exerc. 26).$_*$

* 28) *a*) Let $F \subset M(\Delta)$ be a differential field. One says that a field $F' \supset F$ is an *elementary extension* of F if there is a finite sequence

$$F = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = F' \tag{1}$$

such that for every $j \leqslant n-1$ one has $F_{j+1} = F_j(t_j)$, where one has either $t_j' = a_j'/a_j$ for an $a_j \neq 0$ in F_j (so that one can write $t_j = \log a_j$), or $t_j'/t_j = a_j'$ for an $a_j \in F_j$

(so that $t_j = \exp(a_j)$). An *elementary function* is a function that belongs to an elementary extension of $\mathbf{C}(z)$ (the field of rational functions over $\mathbf{C}$).

For example, the functions

$$\operatorname{Arc}\sin z, \qquad (\log z)^{\log z}, \qquad \left(1 - \exp\left(\frac{1}{\exp\left(\frac{1}{z}\right) - 1}\right)\right)^{\alpha} \qquad (\alpha \in \mathbf{C})$$

are elementary functions.

b) Let $a \in \mathrm{F}$ be such that there are elements u_j $(1 \leqslant j \leqslant m)$ and v in an elementary extension F' of F, and constants $\gamma_j \in \mathbf{C}$ $(1 \leqslant j \leqslant m)$ such that

$$a = \sum_{j=1}^{m} \gamma_j \frac{u_j'}{u_j} + v'. \tag{$*$}$$

Show that then there are elements f_j $(1 \leqslant j \leqslant p)$ and g in F and constants $\beta_j \in \mathbf{C}$ $(1 \leqslant j \leqslant p)$ such that

$$a = \sum_{j=1}^{p} \beta_j \frac{f_j'}{f_j} + g'. \tag{$**$}$$

(Reduce, by induction on the length n of the sequence (1), to the case where $\mathrm{F}' = \mathrm{F}(t)$. By modifying the u_j first show that one can assume the γ_j to be linearly independent over $\mathbf{Q}$. When t is transcendental over F use exerc. 27: if $t' = s'/s$ with $s \in \mathrm{F}$ one must have $u_j \in \mathrm{F}$ and $c \in \mathrm{F}[t]$; by using $(*)$ and arguing by contradiction show that one must have $v = \alpha t + b$ with $\alpha \in \mathbf{C}$ and $b \in \mathrm{F}$. If $t'/t = r'$ with $r \in \mathrm{F}$ show that by replacing v by $v + k.r$ for a suitable integer $k \in \mathbf{Z}$, one can assume that $u_j \in \mathrm{F}$, $v \in \mathrm{F}[t]$; show by contradiction that v is of degree 0, so $v \in \mathrm{F}$. Finally, if t is algebraic over F, embed F' in a Galois extension K of F and consider the transforms of $(*)$ by the Galois group of K over F.)$_*$

* 29) Let f and g be two rational functions of z (elements of $\mathbf{C}(z)$). Show that, for the primitive $\int f(z)\, e^{g(z)}\, dz$ to be an elementary function (exerc. 28) it is necessary and sufficient that there exists a rational function $r \in \mathbf{C}(z)$ such that $f = r' + rg'$. (Put $t = e^g$ and consider the differential field $\mathrm{F} = \mathbf{C}(z, t)$, which is an elementary extension of $\mathbf{C}(z)$ since $t'/t = g'$. Show first that t is transcendental over $\mathbf{C}(z)$; arguing by contradiction, consider a Galois extension K of $\mathbf{C}(z)$ containing t, and the transforms of the equation $t'/t = g'$ by the Galois group of K; one will obtain an equation $g' = u'/u$ with $u \in \mathbf{C}(z)$, and it is impossible for g' to have other than simple poles in $\mathbf{C}$. Applying exerc. 28 one has $ft = \sum_j \gamma_j \dfrac{u_j'}{u_j} + v'$, with $\gamma_j \in \mathbf{C}$, and the u_j and v in F; one can reduce to the case where the γ_j are linearly independent over $\mathbf{Q}$. Then, applying exerc. 27 to the extension $\mathbf{C}(z)(t)$, show that one has $ft = v' + h$, with $h \in \mathbf{C}(z)$ and $v \in \mathbf{C}(z)[t]$. Conclude that v is necessarily of degree 1 in t, and consequently f is equal to the coefficient of t in v'.)

Deduce from this that the primitives $\int e^{z^2}\, dz$ and $\int e^z\, dz/z$ are not elementary functions (*a theorem of Liouville*), by examining in the relation $f = r' - rg'$ the behaviour of the two sides at a neighbourhood of a pole of r.$_*$

30) *a*) If m and n are two integers such that $0 < m < n$ show that

$$\int_0^{+\infty} \frac{x^{m-1}}{1+x^n}\,dx = \frac{\pi}{n \sin \dfrac{m\pi}{n}}.$$

b) Show that for $0 < a < 1$ the integral $\displaystyle\int_0^{+\infty} \frac{x^{a-1}}{1+x}\,dx$ is uniformly convergent for a varying in a compact interval, and deduce from *a*) that

$$\int_0^{+\infty} \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin a\pi}.$$

31) If $\mathrm{I}_{m,n}$ is a primitive of $\sin^m x \cos^n x$ (m and n are arbitrary real numbers), show that

$$\mathrm{I}_{m+2,n} = -\frac{\sin^{m+1} x \cos^{n+1} x}{m+n+2} + \frac{m+1}{m+n+2}\mathrm{I}_{m,n}$$

is a primitive of $\sin^{m+2} x \cos^n x$ if $m+n+2 \neq 0$.

With the aid of this formula recover the formula (29) of III, p. 100 and prove the formula

$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2.4.6\ldots 2n}{1.3.5\ldots(2n+1)} \qquad (n \text{ an integer} \geqslant 0).$$

32) Prove *Wallis' Formula*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}.\frac{2.4.6\ldots 2n}{1.3.5\ldots(2n-1)} = \sqrt{\pi}$$

by using exerc. 31 and the inequality $\sin^{n+1} x \leqslant \sin^n x$ for $0 \leqslant x \leqslant \pi/2$.

33) *a*) Calculate the integrals

$$\int_0^1 \left(1-x^2\right)^n dx, \qquad \int_0^{+\infty} \frac{dx}{\left(1+x^2\right)^n}$$

for n an integer > 0, with the help of exerc. 31.

b) Show that one has

$$1-x^2 \leqslant e^{-x^2} \qquad \text{for } 0 \leqslant x \leqslant 1$$
$$e^{-x^2} \leqslant \frac{1}{1+x^2} \qquad \text{for } x \geqslant 0.$$

c) Deduce from *a*) and *b*) and from Wallis' formula (exerc. 32) that

$$\int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

34) *a*) Show that for $\alpha > 0$ the derivative of

$$\mathrm{I}(\alpha) = \int_0^{+\infty} e^{-\alpha x} \frac{\sin x}{x}\, dx$$

is equal to $-\displaystyle\int_0^{+\infty} e^{-\alpha x} \sin x\, dx$.

b) Deduce that $\displaystyle\int_0^{+\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

35) By differentiating with respect to the parameter and using exerc. 33 *c*), prove the formulae

$$\int_0^{+\infty} e^{-x^2} \cos \alpha x\, dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2}, \qquad \int_0^{+\infty} \frac{1 - e^{-\alpha x^2}}{x^2}\, dx = \sqrt{\pi\,\alpha}$$

$$\int_0^{+\infty} \exp\left(-x^2 - \frac{\alpha^2}{x^2}\right) dx = \frac{\sqrt{\pi}}{2} e^{-2\alpha} \qquad (\alpha > 0).$$

36) Deduce from exerc. 33 *c*) above, and from II, p. 88, exerc. 9, that

$$\int_0^{+\infty} \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

37) Let $\mathbf{f}$ be a regulated vector function on $]0, 1[$, such that the integral $\int_0^{\pi} \mathbf{f}(\sin x)\, dx$ is convergent. Show that the integral $\int_0^{\pi} x\, \mathbf{f}(\sin x)\, dx$ is convergent and that

$$\int_0^{\pi} x\, \mathbf{f}(\sin x)\, dx = \frac{\pi}{2} \int_0^{\pi} \mathbf{f}(\sin x)\, dx.$$

38) Let $\mathbf{f}$ be a regulated vector function for $x \geqslant 0$, continuous at the point $x = 0$, and such that the integral $\displaystyle\int_a^{+\infty} \mathbf{f}(x)\, dx/x$ converges for $a > 0$. Show that for $a > 0$ and $b > 0$ the integral $\displaystyle\int_0^{+\infty} \frac{\mathbf{f}(ax) - \mathbf{f}(bx)}{x}\, dx$ converges and is equal to $\mathbf{f}(0) \log a/b$.

39) Let m be a convex function on $[0, +\infty[$ such that $m(0) = 0$, and p a number such that $-1 < p < +\infty$. Show that the integral $\displaystyle\int_0^{+\infty} x^p \exp(-m'_r(x))\, dx$ is convergent, as is the integral $\displaystyle\int_0^{+\infty} x^p \exp(-m(x)/x)\, dx$, and that

$$\int_0^{+\infty} x^p \exp(-m(x)/x)\, dx \leqslant e^{p+1} \int_0^{+\infty} x^p \exp(-m'_r(x))\, dx.$$

(For $k > 1$ and $A > 0$ note that $m(kx) \geqslant m(x) + (k-1)x\, m_r'(x)$, and deduce the inequality

$$k^{-p-1} \int_0^{kA} x^p \exp(-m(x)/x)\, dx \leqslant \int_0^A x^p \exp\left(-\frac{m(x)}{kx} - \frac{k-1}{k} m_r'(x)\right) dx.$$

Estimate the second integral with the help of Hölder's Inequality (III, p. 115, exerc. 3) then let A tend to $+\infty$ and k tend to 1.)

§ 2.

1) Let $\mathbf{f}$ be a vector function that is n times differentiable on an interval $\mathrm{I} \subset \mathbf{R}$. Prove the formula

$$\mathrm{D}^n\, \mathbf{f}\left(e^x\right) = \sum_{m=1}^{n} \frac{a_m}{m!} e^{mx}\, \mathbf{f}^{(m)}\left(e^x\right)$$

at every point x such that $e^x \in \mathrm{I}$, where the coefficient a_m can be expressed as

$$a_m = \sum_{p=0}^{m} (-1)^p \binom{m}{p} (m-p)^n$$

(method of I, p. 41, exerc. 7, using the Taylor expansion for e^x).

2) Let f be a real function that is n times differentiable at a point x, and $\mathbf{g}$ a vector function n times differentiable at the point $f(x)$. If one puts $\mathrm{D}^n\,(\mathbf{g}(f(x))) = \sum\limits_{k=1}^{n} \mathbf{g}^{(k)}(f(x))\, u_k(x)$, then u_k depends on the function f alone; deduce from this that $u_k(x)$ is the coefficient of t^k in the expansion of $e^{-tf(x)}\mathrm{D}^n(e^{tf(x)})$ (in terms of t).

3) For every *real* $x > 0$ and every *complex* $m = \mu + i\nu$ one puts $x^m = e^{m \log x}$; show that the formula (19) of III, p. 108 remains valid for m complex and $x > -1$ and that the remainder $r_n(x)$ in this formula satisfies the inequalities

$$|r_n(x)| \leqslant \left| \binom{m-1}{n} \frac{m}{\mu} x^n \left[(1+x)^\mu - 1\right] \right| \qquad \text{if } \mu \neq 0$$

$$|r_n(x)| \leqslant \left| \binom{m-1}{n} m x^n \log(1+x) \right| \qquad \text{if } \mu = 0.$$

Generalize the study of the convergence of the binomial series to the case where m is complex.

4) For every real x and every number $p > 1$ prove the inequality

$$|1+x|^p \leqslant 1 + px + \frac{p(p-1)}{2} x^2 + \cdots + \frac{p(p-1)\ldots(p-m+2)}{(m-1)!} x^{m-1} + \frac{p(p-1)\ldots(p-m+1)}{m!} |x|^m + h_p |x|^p$$

where one puts $m = [p]$ (the integer part of p) and

$$h_p = \frac{p(p-1)\ldots(p-m+1)}{(m-1)!} \int_0^1 z^{p-m}(1-z)^{m-1}\, dz.$$

¶ 5) Show that π is irrational, in the following way: if one had $\pi = p/q$ (p and q integers), then on putting $f(x) = (x(\pi - x))^n/n!$, the integral $q^n \int_0^\pi f(x) \sin x \, dx$ would be an integer > 0 (use the formula for integration by parts of order $n + 1$); but show that on the other hand $q^n \int_0^\pi f(x) \sin x \, dx$ tends to 0 as n tends to $+\infty$.

6) Show that on the interval $[-1, +1]$ the function $|x|$ is the uniform of limit of polynomials, by remarking that $|x| = (1 - (1 - x^2))^{1/2}$ and using the binomial series. Deduce another proof of the Weierstrass th. from this (*cf.* II, p. 83, exerc. 20).

¶ 7) Let p be a prime number and $\mathbf{Q}_p$ the field of p-adic numbers (*Gen. Top.*, III, p. 322 and 323, exercs. 23 to 25), let $\mathbf{Z}_p$ be the ring of p-adic integers, and $\mathfrak{p}$ the principal ideal (p) in $\mathbf{Z}_p$.

a) Let $a = 1 + pb$, where $b \in \mathbf{Z}_p$ is an element of the multiplicative group $1 + \mathfrak{p}$; show that when the rational integer m increases indefinitely the p-adic number $\dfrac{(1+pb)^{p^m} - 1}{p^m}$ tends to a limit equal to the sum of the convergent series

$$\frac{pb}{1} - \frac{p^2b^2}{2} + \cdots + (-1)^{n-1}\frac{p^nb^n}{n} + \cdots$$

One denotes this limit by $\log a$.

b) Show that when the $p-$adic number x tends to 0 in $\mathbf{Q}_p$ the number $\dfrac{a^x - 1}{x}$ tends to $\log a$ (use *a*) and the definition of the topology of $\mathbf{Q}_p$).

c) Show that if $p \neq 2$ one has $\log a \equiv pb \pmod{\mathfrak{p}^2}$, and if $p = 2$ then $\log a \equiv 0 \pmod{\mathfrak{p}^2}$, and $\log a \equiv -4b^4 \pmod{\mathfrak{p}^3}$.

d) Show that if $p \neq 2$ (resp. $p = 2$) then $x \mapsto \log x$ is an isomorphism of the multiplicative topological group $1 + \mathfrak{p}$ onto the additive topological group $\mathfrak{p}$ (resp. $\mathfrak{p}^2$); in particular, if e_p is the element of $1 + \mathfrak{p}$ such that $\log e_p = p$ (resp. $\log e_2 = 4$) then the isomorphism of $\mathbf{Z}_p$ onto $1 + \mathfrak{p}$, which is inverse to $x \mapsto \frac{1}{p}\log x$, (resp. $x \mapsto \frac{1}{4}\log x$) is $y \mapsto e_p^y$ (*cf. Gen. Top.*, III, p. 323, exerc. 25).

e) Show that for every $a \in 1 + \mathfrak{p}$ the continuous function $x \mapsto a^x$, defined on $\mathbf{Z}_p$, admits a derivative equal to $a^x \log a$ at every point; deduce from this that the function $\log x$ admits a derivative equal to $1/x$ at every point of $1 + \mathfrak{p}$.

¶ 8) *a*) With the notation of exerc. 7, show that the series with general term $x^n/n!$ is convergent for every $x \in \mathfrak{p}$ if $p \neq 2$, for every $x \in \mathfrak{p}^2$ (but for no $x \notin \mathfrak{p}^2$) if $p = 2$ (determine the exponent of p in the decomposition of $n!$ into prime factors). If $f(x)$ is the sum of this series show that f is a continuous homomorphism of $\mathfrak{p}$ (resp. $\mathfrak{p}^2$) into $1 + \mathfrak{p}^2$. Deduce from this that for every $z \in \mathbf{Z}_p$ one has $f(pz) = e_p^z$ (resp. $f(p^2z) = e_p^z$); in other words,

$$e_p = 1 + \frac{p}{1!} + \frac{p^2}{2!} + \cdots + \frac{p^n}{n!} + \cdots \qquad \text{if } p \neq 2$$

$$e_2 = 1 + \frac{4}{1!} + \frac{4^2}{2!} + \cdots + \frac{4^n}{n!} + \cdots$$

(use exerc. 7 *e*)).

b) For every $a \in 1 + \mathfrak{p}$ and every $x \in \mathbf{Z}_p$ show that $\log(a^x) = x \log a$, and deduce from *a*) and exerc. 7 *d*) that

$$a^x = 1 + \frac{x \log a}{1!} + \frac{x^2(\log a)^2}{2!} + \cdots + \frac{x^n(\log a)^n}{n!} + \cdots$$

c) For every $m \in \mathbf{Z}_p$ show that the continuous function $x \mapsto x^m$, defined on $1 + \mathfrak{p}$, admits a derivative equal to mx^{m-1} (use *b*) and exerc. 7 *e*)).

d) Show that for every $m \in \mathbf{Z}_p$ and every $x \in \mathfrak{p}$ if $p \neq 2$ ($x \in \mathfrak{p}^2$ if $p = 2$) the series with general term $\binom{m}{n} x^n$ is convergent and that its sum is a continuous function of m; deduce from this that this sum is equal to $(1 + x)^m$ by remarking that $\mathbf{Z}$ is (everywhere) dense in $\mathbf{Z}_p$.

¶ 9) With the notation of exerc. 7) one denotes by $\mathbf{O}_2^+(\mathbf{Q}_p)$ (the group of rotations of the space $\mathbf{Q}_p^2$) the group of matrices of the form

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

with elements in $\mathbf{Q}_p$, such that $x^2 + y^2 = 1$, this group being endowed with the topology defined in *Gen. Top.*, VIII, p. 125, exerc. 2.

a) Denote by G_n the subgroup of $\mathbf{O}_2^+(\mathbf{Q}_p)$ formed by the matrices such that $t = y/(1 + x) \in \mathfrak{p}^n$. Show that G_n is a compact group, that $\mathrm{G}_n/\mathrm{G}_{n+1}$ is isomorphic to $\mathbf{Z}/p\mathbf{Z}$, and that the only compact subgroups of G_1 are the groups G_n (*cf. Gen. Top.*, III, p. 323, exerc. 24).

b) Show that G_1 is identical to the subgroup of matrices

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

such that $x^2 + y^2 = 1$, $x \in 1 + \mathfrak{p}^2$ and $y \in \mathfrak{p}$ if $p \neq 2$; to $x \in 1 + \mathfrak{p}^3$ and $y \in \mathfrak{p}^2$ if $p = 2$.

c) Show that the series with general terms $(-1)^n x^{2n}/(2n)!$ and $(-1)^{n-1} x^{2n+1}/(2n + 1)!$ are convergent for every $x \in \mathfrak{p}$ if $\neq 2$, for every $x \in \mathfrak{p}^2$ if $p = 2$; let $\cos x$ and $\sin x$ be the sums of these series. Show that the map

$$x \mapsto \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$$

is an isomorphism of the additive topological group $\mathfrak{p}$ (resp. $\mathfrak{p}^2$) onto the group G_1.

d) If p is of the form $4h + 1$ (h an integer), there exists in $\mathbf{Q}_p$ an element i such that $i^2 = -1$. If with every $z \in \mathbf{Q}_p^*$ one associates the matrix

$$\begin{pmatrix} \frac{1}{2}\left(z + \frac{1}{z}\right) & \frac{1}{2i}\left(z - \frac{1}{z}\right) \\ -\frac{1}{2i}\left(z - \frac{1}{z}\right) & \frac{1}{2}\left(z + \frac{1}{z}\right) \end{pmatrix}$$

one defines an isomorphism of the multiplicative group $\mathbf{Q}_p^*$ onto the group $\mathbf{O}_2^+(\mathbf{Q}_p)$; under this isomorphism the group $1 + \mathfrak{p}$ corresponds to G_1 and one has $\cos px + i \sin px = e_p^{ix}$ (III, p. 126, exerc. 8).

e) If p is of the form $4h+3$ (h an integer) the elements of the matrices of $\mathbf{O}_2^+(\mathbf{Q}_p)$ are necessarily *p-adic integers*. The polynomial X^2+1 is then irreducible in $\mathbf{Q}_p$; let $\mathbf{Q}_p(i)$ be the quadratic extension of $\mathbf{Q}_p$ obtained by adjoining a root i of X^2+1; one endows $\mathbf{Q}_p(i)$ with the topology defined in *Gen. Top.*, VIII, p. 127, exerc. 2. The group $\mathbf{O}_2^+(\mathbf{Q}_p)$ is isomorphic to the multiplicative group N of elements of $\mathbf{Q}_p(i)$ of norm 1, under the isomorphism which associates the matrix $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ with the element $z = x+iy$. Show that there are $p+1$ roots of the equation $x^{p+1}=1$ in $\mathbf{Q}_p(i)$ and that they form a cyclic subgroup R of N (argue as in *Gen. Top.*, III, p. 323, exerc. 24; then show that there are $p+1$ distinct roots of the congruence $x^{p+1} \equiv 1 \pmod{p}$ in $\mathbf{Q}_p(i)$, and, for each root a of this congruence, form the series $(a^{p^{2n}})$). Deduce that the group $\mathbf{O}_2^+(\mathbf{Q}_p)$ is isomorphic to the *product* of the groups R and G_1.

f) Show that, for $p=2$, the group $\mathbf{O}_2^+(\mathbf{Q}_p)$ is isomorphic to the product of the group G_1 and the cyclic group of order 4.

HISTORICAL NOTE
(*Chapters I, II, and III*)

(N.B. Roman numerals refer to the bibliography to be found at the end of this note.)

In 1604, at the apogee of his scientific career, Galileo believed he had demonstrated that in rectilinear motion where the velocity increases in proportion to the distance travelled the law of motion would be the very one ($x = ct^2$) he had discovered in the fall of heavy bodies (III, v. X, p. 115-116). Between 1695 and 1700 there is not a volume of the *Acta Eruditorum* published monthly at Leipzig in which there do not appear memoirs by Leibniz, the Bernoulli brothers, the marquis de l'Hôpital, treating, essentially in the notation which we still use, the most varied problems in differential calculus, integral calculus, and calculus of variations. Thus it is in the interval of almost exactly a century that the infinitesimal calculus, or as the English ended by calling it, the Calculus par excellence ("calculus") was forged; and nearly three centuries of constant use have not yet completely blunted this incomparable instrument.

The Greeks neither possessed nor imagined anything like it. No doubt, if they had known, they would have refused to use it, an algebraic calculus, that of the Babylonians, of which a part of their Geometry is possibly only a transcription, for it is strictly in the domain of geometrical invention that their most brilliant mathematical creativity appears, their method of treating problems which for us are a matter for the integral calculus. Eudoxus, in treating the volume of the cone and of the pyramid, gave the first models of this, which Euclid has transmitted more or less faithfully to us ((I), Book XII, prop. 7 and 10). But, above all, it is to these problems that almost all the work of Archimedes was devoted ((II) and (II bis)); and, by an odd chance, we may still read, in their original text, in the sonorous Dorian dialect in which they are so carefully composed, the greater part of his writings, and, up to the one recently rediscovered, where he expounds the "heuristic" procedures by which he was led to one of his most beautiful results ((II), v. II, p. 425-507). For this is one of the weaknesses of the "exhaustion" of Eudoxus: although an irreproachable method of proof (certain postulates having been agreed) it is not a method of discovery; its application necessarily rests on prior knowledge of the result to be proven; also, as Archimedes said, *"of the results for which Eudoxus first found a proof, concerning the cone and of the pyramid ..., not a small part goes back to Democritus, who was the first to state them without proof"* (*loc. cit.*, p. 430). This circumstance makes a detailed analysis of the work of Archimedes particularly difficult, an analysis which, to tell the truth, seems not to have been undertaken by any modern historian; for in fact we do not know the extent of his awareness of the family connections which unite the various problems with which he dealt (connections which we would express by saying that the same integral reappears in many places in various geometrical aspects) and what importance he might have attributed to them. For example, let us consider the following problems, the first solved by Eudoxus, the others by Archimedes: the volume of the pyramid, the area of a segment of a parabola, the centre of gravity of a triangle, the area of the Archimedean spiral ($\rho = c\omega$ in polar coordinates); these all depend on the integral $\int x^2\,dx$, and, without departing in any way from the spirit of the method of exhaustion, one can reduce them all to the evaluation of "Riemann sums" of the form $\sum_n an^2$. This indeed is how Archimedes treated the spiral ((II), v. II, p. 1-121), using a lemma which amounts to writing

$$N^3 < 3\sum_{n=1}^{N} n^2 = N^3 + N^2 + \sum_{n=1}^{N} n < (N+1)^3.$$

As for the centre of gravity of the triangle, he proved (by exhaustion, using a decomposition into parallel strips) that it lies on each of the medians, so at their point of concurrence ((II), v. II, p. 261-315). For the parabola he gave three procedures: the one, heuristic, designed only to *"give some plausibility to the result"*, reduces the problem to the centre of gravity of the triangle, by an argument from statics in the course of which he does not hesitate to consider the segment of the parabola as the sum of infinitely many line segments parallel to the axis ((II), v. II, p. 435-439); another method relies on a similar principle, but is drawn up in full rigour by exhaustion ((II), v. II, p. 261-315); a last proof, extraordinarily ingenious, but of lesser scope, gives the area sought as the sum of a geometric series, exploiting specific properties of the parabola. Nothing indicates a relation between these problems and the volume of the pyramid; it is even stated ((II), v. II, p. 8) that the problems regarding the spiral have "nothing in common" with certain others regarding the sphere and the paraboloid of revolution, of which Archimedes had occasion to speak in the same introduction, and among which one finds one (the volume of the paraboloid) which reduces to the integral $\int x\,dx$.

As one sees from these examples, the principle of exhaustion is, apart from special tricks, the following: by a decomposition into "Riemann sums" one obtains upper and lower bounds for the quantity under examination, bounds which one compares directly with the expression stated for this quantity, or else with the corresponding bounds for a similar problem which has already been solved. The comparison (which, in the absence of negative numbers, is necessarily done in two parts) is introduced by the ritual phrase: "if not, then, it would be either greater or smaller; suppose, if it were possible, that it were greater, etc.; suppose, if it were possible, that it were smaller, etc.", whence the name "apagogic" or "by reduction to the absurd" ("ἀπαγωγὴ εἰσ ἀδύνατον") which the scholars of the XVIIth century gave to the method. It is in a similar form that the determination of the tangent to the spiral is set out by Archimedes ((II), v. II, p. 62-76), an isolated result, and the only one that we have to cite as an ancient source of "differential calculus" apart from the relatively easy determination of tangents to conics, and a few problems on maxima and minima. If, indeed, in what concerns “integration”, an immense field of research was offered to the Greek mathematicians, not only by the theory of volumes, but also by statics and hydrostatics, they hardly had, lacking kinematics, occasion to tackle differentiation seriously. It is true that Archimedes gives a kinematic definition of his spiral; and, not knowing how he could have been led to knowledge of its tangent, one has the right to ask whether he had any concept of the composition of movements. But in this case would he not have applied so powerful a method to other problems of the same kind? It is more plausible that he must have utilised some heuristic procedure of passing to the limit that the results he knew about conics may have suggested; these, of course, are essentially simpler in nature, since one can construct the points of intersection of a line and a conic, and consequently determine the condition for these points to coincide. As to the definition of the tangent, this latter is conceived as a straight line which, in a neighbourhood of a certain point of the curve, has the curve entirely on one side; its existence is assumed, and it is also assumed that every curve is composed of convex arcs; under these conditions, to prove that a line is tangent to a curve it is necessary to prove certain inequalities, which is of course done with the most complete accuracy.

From the point of view of rigour, the methods of Archimedes leave nothing to be desired; and, even in the XVIIth century, when the most scrupulous mathematicians wished to put a result judged to be particularly delicate entirely beyond doubt, it was an “apagogic” proof that they gave for it ((XI a) and (XII a)). As to their fruitfulness, Archimedes’ œuvre is sufficient witness. But to have the right to see an “integral calculus” there, one would have to exhibit, through the multiplicity of geometric appearances, some sketch of a classification of the problems according to the nature of the underlying “integral”. In the XVIIth century, we shall see, the search for such a classification became, little by little, one of the principal concerns of the geometers; if one does not find a trace in Archimedes, is this not a sign that such speculations would have seemed exaggeratedly “abstract” to him, and that he has deliberately, on the contrary, on each occasion, kept as close as possible to the specific properties of the figure whose study he was pursuing? And must we not conclude that this wonderful œuvre, from which the integral calculus, according to its creators, is entirely drawn, is in a certain way the opposite of the integral calculus?

It is not with impunity, moreover, that one can, in mathematics, let a ditch appear between discovery and proof. In favourable times the mathematician, without abandoning rigour, has only to write down his ideas almost as he conceives them; sometimes he may even hope to do so by dint of a felicitous adjustment of the accepted language and notation. But often he has to resign himself to choosing between incorrect, though perhaps fruitful, modes of exposition, and correct methods which do not permit him to express his thought except by distorting it, and at the cost of a tiring effort. Neither way is free of danger. The Greeks followed the second, and it is there, perhaps, even more than the sterilising effect of the Roman conquest, that one must seek the reason for the surprising arrest of their mathematics almost immediately after its most brilliant flowering. It has been suggested, not implausibly, that the oral teaching of the successors of Archimedes and Appolonius must have contained many a new result without their believing it necessary to inflict on themselves the extraordinary effort required to publish in conformity with the received canons. In any case, these were not the scruples which stopped the mathematicians of the XVIIth century, when, before the crowds of new problems which posed themselves, they searched assiduously in Archimedes' writings for the means to overcome them.

While the great classics of Greek literature and philosophy were all printed in Italy by Aldus Manutius and his followers, and almost all of them before 1520, it was only in 1544 that the first edition of Archimedes, in Greek and Latin appeared [4] , printed by Hervagius at Bâle, no previous Latin version having preceded it; and, far from the mathematicians of this time (absorbed as they were by their algebraic researches) immediately feeling the influence, one had to wait for Galileo and Kepler, both of them astronomers and physicists rather than mathematicians, for this influence to become manifest. From this moment, without break until about 1670, there is no name in the writings of the founders of the Infinitesimal Calculus which recurs so often as that of Archimedes. Many translated him and prepared commentaries on him; all, from Fermat to Barrow, cited him indiscriminately; all claimed to find both a model and a source of inspiration there.

It is true that these declarations, as we shall see, must not all be taken completely literally; here one finds one of the difficulties which hinder a correct interpretation of these writings. The historian must also take account of the organisation of the scientific world at this time, still very defective at the start of the XVIIth century, while towards the end of that century, by the creation of learned societies and scientific journals, by the consolidation and development of the universities, it finished by strongly resembling what we know today. Lacking any periodical up to 1665, the mathematicians had no way of making their work known, other than by writing letters, or by printing a book, most often at their own expense, or at that of some Maecenas if they could find one. Editors and printers capable of work of this sort were rare, sometimes unreliable. After the long delays and the innumerable upsets which a publication of this sort involved, the author had often to face up to interminable controversies, provoked by adversaries who were not always in good faith, and sometimes pursued in a tone of surprising acerbity: for, in the general uncertainty on the very principles of the Infinitesimal Calculus, it was not difficult for anyone to find weak points, or at least obscure and contestable points, in the arguments of their rivals. One understands that in these circumstances many scholars who valued tranquility contented themselves with communicating their methods and results to a few chosen friends. Some, and above all some amateurs of science, such as Mersenne at Paris and later Collins at London, maintained a vast correspondence into every country, of which they communicated extracts here and there, not without mixing into these extracts errors of their own invention. Owners of "methods" which, for lack of concepts and general definitions, they were neither able to draft in the form of theorems nor even to formulate accurately, the mathematicians were reduced to

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[4] *Archimedis Opera quae quidem exstant omnia, nunc primus et gr. et lat. edita* ... Basileae, Jo. Hervagius, 1544, 1 vol. in-fol.

testing them on large numbers of particular cases, believing they could not measure their strength better than by hurling challenges at their colleagues, sometimes accompanied by the publication of their own results in coded language. The studious young travelled, more perhaps than nowadays; and the ideas of a scholar would sometimes spread more as a result of the travels of his pupils than by his own publications, but not without there being yet another cause of misunderstandings. Finally, as the same problems naturally occurred to many mathematicians, some of whom were very distinguished, but had only an imperfect knowledge of the results of the others, claims of priority could not fail to arise incessantly, and it was not unusual for accusations of plagiarism to be added.

Thus it is in the letters and private papers of the scholars of this time, almost as much or even more than in their publications proper, that the historian must seek his documents. But while those of Huygens, for example, have been preserved and made the object of an exemplary publication (XVI), those of Leibniz have not yet been published except in a defective and fragmentary manner, and many others are lost beyond recall. At least the most recent research, founded on the analysis of manuscripts, has put in evidence, in a manner which seems irrefutable, a point which partisan quarrels had almost obscured: it is that every time that one of the great mathematicians of this time has reported on his own work, on the evolution of his thought, on what has influenced him and what not, he has done so in an honest and sincere manner, in all good faith [5]; these precious testimonies, of which we possess quite a large number, can therefore be used with full confidence, and the historian does not have to transform himself into an examining magistrate here. For the rest, most of the questions of priority were totally lacking in sense. It is true that Leibniz, when he adopted the notation dx for the "differential", did not know that Newton had employed $\dot{x}$ for the "fluxion" for about ten years: but what if he had known? To take a more instructive example, who is the author of the theorem $\log x = \int dx/x$, and what is its date? The formula, as we have just written it, is due to Leibniz, for both terms are written in his notation. Leibniz himself, and Wallis, attribute it to Grégoire de Saint-Vincent. This latter, in his *Opus Geometricum* (IX) (it appeared in 1647, but was composed, he said, long before), proves only the equivalent of the following: if $f(a, b)$ denotes the area of the hyperbolic segment $a \leqslant x \leqslant b,\ 0 \leqslant y \leqslant \mathrm{A}/x$, then the relation $b'/a' = (b/a)^n$ implies $f(a', b') = n\, f(a, b)$; to this his pupil and commentator Sarasa almost immediately added the remark [6] that the areas $f(a, b)$ can thus "take the place of logarithms". If he said no more on this, and if Grégoire himself said nothing, was it not because, for the majority of mathematicians of this time, logarithms were "aids to calculation" not yet naturalized with no right to be quoted in mathematics? It is true that Torricelli, in a letter of 1644 (VII bis) spoke of his research on a curve that we would write as $y = a\, e^{-cx},\ \ x \geqslant 0$, adding that Napier (whom moreover he covered in praise) *"pursued only practical*

[5] This applies for example to Torricelli (see (XII), v. VIII, p. 181-194) and to Leibniz (D. MAHNKE, *Abh. Preuss. Akad. der Wiss., 1925* Nr. 1, Berlin, 1926). This is not to say, of course, that a mathematician cannot be subject to illusions on the originality of his ideas; but they are not the greatest who are the most inclined to deceive themselves in this respect.

[6] *Solutio problematis* ... Auctore P. ALFONSO ANTONIO DE SARASA ... Antverpiae, 1649.

arithmetic", while he himself *"drew a speculation in geometry out of it";* and he left a manuscript, evidently prepared for publication, on this curve, though it remained unpublished until 1900 ((VII), v. I, p. 335-347). Descartes, furthermore, had met the same curve about 1639 in connection with "Debeaune's problem" and described it without mentioning logarithms ((X), v. II, p. 514-517). However it may have been, J. Gregory, in 1667, gave, without citing anyone at all ((XVII *a*), reproduced in (XVI bis), p. 407-462), a rule for calculating the areas of hyperbolic segments by means of (decimal) logarithms: this at once implies theoretical knowledge of the connection between the quadrature of the hyperbola and logarithms, and numerical knowledge of the connection between "natural" and "decimal" logarithms. Is it only at this last point that Huygens' claim applies, contesting directly the novelty of Gregory's result (XVI *a*)? This is no more clear to us than to their contemporaries; these in every case had the clear impression that the existence of a link between logarithms and the quadrature of the hyperbola was something known a long time, without being able to refer to anything other than epistolary allusions or even to the book of Grégoire de Saint-Vincent. In 1668, when Brouncker gave series for $\log 2$ and for $\log(5/4)$ (with a meticulous proof of convergence, by comparison with a geometric series)(XIV), he presented them as expressions for corresponding segments of the hyperbolas, and added that the numerical values that he obtained were "in the same ratio as the logarithms" of 2 and of $5/4$. But in the same year, with Mercator (XIII) (or more precisely with the exposition given immediately by Wallis of the work of Mercator (XV bis)), the language changed: since the segments of the hyperbola are proportional to the logarithms, and since it is known that logarithms are defined by their characteristic properties only up to a constant factor, nothing stops one from considering the segments of the hyperbola as logarithms, termed "natural" (in opposition to "artificial" or "decimal" logarithms) or hyperbolic; this last step taken (the series for $\log(1+x)$, given by Mercator, contributed to this), the theorem $\log x = \int dx/x$ was obtained, up to notation, or, rather, it even became the definition. What to conclude, if not that it was by almost imperceptible transitions that this discovery was made, and that a priority dispute in this matter strongly resembles a quarrel between the violin and the trombone over the exact moment when a certain motif appears in a symphony? And to tell the truth, although at the same time other mathematical creations, the arithmetic of Fermat, the kinematics of Newton, bear a strongly individual stamp, it is of the gradual and inevitable unrolling of a symphony, where the "Zeitgeist", at the same time composer and conductor, takes the baton, that the development of the infinitesimal calculus in the XVIIth century reminds one: each executes his part with his own proper timbre, but none is the master of the themes which he makes heard, themes which are almost inextricably entangled by a scholarly counterpoint. It is thus under the form of a thematic analysis that the history has to be written; we shall be content here with a summary sketch, and shall not pretend to minute exactitude[7]. Here in any case are the principal themes which appear under superficial examination:

A) The theme of *mathematical rigour,* contrasting with that of the *infinitely small, indivisibles* or *differentials*. We have seen that both of these held an important place with Archimedes, the first in all of his œuvre, the second in his only treatise on Method, which the XVIIth century did not know, so that if it had been transmitted and not reinvented, it could only have been by the philosophical tradition. The principle of the infinitely small appears, moreover, in two distinct forms, as it concerns "differentiation" or "integration". As to this, suppose first that one is to calculate a plane area: one will divide it into infinitely many infinitely small parallel strips by means of infinitely many equidistant parallels; and each of these strips is a rectangle (even though the finite strips one would obtain using two parallels at a finite distance would not be rectangles). Similarly, a solid of revolution will be decomposed into infinitely many cylinders of the same infinitely small height, by planes perpendicular to the axis [8] ; similar modes of speech might be employed when decomposing an area into triangles by concurrent lines, or reasoning about the length of an arc of a curve as if it were a polygon with infinitely many sides, etc. It is certain that the rare mathematicians who possessed a firm command of Archimedes' methods, such as Fermat, Pascal, Huygens, Barrow, would not, in each particular case, have found any difficulty in replacing the use of this language with rigorous proofs; also they often remark that this language is only a short-hand. *"It would be easy"*, says Fermat, *"to give proofs after the manner of Archimedes; ..., it is enough to have been warned once and for all in order to avoid continual repetitions ... "* ((XI), v. I, p. 257); similarly Pascal: *"thus the one of these methods does not differ from the other save in the manner of speaking"* ((XII *b*), p. 352) [9] ; and Barrow, with his sardonic conciseness: *"longior discursus apagogicus adhiberi possit, sed quorsum?"* (one could prolong this by an apagogic discourse, but to what benefit?) ((XVIII), p. 251). Fermat, it seems, was wary of advancing what he could not so justify, and so condemned himself to not stating any general result except by allusion or under the form of a "method": Barrow, although very careful, was a little less scrupulous. As for the majority of their contemporaries, one can say at the very least that rigour was not their principal concern, and that the name of Archimedes was most often only a tent intended to cover merchandise doubtless of great price, but for which

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[7] In what follows, the attribution of a result to such an author, to such a date, only indicates that this result was known to him at that date (which as often as possible has been verified by examining the original texts); we do not mean to affirm absolutely that this author did not know it earlier, or that he did not receive it from someone else; even less do we want to say that the same result might not have been obtained independently by others, either earlier, or later.

[8] See for example the exposition by Pascal in his "letter to M. de Carcavy" (XII *b*). One will note that, thanks to the prestige of an incomparable language, Pascal manages to create the illusion of perfect clarity, to the point where one of his modern editors goes into ecstasies over "the meticulousness and preciseness in the exactitude of the proof"!

[9] But, in the Letter to Monsieur A. D. D. S.: "*... without stopping, neither at the methods of movements, nor at that of the indivisibles, but following that of the ancients, so that the matter would henceforth be firm and beyond dispute"* ((XII *a*), p. 256).

Archimedes would certainly not have assumed responsibility. All the more so when one deals with differentiation. If the curve to be rectified is assimilated to a polygon with infinitely many sides, it is an "infinitely small" arc of the curve that is assimilated to an "infinitely small" line segment, either the chord, or a segment of the tangent whose existence is assumed; or again it is an "infinitely small" interval of time that one considers, during which (while one is dealing only with velocity) the movement "is" uniform; yet bolder, Descartes, wanting to determine the tangent to the cycloid, which is not covered by his general rule, assimilates curves rolling on one another to polygons, to deduce that "in the infinitely small" the movement can be assimilated to a rotation about the point of contact ((X), v. II, p. 307-338). Here again, a Fermat, who bases his rules for tangents and for maxima and minima on such infinitesimal considerations, is able to justify them in each particular case ((XI *b*); *cf.* also (XI), v. II, *passim,* in particular p. 154-162, and the *Supplément aux Œuvres* (Gauthier-Villars, 1922), p. 72-86); Barrow gave exact proofs for many of his theorems after the manner of the ancients, starting from simple hypotheses of monotonicity and convexity. But it was no longer the moment to pour new wine into old skins. In all that, as we know today, it was the notion of a limit that was being elaborated; and, if one can extract from Pascal, from Newton, from others as well, statements that seem very close to our modern definitions, one need only put them back in their context to perceive the invincible obstacles that hindered rigorous exposition. When, from from the XVIIIth century, some mathematicians concerned for clarity wished to put some order among the confused piles of their riches, such indications, in the writings of their predecessors, were precious to them; when d'Alembert, for example, explained that there is nothing in differentiation other than the notion of a limit, and defined this accurately (XXVI), one can believe that he was guided by Newton's considerations on the "first and last reasons of vanishing quantities" (XX). But, so far as the XVIIth century is concerned, it is very necessary to state that the way was not open to modern analysis except when Newton and Leibniz, turning their backs on the past, agreed provisionally to seek the justification of new methods, not in rigorous proofs, but in the fruitfulness and coherence of the results.

B) *Kinematics*. Archimedes, one has seen, had already given a kinematical definition of his spiral; and in the middle ages there developed (but, without evidence to the contrary, without infinitesimal considerations) a rudimentary theory of the variation of quantities as a function of time, and of their graphical representation, which one can perhaps trace back to Babylonian astronomy. But it was of the greatest importance for the mathematics of the XVIIth century that, from the beginning, the problems of differentiation arose, not only regarding tangents, but also regarding velocities. Galileo, nevertheless, ((III) and (III bis)), investigating the law of the velocity in the fall of heavy bodies (after having obtained the law for distances $x = at^2$, from experiments on an inclined plane), does not proceed by differentiation: he makes various hypotheses about the velocity, first $v = dx/dt = cx$ ((III), v. VIII, p. 203), then later $v = ct$ (*id.,* p. 208), and seeks to recover the law of distances by reasoning, in a rather obscure way, about the graph of the velocity as a function of the time; Descartes (in 1618) argued similarly, about the law $v = ct$, but as a true mathematician and with as much clarity as the language of indivisibles permits [10] ((X), v. X, p. 75-78); in both of these, the graph of the velocity (in this instance a straight line) plays the principal rôle, and one may ask up to what point they were aware of the proportionality between the distances travelled and the areas contained between the axis of time and the curve of the velocities; but it is difficult to affirm anything on this point, even though Descartes' language seems to imply an awareness of the fact in question (which certain historians would like to trace back to the middle ages [11]), although Galileo makes no clear allusion to it. Barrow stated it explicitly in 1670 ((XVIII), p. 171); perhaps it was no novelty to anyone at this time, and Barrow did not present it as such; but, no more for this result than for any another, should one try to fix a date too precisely. As for the hypothesis $v = cx$, also envisaged by Galileo, he contented himself (*loc. cit.*) with proving that it is untenable (or, in modern language, that the equation $dx/dt = cx$ has no solution $\neq 0$ which vanishes for $t = 0$), by an obscure argument which Fermat later ((XI), v. II, p. 267-276) took the trouble to develop (and which comes close to saying that, $2x$ being a solution if x is, $x \neq 0$ would be contrary to the physically evident uniqueness of the solution). But it is this same law $dx/dt = cx$ which, in 1614, served Napier to introduce his logarithms, for which he gave a kinematical definition (IV), and, in our notation, would be written as follows: if, on two straight lines, two moving bodies travel according to the laws $dx/dt = a,\ dy/dt = -ay/r,\ x_0 = 0,\ y_0 = r$, then one says that x is the "logarithm" of y (in modern notation, $x = r\log(r/y)$). We have seen that the solution curve of $dy/dx = c/x$ appeared in 1639 with Descartes, who described it kinematically ((X), v. II, p. 514-517); it is true that he classified all non-algebraic curves rather disdainfully as "mechanical", and claimed to exclude them from geometry; but happily this taboo, against which Leibniz, much later, still believed he had to protest vigorously, was not observed by his contemporaries, nor by Descartes himself. The cycloid, and the logarithmic spiral, appeared, and were studied eagerly, and their study powerfully helped the interpenetration of geometric and kinematical methods. The principle of composition of movements, and more precisely of composition of velocities, was at the basis of the theory of movement of projectiles expounded by Galileo in the chef-d'œuvre of his old age, the *Discorsi* of 1638 (((III), v. VIII, p. 268-313), a theory which thus contains implicitly a new determination of the tangent to a parabola; if Galileo did not remark this expressly, Torricelli, on the other hand (VII), v. III, p. 103-159) insisted on this point, and founded on this very principle a general method for determining the tangents to curves for which a kinematical definition could be given. It is true that he had been forestalled in this, by several years, by Roberval (VIII *a*), who said he had been led to this method by the study of the cycloid; this same problem of the tangent to

[10] Decartes even adds an interesting geometrical argument by which he deduces the law $x = at^2$ from the hypothesis $dv/dt = c$. On the other hand it is ironic, ten years later, to see him muddle his notes, and copy for Mersenne's use an incorrect argument on the same question, where the graph of the velocity as a function of time is confused with the graph as a function of the distance travelled ((X), v. I, p. 71).

[11] H. WIELEITNER, Der "Tractatus de latitudinibus formarum" des Oresme, *Bibl. Mat.* (III), v. 13 (1912), p. 115-145.

the cycloid gave Fermat, besides, the opportunity to demonstrate the power of his method of differentiation ((XI *b*), p. 162-165), while Descartes, unable to apply his algebraic method here, invented the instantaneous centre of rotation for the occasion ((X), v. II, p. 307-338).

But, as the infinitesimal calculus developed, kinematics ceased to be a separate science. One sees more and more that in spite of Descartes the algebraic curves and functions have nothing, from the "local" point of view, that of the infinitesimal calculus, to distinguish them from other much more general ones; the functions and curves with a kinematical definition are functions and curves like the others, amenable to the same methods; and the variable "time" is merely a parameter whose temporal aspect is purely a matter of language. Thus, with Huygens, even when he dealt with mechanics, it was geometry that dominated (XVI *b*); and Leibniz did not give time any privileged rôle in his calculus. In contrast, Barrow devised, from the simultaneous variation of various quantities as a function of an independent universal variable conceived as a "time", the foundation of an infinitesimal calculus with a geometric tendency. This idea, which must have come to him when he sought to recover the method of composition of movements, whose existence he knew only by hearsay, is expounded in detail, in very clear and very strong terms in the first three of his *Lectiones Geometricae* (XVIII); there, for example, he shows carefully that if a moving point has as projections on two rectangular axes AY, AZ, moving points one of which moves with a constant velocity a and the other with a velocity v that increases with time, then the trajectory has a tangent with gradient equal to v/a, and is concave towards the direction of increasing Z. In the rest of the *Lectiones* he pursued these ideas very far, and although he indulged the affectation of drafting them from beginning to end in a form as geometric and little algebraic as possible, one may see there, with Jakob Bernoulli ((XXIII), v. I, p. 431 and 453), the equivalent of a good part of the infinitesimal calculus of Newton and Leibniz. Exactly the same ideas served as the point of departure for Newton ((XIX *c*) and (XX)): his "fluentes" are various quantities, functions of a "time" which is only a universal parameter, and the "fluxions" are their derivatives with respect to "time"; the possibility that Newton permits himself of changing the parameter at need is equally present in the work of Barrow, though used less flexibly by him [12]. The language of fluxions, adopted by Newton and imposed by his authority on the English mathematicians of the following century, thus represents the last outpost, for the period that we treat, of the kinematical methods whose true rôle was now over.

[12] On the relations between Barrow and Newton, see OSMOND, *Isaac Barrow, His life and time,* London, 1944. In a letter of 1663 (*cf.* ST. P. RIGAUD,*Correspondence of scientific men ...,* Oxford, 1841, vol. II, p. 32-33), Barrow speaks of his thoughts, already old, on the composition of movements, which had led him to a very general theorem on tangents (if it is that of *Lect. Geom.,* Lect. X (XVIII), p. 247), which is so general that it actually contains as a particular case all that had been done up to then on the subject). On the other hand, Newton was the pupil of Barrow in 1664 and 1665, but said he had independently obtained his rule for deducing a relation between their "fluxions" from a relation between "fluentes". It is likely that Newton had taken the general idea of quantities varying as a function of time, and of their velocities of variation, from Barrow's teaching, concepts which his thoughts on dynamics (to which Barrow must have remained a total stranger) soon helped clarify.

C) *Algebraic geometry.* This is a parasitic theme, a stranger to our subject, which intrudes itself by the fact that Descartes, with his systematic attitude, claimed to make algebraic curves the exclusive object of geometry ((X), v. VI, p. 390); also, it was a method of algebraic geometry, and not like Fermat, a method from the differential calculus, that he gave for the determination of tangents. The results bequeathed by the ancients on the intersection of a line and a conic, the thoughts of Descartes himself on the intersection of two conics, and the problems which reduce to them, would have led him completely naturally to the idea of taking the coincidence of two intersections as the criterion for contact: we know today that in algebraic geometry this is the correct criterion, and of such great generality that it is independent of the concept of limit and of the nature of the "base field". Descartes first applied it in a not very convenient way, trying to make two intersections of the curve under study and a circle having its centre on Ox coincide at a given point ((X), v. VI, p. 413-424); his pupils, van Schooten, Hudde, substituted a line for the circle and obtained the gradient of the tangent to the curve

$$\mathrm{F}(x, y) = 0$$

in the form $-\mathrm{F}'_x/\mathrm{F}'_y$, the "derived polynomials" F'_x, F'_y being defined by their formal rule of formation ((X bis), v. I, p. 147-344 and (XXII), p. 234-237); de Sluse also arrived at this result at about the same time ((XXII), p. 232-234). Of course the clear-cut distinctions which we note here, and which alone give a meaning to the controversy between Descartes and Fermat, could not have existed in any way in the minds of mathematicians of the XVIIth century: we have mentioned them only to clarify one of the most curious episodes of the history which concerns us, and to remark the almost immediate complete eclipse of algebraic methods, for the time being absorbed by differential methods.

D) *Classification of problems.* This theme, we have seen, appears to be absent from Archimedes' œuvre, and it was immaterial to him whether he solved a problem directly or reduced it to a problem already treated. In the XVIIth century the problems of differentiation first appear under three distinct aspects: velocities, tangents, maxima and minima. As to these last, Kepler (V) made the observation (which one already finds in Oresme [13] and which had not escaped even the Babylonian astronomers) that the variation of a function is particularly slow in the neighbourhood of a maximum. Fermat, already before 1630 ((XI bis); *cf.* (XI), v. II, p. 71) inaugurated his infinitesimal method *à propos* these problems, which in modern language amounts to examining the first two terms (the constant term and the first order term) of the Taylor expansion, and to writing that the second vanishes at an extremum; he went on to extend his method to the determination of tangents, and even applied it to the search for points of inflection. If one takes account of what has been said above about kinematics, one sees that the unification of these three types of problem regarding the first derivative was accomplished quite early. As for problems regarding

[13] H. WIELEITNER, Der "Tractatus de latitudinibus formarum" des Oresme, *Bibl. Mat.* (III), v. 13 (1912), p. 115-145, in particular p. 141.

the second derivative, they do not appear until later, above all with the works of Huygens on the evolute of a curve (published in 1673 in his *Horologium Oscillatorium* (XVI b)); at this time, Newton, with his fluxions, was already in possession of all the analytical tools needed to solve such problems; and, despite all the geometric talent which Huygens expended on it (and of which later differential geometry would profit at its beginning), they did not serve for anything else, during the period which we treat, than to allow the new analysis to proclaim the power of its tools.

As to integration, this appeared with the Greeks as the calculation of areas, volumes, moments, as the calculation of the perimeter of the circle and the areas of spherical segments; to which the XVIIth century added the rectification of curves, the calculation of the area of surfaces of revolution, and (with the work of Huygens on the compound pendulum (XVI *b*)) the calculation of moments of inertia. It was first a matter of recognising the connections between all these problems. For areas and volumes the first immense step was taken by Cavalieri, in his Geometry of Indivisibles (VI *a*). There he stated, and claimed to prove, more or less the following principle: if two plane areas are such that every line parallel to a given direction cuts them in segments which are in a constant ratio, then the areas are in the same ratio; an analogous principle is stated for the volumes cut by planes parallel to a fixed plane in areas whose measures are in a constant ratio. It is likely that these principles were suggested to Cavalieri by theorems such as those of Euclid (or rather of Eudoxus) on the ratio of the volumes of pyramids of the same height, and that before stating them in a general manner he had first verified their validity on a great number of examples taken from Archimedes. He "justified" them by employing a language on whose legitimacy one sees him question Galileo in a letter of 1621, although already in 1622 he used it without hesitation ((III), v. XIII, p. 81 and 86) and of which this is the essential part. Consider for example two areas

$$\text{the one } 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant f(x), \qquad \text{the other } 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant g(x);$$

the sums of the ordinates $\sum\limits_{k=0}^{n-1} f(ka/n)$, $\sum_{k=0}^{n-1} g(ka/n)$ are in a ratio which, for n sufficiently large, is as close as one requires to the ratio of the two areas, and it would not even be difficult to demonstrate this by exhaustion when f and g are monotone; Cavalieri passed to the limit, made $n = \infty$, and spoke of "the sum of all the ordinates" of the first curve, which is in a ratio to the analogous sum for the second curve rigorously equal to the ratio of their areas; the same for volumes; this language was universally adopted later, even by authors, like Fermat, who had the clearest awareness of the precise facts which it covers. It is true that subsequently many mathematicians, such as Roberval (VIII *a*) and Pascal (XII *b*), preferred to see, in the ordinates of the curve of which one forms the "sum", not line segments like Cavalieri, but rectangles of the same infinitely small width, which is no great advance from the point of view of rigour (whatever Roberval says), but perhaps keeps the imagination from going off the rails too easily. In any case, and since one deals only with ratios, the expression "sum of all the ordinates" of the curve $y = f(x)$ or, in short, "all the ordinates" of the curve, is, in the final analysis, as it also appears in the writings of Pascal, the exact equivalent of the Leibnizian $\int\ y\,dx$.

From the language adopted by Cavalieri, the principles stated above follow inevitably, and also immediately imply consequences which we shall state in modern notation, having understood that $\int f\,dx$ simply means the area contained between Ox and the curve $y = f(x)$. First, every plane area cut by every line $x =$ constant in segments the sum of whose lengths is $f(x)$, is equal to $\int f\,dx$; the same is true for every volume cut by each plane $x =$ constant in an area of measure $f(x)$. Further, $\int f\,dx$ depends linearly on f; we have $\int (f+g)\,dx = \int f\,dx + \int g\,dx$, $\int cf\,dx = c\int f\,dx$. In particular, all problems of areas and volumes are reduced to quadratures, that is to say, to the calculation of areas of the form $\int f\,dx$; and, what is perhaps novel and more important, one must consider as equivalent two problems that depend on the same quadature, and one has the tools to decide, in each case, if this is so. The Greek mathematicians never attained (or perhaps would never have agreed to attain) such a degree of "abstraction". Thus ((VI), p. 133) Cavalieri "demonstrates" quite easily that two similar volumes are in the ratio of the cube of the ratio of similarity, whereas Archimedes stated this conclusion, for quadrics of revolution and segments of them, only at the end of his theory of these solids ((II), v. p. 258). But to reach this point it was necessary to throw Archimedean rigour overboard.

One thus had there the means of classifying problems, at least provisionally, according to the degree of real or apparent difficulty presented by the quadratures to which they reduce. This is where the algebra of the time served as a model: for in algebra too, and in the algebraic problems that arose in geometry, although the Greeks were interested only in the solutions, the algebraists of the XVIth and XVIIth century had begun to turn their attention principally to the classification of problems according to the nature of the tools which might serve to solve them, thus anticipating the modern theory of algebraic extensions; and they had not only proceeded to a first classification of problems according to the degree of the equation on which they depend, but had already posed difficult questions on possibility: the possibility of solving every equation in radicals (in which many no longer believed), etc. (see the Historical Note to Book II, chap. V); they preoccupied themselves also with reducing all the problems of a given degree to a form of geometric type. Likewise in the Integral Calculus the principles of Cavalieri put him in a position to recognise immediately that many of the problems solved by Archimedes reduce to the quadratures $\int x^n\,dx$ for $n = 1, 2, 3$; and he devised an ingenious method of effecting this quadrature for as many values of n as one wishes (the method amounts to observing that $\int_0^{2a}\ x^n\,dx = c_n\,a^{n+1}$ by homogeneity, and to writing

$$\int_0^{2a} x^n\,dx = \int_{-a}^{a} (a+x)^n\,dx = \int_0^{a} \big((a+x)^n + (a-x)^n\big)\,dx,$$

from which, on expanding, one obtains a recurrence relation for the c_n) ((VI a), p. 159 and (VI b), p. 269-273). But Fermat had already gone much further, showing first (before 1636) that $\displaystyle\int_0^a x^n\,dx = \frac{a^{n+1}}{n+1}$ for n a positive integer ((XI), v. II, p. 83), by means of a formula for the sums of powers of the first N integers (a process copied from the quadrature of the spiral by Archimedes), then by extending this formula to all rational $n \neq -1$ ((XI), v. I, p. 195-198); he did not draft a proof of this last result (communicated to Cavalieri in 1644) until much later, after reading the writings of Pascal on integration [14] (XI *c*).

These results, joined to geometric considerations which took the place of change of variables and integration by parts, already allowed one to solve a large number of problems that reduce to elementary quadratures. Beyond that, one first encountered the quadrature of the circle and the hyperbola: since it is above all with "indefinite integrals" that one dealt at this time, the solutions of these problems, in modern terms, is furnished respectively by the inverse trigonometric functions and by the logarithm; these were given geometrically, and we have seen how the latter was introduced little-by-little into analysis. These quadratures formed the subject of numerous works, by Grégoire de St.-Vincent (IX), Huygens ((XVI *c*) and (XVI *d*)), Wallis (XV *a*), Gregory (XVII *a*); the first believed he had effected the quadrature of the circle, the last that he had proved the transcendence of π; they developed processes of indefinite approximation of the circular and logarithmic functions, some with a theoretical slant, others oriented towards numerical calculus, which would soon, with Newton ((XIX *a*) and (XIX *b*)), Mercator (XIII), J. Gregory (XVII bis), then Leibniz (XXII), come to general methods of series expansion. In every case, the conviction was born, little by little, of the "impossibility" of the quadratures in question, that is to say, of the nonalgebraic character of the functions that they defined; and at the same time, it became usual to consider that a problem was solved so far as its nature permitted, when it had been reduced to one of the "impossible" quadratures. This is the case, for example, of problems on the cycloid, solved by the trigonometric functions, and the rectification of the parabola, reduced to the quadrature of the hyperbola.

The problems of rectification, of which we have just cited two of the most famous, had a particular importance, since they formed a natural geometric transition between differentiation, which they presuppose, and integration under which they come; one can associate with them the problems on the area of surfaces of revolution. The ancients had treated only the case of the circle and of the sphere. These questions appeared only very late in the XVIIth century; it seems that the difficulty, insurmountable at the time, of the rectification of the ellipse (considered the simplest curve after the circle) had discouraged effort. Kinematical methods gave some purchase on these problems, and allowed Roberval (VIII *b*) and Torricelli ((VII), v. III, p. 103-159), between 1640 and 1645, to obtain results on the arcs of spirals; but it is only in the years preceding 1660 that they became the order of the day; the cycloid was rectified by Wren in 1658 ((XV), v. I, p. 533-541); a little later the curve $y^3 = ax^2$ by various authors ((XV), v. I, p. 551-553; (X bis), p. 517-520; (XI *d*)), and several others too ((XI), v. I, p. 199; (XVI), v. II, p. 224) reduced the rectification of the parabola to the

[14] It is remarkable that Fermat, so scrupulous, used the additivity of the integral, without a word to justify it, in the applications which he gave of his general results: did he base himself on the piecewise monotonicity, implicitly assumed, of the functions under study, from which it is indeed not difficult to justify the additivity by exhaustion? or was he already, despite himself, swept along by the language he employed?

quadrature of the hyperbola (that is to say, to an algebraico-logarithmic function). This last example is the most important, for it is a particular case of the general principle that rectification of the curve $y = f(x)$ is nothing else than the quadrature of $y = \sqrt{1 + (f'(x))^2}$; and it is indeed from this principle that Heurat deduced it. It is no less interesting to follow the attempts of the aging Fermat, in his work on the curve $y^3 = ax^2$ (XI d); to the curve $y = f(x)$ of arc $s = g(x)$ he associated the curve $y = g(x)$, and determined the tangent to this from the tangent to the first (in modern language, he showed that their gradients $f'(x)$, $g'(x)$ are linked by the relation

$$\bigl(g'(x)\bigr)^2 = 1 + \bigl(f'(x)\bigr)^2\bigr);$$

one thinks oneself very near to Barrow, and one need only combine this result with that of Heurat (which is more or less what Gregory did in 1668 ((XVII bis), p. 488-491)) to obtain the relation between tangents and quadratures; but Fermat stated only that if, for two curves each referred to a system of rectangular axes, the tangents at points with same abscissa always have the same gradient, then the curves are equal, or, in other words, that the knowledge of $f'(x)$ determines $f(x)$ (up to a constant); and he justified this assertion only by an obscure argument of no probative value.

Less than ten years later, the *Lectiones Geometricae* of Barrow (XVIII) had appeared. From the outset (Lect. I), he states in principle that, in a rectilinear movement, the distances are proportional to the areas $\int_0^t v\,dt$ contained between the time axis and the graph of the velocity. One would expect him to deduce the link between the derivative conceived as the gradient of the tangent and the integral conceived as an area from this, and from his kinematical method already cited on the determination of tangents; but there is none of that, and he shows later, in a purely geometric manner ((XVIII), Lect. X, §11, p. 243) that, if two curves $y = f(x)$, $\mathrm{Y} = \mathrm{F}(x)$ are such that the ordinates Y are proportional to the areas $\int_a^x y\,dx$ (that is to say, if $c\mathrm{F}(x) = \int_a^x f(x)\,dx$), then the tangent to $\mathrm{Y} = \mathrm{F}(x)$ cuts Ox at the point with abscissa $x - \mathrm{T}$ determined by $y/\mathrm{Y} = c/\mathrm{T}$; the proof is, moreover, perfectly correct, starting from the explicit hypothesis that $f(x)$ is monotone: and it is stated that the sense of variation of $f(x)$ determines the sense of the concavity of $\mathrm{Y} = \mathrm{F}(x)$. But one may note that this theorem is somewhat lost among a crowd of others, of which many are very interesting; the unwarned reader is tempted only to see a means of solving by quadrature the problem $\mathrm{Y}/\mathrm{T} = f(x)/c$, that is to say, a particular problem of determining a curve from information about its tangent (or, as we would say, a differential equation of a particular kind); and all the more so since the applications which Barrow gave of it concern, before all, problems of the same kind (that is to say, differential equations integrable by "separation of variables"). The geometric language which Barrow imposed on himself is here a cause why the link between differentiation and integration, so clear where kinematics was concerned, was somewhat obscured.

On the other hand, various methods had taken shape, for reducing some problems of integration to others, and to "solving" them, or at least to reducing them to the already classified "impossible" problems. In its simplest geometrical form, integration by parts consists of writing the area bounded by Ox, Oy, and an arc of the monotone curve $y = f(x)$ joining a point $(a, 0)$ of Ox to a point $(0, b)$ of Oy, as $\int_0^a y\,dx = \int_0^b x\,dy$; and it is frequently used implicitly. The following generalization appears, already well hidden, in Pascal ((XII c), 287-288): $f(x)$ being as above, let $g(x)$ be a function $\geqslant 0$, and let $\mathrm{G}(x) = \int_0^x g(x)\,dx$; then one has $\int_0^a y\,g(x)\,dx = \int_0^b \mathrm{G}(x)\,dy$, which he proved ingeniously by evaluating the volume of the solid $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant f(x)$, $0 \leqslant z \leqslant g(x)$ in two ways; the particular case $g(x) = x^n$, $\mathrm{G}(x) = \dfrac{x^{n+1}}{n+1}$ plays an important rôle, both in Pascal (*loc. cit.*, p. 289-291) and in Fermat ((XI), v. I, p. 271); the latter (whose work bears the significant title *"Transmutation et émendation des équations des courbes, et ses applications variées à la comparaison des espaces curvilignes entre eux et avec les espaces rectilignes* ... ") did not prove this, doubtless because he did not judge it useful to repeat what Pascal had just published. These theorems on "transmutation", where we would see a combination of integration by parts and of change of variables, take the place of this to a certain extent, for it was not introduced until much later; it is indeed contrary to the mode of thought of the time, still too geometric and too little analytic, to allow the use of variables others than those imposed by the figure, that is to say, one or other of the coordinates (or sometimes polar coordinates), or the arc length of the curve. It is thus that we find results in Pascal (XII d) which, in modern notation, can be written, putting $x = \cos t$, $y = \sin t$, and for particular functions $f(x)$:

$$\int_0^1 f(x)\,dx = \int_0^{\pi/2} f(x) y\,dt,$$

and, in J. Gregory ((XVII bis), p. 489), for a curve $y = f(x)$ and its arc s, $\int y\,ds = \int z\,dx$, with $z = y\sqrt{1 + y'^2}$. It is only in 1669 that we see Barrow in possession of the general theorem on change of variables ((XVIII), p. 298-299); his statement, geometric as always, amounts to the following: let x and y be linked by a monotone relation, and let p be the gradient of this relation at the point (x, y); then, if the functions $f(x)$, $g(y)$ are such that $f(x)/g(y) = p$ for every pair of corresponding values (x, y), then the areas $\int f(x)\,dx$, $\int g(y)\,dy$, taken between corresponding limits, are equal; and conversely, if these areas are always equal (f and g being implicitly assumed to be of constant sign), then $p = f(x)/g(y)$; the converse naturally serves to apply the theorem to the solution of differential equations (by "separation of variables"). But Barrow inserted the theorem only in an appendix (Lect. XII, app. III, theor. IV), where, observing that many of his previous results are only particular cases, he excused himself as having discovered it too late to make more use of it.

Thus, around 1670, the situation was the following. One knew how to treat problems which reduce to the first derivative by uniform procedures, and Huygens had treated geometric questions which reduce to the second derivative. One knew how to reduce all problems of integration to quadratures; one had various techniques, of a geometric aspect, for reducing quadratures one to another, in the badly classified cases, and one was accustomed, from this point of view, to the handling of the trigonometric and logarithmic functions; one had become aware of the link between differentiation and integration; one had begun to tackle the "method of inverse tangents", a name given at this time to problems which reduce to differential equations of the first order. The sensational discovery of the series $\log(1+x) = -\sum_{1}^{\infty}(-x)^n/n$ by Mercator had just opened totally new perspectives on the application of series, and principally of power series, to problems that had been called "impossible". On the other hand, the ranks of the mathematicians had been very much thinned: Barrow had resigned his professorial chair for that of a preacher; Huygens apart (who had almost all his mathematical œuvre behind him, having obtained already all the principal results of the *Horologium Oscillatorium* which he was setting himself to edit definitively), only Newton at Cambridge, and J. Gregory isolated at Aberdeen were active; and Leibniz would soon join them with neophyte ardour. All three, Newton already from 1665, J. Gregory from the publication of Mercator in 1668, Leibniz from about 1673, devoted themselves principally to the topic of the day, the study of power series. But, from the point of view of the classification of problems, the principal effect of the new methods seemed to be to obliterate all distinctions between them; and indeed Newton, more analyst than algebraist, did not hesitate to announce to Leibniz in 1676 (XXII), p. 224) that he knew how to solve all differential equations [15] ; to which Leibniz responded ((XXII), p. 248-249) that he, on the contrary, was concerned to obtain the solution in finite terms whenever he could "assuming the quadratures", and also to know whether every quadrature could be reduced to those of the circle and hyperbola as was claimed in most of the cases already studied; in this connection he recalled that Gregory believed (with reason, as we know today) that the rectification of the ellipse and hyperbola were not reducible to the quadratures of the circle and hyperbola; and Leibniz asked up to what point the method of series, such as Newton employs, could give replies to these questions. Newton, for his part ((XXII), p. 209-211), declared himself in possession of criteria, which he did not indicate, for deciding, apparently by examining the series, on the "possibility" of certain quadratures (in finite terms), and gave a (very interesting) example of a series for the integral $\int x^a\,(1+x)^\beta\,dx$.

One sees the immense progress made in less than ten years: the questions of classification were already stated in these letters in fully modern terms; if the one that Leibniz raised was solved in the XIXth century by the theory of abelian integrals, the other, on the possibility of reducing a given differential equation to quadratures, is still open despite important recent work. If this was so, it was that Newton and Leibniz, each on his own account, had already reduced the fundamental operations of the infinitesimal calculus to an algorithm; it sufficed to write, in the notation used by one or the other, a problem of quadrature or a differential equation, for its algebraic structure to appear immediately, disengaged from its geometric coating; the methods

[15] In the course of the exchange of letters, which was not done directly between the interested parties, but officially through the secretary of the Royal Society as intermediary, Newton "staked his claim", stating his method as follows: $5accdae\ 10effh\ 12i\ \ldots rrrsssssttuu$, an anagram in which one finds encapsulated the method of solution by a power series with undetermined coefficients ((XXII), p. 224).

of "transmutation" also can be written in simple analytic terms; the problems of classification are stated in a precise fashion. Mathematically speaking, the XVIIth century had reached its end.

E) *Interpolation* and the *calculus of differences*. This theme (from which we do not separate the study of the *binomial coefficients*) appeared early and continued through the century, for reasons simultaneously theoretical and practical. One of the great tasks of the time was in fact the calculation of trigonometric, logarithmic, and nautical tables, made necessary by the rapid progress of geography, navigation, theoretical and practical astronomy, of physics, of celestial mechanics; and many of the most eminent mathematicians, from Kepler to Huygens and Newton, participated, either directly, or by theoretical research into the most efficient processes of approximation.

One of the first problems, in the use and even the preparation of tables, was that of interpolation; and as the precision of calculations increased, it was perceived in the XVIIth century that the ancient procedure of linear interpolation loses its validity as the first differences (differences between successive values in the table) cease to be perceptibly constant; thus one sees Briggs, for example [16], make use of differences of higher order, and even of rather high order, in the calculation of logarithms. Later, we see Newton ((XIX *d*) and (XX), Book III, Lemma 5) [17] and J. Gregory ((XVII bis), p. 119-120), each on his own, pursue in parallel their research on interpolation and on power series; both arrived, moreover by different methods, on the one hand at the formula for interpolation by polynomials, called "Newtonian", and on the other at the binomial series ((XVII bis), p. 131; (XXII), p. 180) and at the principal power series expansions of classical analysis ((XVII bis); (XIX *a* and *d*) and (XXII), p. 179-192 and 203-225); it is hardly in doubt that these two lines of research reacted on one another, and were intimately linked also in the mind of Newton to the discovery of the principles of the infinitesimal calculus. Great concern for practical numerical work appears in Gregory as in Newton, in construction and usage of tables, in numerical calculation of series and integrals; in particular, although one does not find any careful proof of convergence, of the kind of that of Lord Brouncker cited above, both make constant mention of the convergence of their series from the practical point of view of their aptness for calculation. This is again how we see Newton, in reply to a question posed by Collins for practical purposes [18] apply the so-called Euler-Maclaurin summation method to the approximate calculation of $\displaystyle\sum_{p=1}^{N} \frac{1}{n+p}$ for large values of N.

One also encounters early on the calculation of values of a function starting from the differences, employed as a practical procedure for integration, and even, might one say, of integration of a differential equation. Thus Wright, in 1599, having, with

[16] H. BRIGGS, *Arithmetica logarithmica,* London, 1624, chap. XIII.

[17] See also D.C. FRASER, Newton's Interpolation Formulas, *Journ. Inst. Actuaries,* v. 51 (1918), p. 77-106 and p. 211-232, and v. 58 (1927), p. 53-95 (articles reprinted as a booklet, London (undated)).

[18] *Cf.* ST. P. RIGAUD, *Correspondence of scientific men* ... Oxford, 1841, v. II, p. 309-310.

a view to nautical tables, to solve a problem which we would write as

$$\frac{dx}{dt} = \sec t = \frac{1}{\cos t},$$

proceeded by adding the values of $\sec t$, by successive intervals of a second of arc [19] obtaining naturally, to all intents and purposes, a table of values of $\log \tan \left(\pi/4+t/2\right)$; and this coincidence, observed since the calculation of the first tables of $\log \tan t$, remained unexplained until the integration of $\sec t$ by Gregory in 1668 ((XVII c) and (XVII bis) p. 7 and 463).

But these questions also have a purely theoretical and even arithmetical aspect. Let us agree to denote by $\Delta^r x_n$ the sequences of successive differences of a sequence $\left(x_n\right)_{n\in\mathrm{N}}$, defined by recurrence by means of $\Delta x_n = x_{n+1} - x_n$, and

$$\Delta^r x_n = \Delta\left(\Delta^{r-1} x_n\right),$$

and to denote by S^r the inverse operation of Δ and its iterates, thus putting $y_n = \mathrm{S}x_n$ if $y_0 = 0$, $\Delta y_n = x_n$, and $\mathrm{S}^r x_n = \mathrm{S}\left(\mathrm{S}^{r-1} x_n\right)$; one has $\mathrm{S}^r x_n = \sum_{p=0}^{n-r} \binom{n-p-1}{r-1} x_p$, and, in particular, if $x_n = 1$ for all n, one has $\mathrm{S}x_n = n$, and the sequences $\mathrm{S}^2 x_n$ and $\mathrm{S}^3 x_n$ are those of the "triangular" and "pyramidal" numbers already studied by the Greek arithmeticians, and in general $\mathrm{S}^r x_n = \binom{n}{r}$ for $n \geqslant r$ (and $\mathrm{S}^r x_n = 0$ for $n < r$); these sequences were introduced, from this point of view, certainly as early as the XVIth century; they appeared of their own accord also in the combinatorial problems, which, either on their own, or in connection with probabilities, played rather a large rôle in the mathematics of the XVIIth century, for example, with Fermat and Pascal, then with Leibniz. They also appeared in the expression for the sum of the m^{th} powers of the first N integers, whose calculation, as we have seen, underlay the integration of $\int x^m\,dx$ for m an integer, by the first method of Fermat ((XI), v. II, p. 83). This was how Wallis proceeded in 1655 in his *Arithmetica Infinitorum* (XV a) not knowing the (unpublished) work of Fermat, and also, he said, in ignorance of the method of indivisibles except by reading Torricelli; it is true that Wallis, keen to finish, did not delay himself with meticulous research: once he had achieved the result for the first integral values of m he supposed it true "by induction" for every integer m, passed correctly to the case $m = 1/n$ for integer n, then by an "induction" yet more summary than the first, to arbitrary rational m. But the interest and originality of his work is that he raised himself progressively from there to the study of the "Eulerian" integral $\mathrm{I}(m,n) = \int_0^1 (1 - x^{1/m})^n\,dx$ (of which the value, for m and $n > 0$ is $\Gamma(m+1)\,\Gamma(n+1)/\Gamma(m+n+1)$) and other similar integrals, drew up the table of values of $1/\mathrm{I}(m,n)$ for integers m and n, which is none other than that of the integers $\binom{m+n}{n}$, and, by methods almost identical to those

[19] ED. WRIGHT, *Table of Latitudes,* 1599 (*cf. Napier Memorial Volume,* 1914, p. 97).

which one uses today in expounding theory of the Γ function, ended at the infinite product for $\mathrm{I}(\frac{1}{2}, \frac{1}{2}) = \pi/4 = \left(\Gamma(\frac{3}{2})\right)^2$; it was not difficult, besides, to render his method correct by integrations by parts and very simple changes of variables, and by the consideration of $\mathrm{I}(m, n)$ for all real values of m and n, which he could hardly have thought of, but which the Newtonian analysis would soon make possible. In any case it was the "interpolation" effected by Wallis of the integers $\binom{m+n}{n}$ to non-integral values of m (more precisely to values of the form $n = p/2$, with p an odd integer) which served as a point of departure for Newton when he started out ((XXII), p. 204-206), leading him, first to study the particular case $(1-x^2)^{p/2}$, to the binomial series, then to the introduction of x^a (thus denoted) for every real a, and to the differentiation of x^a by means of the binomial series; all this without a great effort to obtain proofs or even rigorous definitions; further, a remarkable innovation, was that from knowledge of the derivative of x^a he deduced $\int x^a\,dx$ for $a \neq -1$ ((XIX a) and (XXII), p. 225). For the rest, although he was soon in possession of much more general methods for expansion in power series, such as the so-called Newton polygonal method (for algebraic functions) ((XXII), p. 221) and that of undetermined coefficients, he returned many times later, with a sort of predilection, to the binomial series and its generalizations; it was from there, for example, that he seems to have derived the expansion of $\int x^a(1+x)^\beta\,dx$ mentioned above ((XXII), p. 209).

The evolution of these ideas on the continent, however, was very different, and much more abstract. Pascal had drawn close to Fermat in the study of the binomial coefficients (from which he formed and named the "arithmetic triangle") and their use in the calculus of probabilities and the calculus of differences; when he tackled integration he introduced the same ideas there. Like his predecessors, when he used the language of indivisibles, he conceived the integral $\mathrm{F}(x) = \displaystyle\int_0^x f(x)\,dx$ as the value of the ratio of "the sum of all the ordinates of the curve"

$$\mathrm{S}\left(f\left(\frac{n}{N}\right)\right) = \sum_{0 \leqslant p < \mathrm{N}x} f\left(\frac{p}{N}\right),$$

to the "unit" $\mathrm{N} = \sum_{0 \leqslant p < \mathrm{N}} 1$ for $\mathrm{N} = \infty$ ((XII b), p. 352-355) (or, when he abandoned this language for the correct language of exhaustion, as the limit of this ratio as N increases indefinitely). But, having the problems of moments in view, he observed that, when dealing with discrete masses y_i distributed at equidistant intervals, the calculation of the total mass comes back to the operation $\mathrm{S}y_n$ defined above, and the calculation of the moment to the operation $\mathrm{S}^2\, y_n$; and, by analogy, he iterated the operation $\int$ to form what he called the "triangular sums of the ordinates", thus, in our language, the limits of the sums $\mathrm{N}^{-2}\,\mathrm{S}^2\left(f(n/\mathrm{N})\right)$, that is to say the integrals $\mathrm{F}_2(x) = \int_0^x \mathrm{F}(x)\,dx$; a new iteration gave him the "pyramidal sums" $\mathrm{F}_3(x) = \int_0^x \mathrm{F}_2(x)\,dx$, the limits of $\mathrm{N}^{-3}\,\mathrm{S}^3\left(f(n/\mathrm{N})\right)$; the context makes clear that it was not for lack of originality either in his thought or his language that he stopped here, but only because he anticipated using only these, whose systematic employment is at the base of a good part of his results, and for which he immediately proved the properties which we would write as $F_2(x) = \int_0^x (x-u)\, f(u)\, du$, $F_3(x) = \frac{1}{2}\int_0^x (x-u)^2 f(u)\, du$ ((XII *b*), p. 361-367); all this without writing a single formula, but in a language so transparent and so precise that one can immediately transcribe it into formulae as we have just done. With Pascal, as with his predecessors, but in a much clearer and systematic manner, the choice of the independent variable (which is always one of the coordinates, or else the arc of a curve) is implicit in the convention which fixes the equidistant (though "infinitely close") points of subdivision of the interval of integration; these points, according to the case, are either on Ox or on Oy or on the arc of the curve, and Pascal takes care never to leave any ambiguity on this subject ((XII *b*), p. 368-369). When he has to change variable he does so by means of a principle which amounts to saying that the area $\int f(x)\, dx$ can be written $\mathrm{S}\big(f(x_i)\,\Delta x_i\big)$ for every subdivision of the interval of integration into "infinitely small" intervals Δx_i, equal or not ((XII *d*), p. 61-68).

As one sees, we are already very close to Leibniz; and it was, one might say, a lucky accident that he, when he wished to be initiated into modern mathematics, had met Huygens, who immediately put the writings of Pascal into his hands ((XXII), p. 407-408); he was particularly prepared by his thoughts on combinatorial analysis, and we know that he made a profound study of it, as is reflected in his œuvre. In 1675, we see him transcribe the theorem of Pascal cited above, in the form $\mathrm{omn}(x\omega) = x.\mathrm{omn}\,\omega - \mathrm{omn}(\mathrm{omn}\,\omega)$, where omn ω is an abbreviation for the integral of ω taken from 0 to x, for which Leibniz, several days later, substituted $\int \omega$ (the initials of "summa omnium ω") at the same time as he introduced d for the infinitely small "difference", or as he would soon say, the differential ((XXII), p. 147-167). Conceiving these "differences" as quantities comparable among themselves though not with finite quantities, he nevertheless most often took, explicitly or not, the differential dx of the independent variable x as unity, $dx = 1$ (which amounts to identifying the differential dy with the derivative dy/dx), and at first omitted it from his notation for the integral, which thus appeared as $\int y$ rather than as $\int y\, dx$; but he scarcely delayed introducing this latter, and kept to it systematically once he perceived its invariant character with respect to the choice of independent variable, which dispenses with having this choice constantly present in mind [20] ; and he showed no little satisfaction when he returned to the study of Barrow, whom he had neglected until then, in noting that the general theorem on change of variable, of which Barrow makes so much, follows immediately from his own notation ((XXII), p. 412). Moreover in all this he kept very close to the calculus of differences, from which his differential calculus was deduced by a passage to the limit which of course he would have had to take great pains to justify rigorously; and in what followed he insisted deliberately on the fact that his principles apply equally well to one and to the other. He cited Pascal expressly, for example, when, in his correspondence with

[20] *"J'avertis qu'on prenne garde de ne pas omettre dx ... faute fréquemment commise, et qui empêche d'aller de l'avant, du fait qu'on ôte par là à ces indivisibles, comme ici dx, leur généralité ... de laquelle naissent d'innombrables transfigurations et équipollences de figures."* ((XXI *b*), p. 233).

Johann Bernoulli ((XXI), v. III, p. 156), referring to his first researches, he gave a formula from the calculus of differences which is a special case of that of Newton, and from it deduced by "passing to the limit" the formula $y = \sum_{1}^{\infty} (-1)^{n-1} \frac{d^n y}{dx^n} \frac{x^n}{n!}$ (where y is a function vanishing for $x = 0$, and the $d^n y/dx^n$ are its derivatives for the value x of the variable), a formula equivalent to a similar one which Bernoulli had just communicated to him ((XXI), v. III, p. 150) and (XXIV), v. I, p. 125-128), and which the latter proved by successive integrations by parts. This formula, as one sees, is very close to the Taylor series; and it was the same argument as Leibniz, by passage to the limit starting from the calculus of differences, that Taylor rediscovered in 1715 to obtain "his" series,[21] without however making great use of it.

F) One will already have seen, implicit in the evolution described above, the progressive *algebraisation* of the infinitesimal calculus, that is to say its reduction to an *operational calculus* endowed with a uniform system of notation of algebraic character. As Leibniz had indicated many times with perfect clarity ((XXI *b*) p. 230-233), it was a matter of doing for the new analysis what Viète had done for the theory of equations, and Descartes for geometry. To understand the need for this, one has only to read a few pages of Barrow; at no time can one cope without having under one's eyes a sometimes complicated figure, described meticulously beforehand; he needs no fewer than 180 figures for the 100 pages (Lect. V-XII) which form the essence of his work.

There could hardly be a question of algebraisation, it was true, before some unity had appeared across the multiplicity of geometric appearances. However, Grégoire de St. Vincent (IX) had already introduced (under the name *"ductus plani in planum"*) a sort of law of composition which amounts to using systematically the integrals $\int_a^b f(x)g(x)\,dx$, considered as volumes of solids $a \leqslant x \leqslant b$, $0 \leqslant y \leqslant f(x)$, $0 \leqslant z \leqslant g(x)$; but he was far from drawing the consequences which Pascal later deduced, as one has seen, from studying the same solid. Wallis in 1655, and Pascal in 1658, forged, each in his own way, languages of algebraic character, in which, without writing any formula, they drafted statements which one can transcribe immediately into formulae of the integral calculus as soon as one has understood the mechanism. The language of Pascal is particularly clear and precise; and, if one does not understand why he refused to use the algebraic notation, not only of Descartes, but even that of Viète, one can only admire the *tour de force* which he accomplished, and which his mastery of language alone made possible.

But let several years pass and all changes. Newton was the first to conceive the idea of replacing all operations of a geometric character, of contemporary infinitesimal

[21] B. TAYLOR, *Methodus Incrementorum directa et inversa,* Lond., 1715. For the calculus of differences Taylor could naturally rely on the results of Newton, contained in a famous lemma of the *Principia* ((XX), Book III, lemma 5) and published more fully in 1711 (XIX *d*). As to the idea of passing to the limit, it seems to be typically Leibnizian; and one could hardly believe in the originality of Taylor on this point if one did not know many examples from all periods of disciples ignorant of all apart from the writings of their master and patron. Taylor cited neither Leibniz nor Bernoulli; but the Newton-Leibniz controversy was raging, Taylor was secretary of the Royal Society, and Sir Isaac was its all-powerful president.

analysis, by a single analytic operation, differentiation, and by the solution of the inverse problem; an operation which of course the method of power series allowed him to execute with extreme facility. Borrowing his language, we have seen that, with the fiction of a universal "temporal" parameter, he styles as "fluentes" the variable quantities as a function of this parameter, and their derivatives as "fluxions". He does not seem to have attached a particular importance to notation, and his devotees later vaunted the absence of a systematic notation as an advantage; nevertheless, for his personal use, he soon adopted the habit of denoting the fluxion by a point, thus dx/dt by $\dot{x}$, d^2x/dt^2 by $\ddot{x}$, etc. As to integration, it seems that Newton, just like Barrow, had never envisaged it save as a problem (to find the fluent knowing the fluxion, to solve $\dot{x} = f(t)$), and not as an operation; also he has no name for the integral, nor, it seems, a standard notation (except sometimes a square, $\left|\overline{f(t)}\right.$ or $\Box f(t)$ for $\int f(t)\,dt$). Was it because he was reluctant to give a name and a sign to what was not defined in a unique manner, but only up to an additive constant? For lack of a text one can only ask the question.

As much as Newton was empirical, correct, circumspect in his greatest boldnesses, so Leibniz was systematic, a generalizer, adventurous innovator and sometimes boastful. From his youth he had in his head the idea of a "characteristic" or universal symbolic language, which would be to the total of human thought what algebraic notation is to algebra, where every name or sign would be the key to all the qualities of the thing signified, and which one could not use correctly without as a result reasoning correctly. It is easy to treat such a project as chimerical; yet it is not an accident that its author was the very man who would soon recognise and isolate the fundamental concepts of the infinitesimal calculus, and endow it with its more or less definitive notation. We have already witnessed above its birth, and observed the care with which Leibniz, who seemed aware of his mission, modified them progressively to ensure the simplicity and above all the invariance that he sought (XXI *a* and *b*). What is important to remark here is the clear concept of $\int$ and of d, of the integral and of the differential, as mutually inverse operators, from when he introduced them (knowing nothing yet of the ideas of Newton). It is true that in proceeding in this way he could not escape from the ambiguity inherent in the indefinite integral, which was the weak point of his system, over which he skates adroitly, as do his successors. But what strikes one, from the first appearance of the new symbols, is to see Leibniz immediately busy formulating their rules of use, asking if $d(xy) = dx\,dy$ ((XXII), p. 16-166), and answering himself in the negative, then coming progressively to the correct rule (XXI *a*), which he would later generalize to his famous formula for $d^n(xy)$ ((XXI), v. III, p. 175). Of course, at the moment that Leibniz was groping in this way, Newton had already known for ten years that $z = xy$ implies $\dot{z} = \dot{x}y + x\dot{y}$; but he never took the trouble to say so, not seeing in it anything other than a special case, not worthy of naming, of his rule for differentiating a relation $\mathrm{F}(x, y, z) = 0$ among fluents. On the contrary, the principal concern of Leibniz was not to make his methods serve for the solution of such concrete problems, nor even to deduce them from rigorous and impregnable principles, but above all to set up an algorithm resting on the formal manipulation of a few simple rules. It was in this spirit that he improved algebraic notation by the use of parentheses, that he progressively adopted $\log x$ or lx for the logarithm [22], and that he insisted on the "exponential calculus", that is the systematic consideration of exponentials a^x, x^x, x^y, where the exponent is a variable. Above all, while Newton did not introduce fluxions of higher order except as strictly necessary in each concrete case, Leibniz early oriented himself towards the creation of an "operational calculus" by the iteration of d and of $\int$; little-by-little becoming aware of the analogy between multiplication of numbers and the composition of operators in his calculus, he adopted, with happy audacity, the notation for exponents to write the iterates of d, thus writing d^n for the n^{th} iterate ((XXII), p. 595 and 601 [23], and (XXI), v. V, p. 221 and 378) and even d^{-1}, d^{-n} for $\int$ and its iterates ((XXI), v. III, p. 167); and he even tried to give a meaning to d^α for arbitrary real α ((XXI), v. II, p. 301-302, and v. III, p. 228).

This is not to say that Leibniz was not also interested in the applications of his calculus, well knowing (as Huygens often repeated to him ((XXII), p. 599)) that they are the touchstone; but he lacked the patience to go deeply into them, and above all he sought the opportunity to formulate new general rules. It was thus that in 1686 (XXI c) he treated the curvature of curves, and the osculating circle, to end up in 1692 (XXI d) with the general principles of the contact of plane curves [24]; and in 1692 (XXI e) and 1694 (XXI f) he set out the bases for the the theory of envelopes; concurrently with Johann Bernoulli, he effected in 1702 and 1703 the integration of rational fractions by decomposition into simple elements, but first in a formal manner and without fully appreciating the circumstances which accompany the presence of complex linear factors in the denominator (XXI, g and h). It was thus again that one August day in 1697, meditating in his carriage on questions of the calculus of variations, he had the idea for the rule for differentiation with respect to a parameter under the $\int$ sign, and enthusiastically sent it off immediately to Bernoulli ((XXI, v. III, p. 449-454). But when he got there, the fundamental principles of his calculus had been established a long time, and their use had begun to spread: the algebraisation of infinitesimal analysis was an accomplished fact.

G) The notion of *function* was introduced and clarified in many ways during the XVIIth century. All kinematics rests on an intuitive, in some way experimental, idea of quantities that vary in time, that is, functions of time, and we have already seen how one thus comes to a function of a parameter, such as appears with Barrow, and, under

[22] But he had no symbol for the trigonometric functions, nor (lacking a symbol for e) for the "the number whose logarithm is x".

[23] . . . *"c'est à peu près comme si, au lieu des racines et puissances, on vouloit toujours substituer des lettres, et au lieu de* xx, *ou* x^3, *prendre* m, *ou* n, *après avoir déclaré que ce doivent estre les puissances de la grandeur* x. *Jugés, Mons., combien cela embarasseroit. Il en est de mesme de* dx *ou de* ddx, *et les differences ne sont pas moins des affections des grandeurs indéterminées dans leur lieux, que les puissances sont des affections d'une grandeur prise à part. Il me semble donc qu'il est plus naturel de les désigner en sorte qu'elles fassent connoistre immédiatement la grandeur dont elles sont les affections."*

[24] He first committed a singular error on that point, believing that the "kissing circle" (the osculating circle) has four points in common with the curve at the point of contact; he surrendered later on this matter, only with difficulty, to the objections of the Bernoulli brothers ((XXI), v. III, p. 187-188, 201-202 and 207).

the name fluent, with Newton. The notion of an "arbitrary curve" appeared often, but was rarely made precise; it could be that it was often thought of in a kinematic or in any case experimental form, and without judging it necessary for a curve to be susceptible of a geometric or analytic characterisation in order to serve as an object of argument; so it was, in particular (for reasons which we are better able to understand today) when it concerns integration, for example with Cavalieri, Pascal and Barrow; the latter, reasoning about the curve defined by $x = ct,\ y = f(t)$, with the hypothesis that dy/dt should be increasing, even said expressly that *"it does not matter"* that dy/dt increase *"regularly according to some law, or even irregularly"* ((XVIII), p. 191) that is, as we would say, be susceptible or not to an analytic definition. Unfortunately this clear and fruitful idea, which would, suitably clarified, reappear in the XIXth century, could not then fight against the confusion created by Descartes, when the latter had, in the first place, banned from "geometry" all curves not susceptible of a precise analytic definition, and in the second place, restricted the admissible procedures of formation in such a definition to algebraic operations alone. It is true that, on this last point, he was not followed by the majority of his contemporaries; little-by-little, and often by very subtle detours, the various transcendental operations, the logarithm, the exponential, the trigonometric functions, quadratures, solution of differential equations, passage to the limit, summation of series, became established, though it is not easy in each case to mark the precise moment when the step forward was made; and, moreover, the first step forward was often followed by a step back. For the logarithm, for example, one must consider as important stages the appearance of the logarithmic curve ($y = a^x$ or $y = \log x$ according to the choice of axes), of the logarithmic spiral, the quadrature of the hyperbola, the series expansion of $\log(1+x)$, and even the adoption of the symbol $\log x$ or lx. In what concerns the trigonometric functions, which in a certain sense reach back to antiquity, it is interesting to observe that the sinusoid did not first appear as defined by an equation $y = \sin x$, but with Roberval ((VIII *a*), p. 63-65), as a "companion of the cycloid" (so as a case of the curve

$$y = \mathrm{R}\left(1 - \cos\frac{x}{\mathrm{R}}\right)\Big),$$

that is, as an auxiliary curve whose definition is derived from that of the cycloid. To encounter the general notion of an analytic expression we must go to J. Gregory, who defined it in 1667 ((XVI bis), p. 413), as a quantity which was obtained from other quantities by a succession of algebraic operations *"or by any other imaginable operation"*; he attempted to make this notion precise in his preface ((XVI bis), p. 408-409), explaining the necessity of adding to the five operations of algebra [25] a sixth operation, which, when all is said and done, is none other than passing to the limit. But these interesting thoughts were soon forgotten, submerged in the torrent of series expansions discovered by Gregory himself, by Newton, and others; and the prodigious success of this last method created a lasting confusion between functions susceptible of an analytic definition, and functions expandable in power series.

[25] These are the four rational operations, and the extraction of roots of any order: J. Gregory never ceased to believe in the possibility of solving equations of all degrees by radicals.

As for Leibniz, he seemed to hold to the Cartesian point of view, widened by the explicit adjunction of quadratures, and by the implicit adjunction of the other operations familiar to the analysis of his time: summation of power series, solution of differential equations. Likewise, Johann Bernoulli, when he wanted to consider an arbitrary function of x, introduced it as *"a quantity formed in any way starting from x and the constants"* ((XXI), v. II, p. 150), sometimes specifying that it concerns a quantity formed *"in an algebraic or transcendental manner"* ((XXI), v. II, p. 324); and, in 1698, he agreed with Leibniz to call such a quantity a "function of x" ((XXI), v. III, p. 507-510 and p. 525-526) [26]. Leibniz had already introduced the words "constant", "variable", "parameter", and refined, *à propos* envelopes, the notion of a family of curves depending on one or more parameters (XXI *e*). The questions of notation were clarified also in the correspondence with Johann Bernoulli: the latter readily writes X or ξ, for an arbitrary function of x ((XXI), v. III, p. 531); Leibniz approved, but also proposed $x^{\overline{|1|}}$, $x^{\overline{|2|}}$, where we would write $f_1(x)$, $f_2(x)$; and he proposed, for the derivative dz/dx of a function z of x, the notation dx (in contrast to dz, which is the differential) while Bernoulli wrote Δz ((XXI), v. III, p. 537 and 526).

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Thus, with the century, the heroic epoch was over. The new calculus, with its notions and notation, was established, in the form which Leibniz had given it. The first disciples, Jakob and Johann Bernoulli, were rivals in discovery with the master, browsing through the rich expanses to which he had shown them the way. The first treatise on differential and integral calculus was written in 1691 and 1692 by Johann Bernoulli [27], for the benefit of a marquis who showed himself an apt pupil. It hardly mattered that Newton decided at last, in 1693, to publish a parsimoniously brief glimpse of his fluxions ((XV), v. II, p. 391-396); if his *Principia* provided food for thought for more than a century, on the terrain of the infinitesimal he had been caught up with, and on many points overtaken.

[26] Until then, and already in a manuscript of 1673, Leibniz had employed this word as an abbreviation to denote a quantity "remplissant telle ou telle fonction" for a curve, for example the length of the tangent or of the normal (bounded by the curve and Ox), or even the subnormal, the subtangent, etc., thus, in sum, a function of a variable point on a curve, with a geometrico-differential definition. In the same manuscript of 1673 the curve is assumed to be defined by a relation between x and y, *"donnée par une équation"*, but Leibniz adds that *"il n'importe pas que la courbe soit ou non géométrique"* (that is to say, in our language, algebraic) (*cf.* D. MAHNKE, *Abh. Preuss. Akad. der Wiss.*, 1925, Nr. 1, Berlin, 1926).

[27] The part of this treatise on the integral calculus was published only in 1742 ((XXIV), v. III, p. 385-558); that which deals with the differential calculus was rediscovered and published only recently (XXIV bis); it is true that the Marquis de l'Hôpital had published it in French, slightly recast, under his own name, as Bernoulli remarked bitterly in his letters to Leibniz.

The weaknesses of the new system are nevertheless visible, at least to our eyes. Newton and Leibniz, abolishing at one blow a tradition of two millennia, accorded the primordial rôle to differentiation and reduced integration to be only its inverse; it would take all the XIXth century, and part of the XXth, to reestablish a fair equilibrium, by putting integration at the basis of the general theory of functions of a real variable and of its modern generalizations (see the Historical Notes to the Book on Integration). This reversal of point of view was also responsible for the excessive, almost exclusive, rôle, given to the indefinite integral at the expense of the definite integral, already seen in Barrow, and above all in Newton and Leibniz: there also the XIXth century had to put things back in place. Finally, the characteristically Leibnizian tendency to formal manipulation of symbols continued to accentuate itself throughout the XVIIIth century, well beyond what the resources of analysis at this time could justify. In particular, one must recognise that the Leibnizian notion of differential had, to tell the truth, no meaning; at the start of the XIXth century it fell into a disrepute from which it has been raised only little-by-little; and, if the employment of first differentials has been completely legitimised, the differentials of higher order, although so useful, have to this day not yet been truly rehabilitated.

However it may be, the history of the differential and integral calculus, starting from the end of the XVIIth century, divides into two parts. The one relates to the applications of this calculus, always more rich, numerous and varied. To the differential geometry of plane curves, to differential equations, to power series, to the calculus of variations, referred to above, were added the differential geometry of skew curves, then of surfaces, multiple integrals, partial differential equations, trigonometric series, the study of numerous special functions, and many other types of problems, whose history will be expounded in the Books devoted to them. We treat here only the works which have contributed to fine-tune, deepen, and consolidate the very principles of the infinitesimal calculus, in what concerns functions of a real variable.

From this point of view, the great treatises of the middle of the XVIIIth century offer few novelties. Maclaurin in Scotland [28], Euler on the continent (XXV *a* and *b*), remained faithful to the traditions to which each was the heir. It is true that the first exerted himself to clarify the Newtonian concepts a little [29], while the second, pushing the Leibnizian formalism to its extreme, was content, like Leibniz and Taylor, to let the differential calculus rest on a very obscure passage to the limit starting from the calculus of differences, a calculus of which he gave a very careful exposition. But above all Euler completed the work of Leibniz in introducing the notation still in use today for e, i, and the trigonometric functions, and in spreading the notation π. On the other hand, and if most often he made no distinction between functions and analytic expressions, he insisted, *à propos* trigonometric series and the problem of vibrating strings, on the necessity of not restricting oneself to the functions so

[28] C. MACLAURIN, *A complete treatise of fluxions,* Edinburgh, 1742.

[29] They much needed, indeed, to be defended against the philosophico-theologico-humoristic attacks of the famous Bishop Berkeley. According to him, one who believed in fluxions could not find it difficult to have faith in the mysteries of religion: an argument *ad hominem,* lacking neither logic nor piquancy.

defined (and which he qualified as "continuous"), but to consider too, should the case arise, arbitrary, or "discontinuous", functions, given experimentally by one or more arcs of a curve ((XXV c), p. 74-91). Finally, although this goes a little outside our framework, it is impossible not to mention here his extension of the exponential function to the complex plane, whence he obtained his celebrated formulae linking the exponential with the trigonometric functions, as well as the definition of the logarithm of a complex number; there one finds elucidated definitively the famous analogy between the logarithm and the inverse circular functions, or, in the language of the XVIIth century, between the quadratures of the circle and of the hyperbola, already observed by Grégoire de St. Vincent, made precise by Huygens and above all by Gregory, and which, with Leibniz and Bernoulli, appeared in the formal integration of $\dfrac{1}{1+x^2} = \dfrac{i}{2(x+i)} - \dfrac{i}{2(x-i)}$.

D'Alembert, meanwhile, the enemy of all mystique in mathematics as elsewhere, had, in remarkable articles (XXVI), defined with the greatest clarity the notions of limit and of derivative, and maintained forcefully that at bottom this is the whole "metaphysics" of the infinitesimal calculus. But this wise counsel did not have an immediate effect. The monumental work of Lagrange (XXVII) represents an attempt to found analysis on one of the most arguable of the Newtonian concepts, that which confuses the notions of an arbitrary function and of a function expandable in a power series, and to derive the notion of differentiation from that (by considering the coefficient of the term of first order in the series). Of course, a mathematician of the calibre of Lagrange could not fail to obtain important and useful results at that point, like, for example (and in a manner which was in reality independent of the point of departure just indicated) the general proof of the Taylor formula with remainder expressed as an integral, and its evaluation by the mean value theorem; besides, Lagrange's work was at the origin of Weierstrass' method in the theory of functions of a complex variable, as well as the modern algebraic theory of formal series. But, from the point of view of its immediate purpose, it represented a retreat rather than a progress.

With the teaching works of Cauchy, on the contrary (XXVIII), one finds oneself again on solid ground. He defined a function essentially as we do today, although in language still a little vague. The notion of limit, fixed once and for all, was taken as the point of departure; those of a continuous function (in the modern sense) and of the derivative are deduced immediately, as well as their principal elementary properties: and the existence of the derivative, instead of being an article of faith, becomes a question to study by the ordinary methods of analysis. Cauchy, to tell the truth, hardly interested himself in this; and on the other hand, if Bolzano, having come to the same principles, constructed an example of a continuous function not having a finite derivative at any point (XXIX), this example was not published, and the question was not settled publicly until Weierstrass, in a work of 1872 (and in his course from 1861) (XXXII).

As regards integration, the work of Cauchy represents a return to the sound traditions of antiquity and the first part of the XVIIth century, but relying on still inadequate technical means. The definite integral, for a long time relegated to the second rank, again became the primordial notion, for which Cauchy adopted definitively the notation $\int_a^b f(x)\,dx$ proposed by Fourier (in place of the unwieldy $\int f(x)\,dx \begin{bmatrix} x = b \\ x = a \end{bmatrix}$ sometimes employed by Euler); and, to define it, Cauchy returned to the method of exhaustion, or as we would say, to "Riemann sums" (which it would be better to call Archimedes sums, or Eudoxus sums). It is true that the XVIIth century never judged it apposite to subject the notion of area, which to it appeared at least as clear as that of an incommensurable real number, to critical examination; but the convergence of the "Riemann" sums towards the area under the curve, if only a monotone or piecewise monotone curve, was an idea familiar to all authors concerned with rigour in the XVIIth century, such as Fermat, Pascal, Barrow; and J. Gregory, particularly well prepared by his thoughts on passage to the limit and his familiarity with an already very abstract form of the principle of "nested intervals", had even drafted, it would appear, a careful proof, which remained unpublished ((XVII bis), p. 445-446), and could have served Cauchy almost without change, had he known it [30]. Unfortunately for him, Cauchy claimed to prove the existence of the integral, that is the convergence of the "Riemann sums", for an arbitrary continuous function; and his proof, though correct if based on the theorem on the uniform continuity of continuous functions on a closed interval, was denuded of all probative value for lack of this concept. Nor did Dirichlet seem to notice the difficulty when he composed his celebrated memoirs on trigonometric series, since he cited the theorem in question as "easy to prove" ((XXX), p. 136); it is true that, in the final analysis, he applied it only to bounded piecewise monotone functions; Riemann, more circumspect, mentions only these last, when it is a matter of using his necessary and sufficient condition for the convergence of the "Riemann sums" ((XXXI), p. 227-271). Once the theorem on uniform continuity had been established by Heine (*cf. Gen. Top.*, II, Historical Note, p. 217), the question of course no longer offered any difficulty; and it was easily resolved by Darboux in 1875 in his memoir on the integration of discontinuous functions (XXXIII), a memoir where he happens to be in agreement on many points with the important researches of P. du Bois-Reymond, which appeared about the same time. As a result, one finds proved for the first time, but this time definitively, the linearity of the integral of continuous functions. On the other hand, the notion of uniform convergence of a sequence or a series, introduced by Seidel, among others, in 1848, and put to good use by Weierstrass (*cf. Gen. Top.*, X, Historical Note, p. 347), had made it possible to give a solid basis, under conditions a little too restrictive, it is true, to the integration of series term-by-term, and differentiation under the $\int$ sign, while awaiting the modern theories of which we do not have to speak here, and which clarify these questions in a provisionally definitive manner.

We have thus reached the final stage of the classical infinitesimal calculus, that represented by the great Treatises on Analysis of the end of the XIXth century; from our point of view, that of Jordan (XXXIV) occupies the most eminent place among

[30] This at least is what the summary given by Turnbull from the manuscript would indicate.

them, in part for aesthetic reasons, but also because, if it constitutes an admirable exposition of the results of classical analysis, it announces modern analysis in many ways, and prepares the way for it. After Jordan came Lebesgue, and one enters into the subject of another Book of the present work.

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CHAPTER IV
Differential equations

§ 1. EXISTENCE THEOREMS

1. THE CONCEPT OF A DIFFERENTIAL EQUATION

Let I be an interval contained in $\mathbf{R}$, not reducing to a single point, E a *topological vector space* over $\mathbf{R}$, and A and B two open subsets of E. Let $(\mathbf{x}, \mathbf{y}, t) \mapsto \mathbf{g}(\mathbf{x}, \mathbf{y}, t)$ be a continuous map of $A \times B \times I$ into E; to every *differentiable* map $\mathbf{u}$ of I into A whose derivative takes its values in B we associate the map $t \mapsto \mathbf{g}(\mathbf{u}(t), \mathbf{u}'(t), t)$ of I into E, and denote it by $\tilde{\mathbf{g}}(\mathbf{u})$; so $\tilde{\mathbf{g}}$ is defined on the set $\mathcal{D}(A, B)$ of differentiable functions of I into B whose derivatives have their values in B. We shall say that the equation $\tilde{\mathbf{g}}(\mathbf{u}) = 0$ is a *differential equation* in $\mathbf{u}$ (relative to the *real* variable t); a *solution* of this equation is also called an *integral* of the differential equation (on the interval I); it is a differentiable map of I into A, whose derivative takes values in B, such that $\mathbf{g}(\mathbf{u}(t), \mathbf{u}'(t), t) = 0$ for *every* $t \in I$. By abuse of language we shall write the differential equation $\tilde{\mathbf{g}}(\mathbf{u}) = 0$ in the form

$$\mathbf{g}(\mathbf{x}, \mathbf{x}', t) = 0,$$

on the understanding that $\mathbf{x}$ belongs to the set $\mathcal{D}(A, B)$.

For example, for $I = E = \mathbf{R}$ the relations

$$x' = 2t, \qquad tx' - 2x = 0, \qquad x'^2 - 4x = 0, \qquad x - t^2 = 0$$

are differential equations, all four of which admit the function $x(t) = t^2$ as a solution.

In this chapter we consider in principle only the case where E is a *complete normed space* over R, and where the differential equations are of the specific form

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \tag{1}$$

("explicit equations in $\mathbf{x}'$"), where $\mathbf{f}$ denotes a function defined on $I \times H$ with values in E, and H is an *open* nonempty subset of E. We shall, moreover, widen a little the concept of a *solution* (or *integral*) of such an equation (on the interval I): we shall say that a function $\mathbf{u}$, defined and continuous on I, with values in H, is a solution (or integral) of the equation (1) if there exists a *countable* subset A of I such that at every point t of the complement of A in I the function $\mathbf{u}$ admits a derivative $\mathbf{u}'(t)$

such that $\mathbf{u}'(t) = \mathbf{f}(t, \mathbf{u}(t))$. If $\mathbf{u}$ is differentiable and satisfies the relation above for *every* point $t \in \mathrm{I}$ we shall say that it is a *strict* solution of the equation (1) on I.

In the particular case of a differential equation of the form $\mathbf{x}' = \mathbf{f}(t)$, where $\mathbf{f}$ is a map of I into E, the solutions in the above sense are the *primitives* of the function $\mathbf{f}$ (II, p. 51), and the strict solutions are the *strict primitives*.

When E is a *product* of complete normed spaces E_i $(1 \leqslant i \leqslant n)$, one can write $\mathbf{x} = \big(\mathbf{x}_i\big)_{1 \leqslant i \leqslant n}$ and $\mathbf{f} = \big(\mathbf{f}_i\big)_{1 \leqslant i \leqslant n}$, where $\mathbf{x}_i$ is a map from I into E_i and $\mathbf{f}_i$ is a map from $\mathrm{I} \times \mathrm{H}$ into E_i; the equation (1) is then equivalent to the *system of differential equations*

$$\mathbf{x}_i' = \mathbf{f}_i(t, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \qquad (1 \leqslant i \leqslant n). \tag{2}$$

The most important case is that where the E_i are equal to $\mathbf{R}$ or to $\mathbf{C}$; one then says that (2) is a system of *scalar* differential equations.

One may reduce the study of relations of the form

$$\mathbf{x}^{(n)} = \mathbf{f}\big(t, \mathbf{x}, \mathbf{x}', \ldots, \mathbf{x}^{(n-1)}\big) \tag{3}$$

to that of the system (2), where $\mathbf{x}$ is an n times differentiable vector function on I; for on putting $\mathbf{x}_1 = \mathbf{x}$, and $\mathbf{x}_p = \mathbf{x}^{(p-1)}$ for $2 \leqslant p \leqslant n$, the relation (3) is equivalent to the system

$$\begin{cases} \mathbf{x}_i' = \mathbf{x}_{i+1} & (1 \leqslant i \leqslant n-1) \\ \mathbf{x}_n' = \mathbf{f}(t, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n). \end{cases} \tag{4}$$

A relation of the form (3) is called a *differential equation of order n* (explicitly resolved for $\mathbf{x}^{(n)}$); in contrast, equations of the form (1) are called differential equations of *first order.*

Similarly one may reduce any "system of differential equations" of the form

$$\mathrm{D}^{n_i}\,\mathbf{x}_i = \mathbf{f}_i(t, \mathbf{x}_1, \mathrm{D}\mathbf{x}_1, \ldots, \mathrm{D}^{n_1-1}\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathrm{D}\mathbf{x}_p, \ldots, \mathrm{D}^{n_p-1}\mathbf{x}_p) \tag{5}$$

$(1 \leqslant i \leqslant p)$ to a system of the form (2), where $\mathbf{x}_i$ is function on I which is n_i times differentiable on I (for $1 \leqslant i \leqslant p$).

2. DIFFERENTIAL EQUATIONS ADMITTING SOLUTIONS THAT ARE PRIMITIVES OF REGULATED FUNCTIONS

Recall (II, p. 54, def. 3) that a vector function $\mathbf{u}$ defined on an interval $\mathrm{I} \subset \mathbf{R}$ is said to be *regulated* if it is the uniform limit of step functions on every compact subset of I; an equivalent condition is that at every interior point of I the function $\mathbf{u}$ has a right and a left limit, and also a right limit at the left-hand end point of I and a left limit at the right-hand endpoint of I, when these two points belong to I (II, p. 54, th. 3). In this chapter we shall restrict ourselves to differential equations (1) for which every solution is a *primitive of a regulated function* on I. This condition is clearly satisfied if, for every *continuous* map $\mathbf{u}$ of I into H, the function $\mathbf{f}(t, \mathbf{u}(t))$ is *regulated* on I; the following lemma gives a sufficient condition for this:

Lemma 1. Let $\mathbf{f}$ *be a map from* $\mathrm{I} \times \mathrm{H}$ *into* E *such that, on writing* $\mathbf{f}_{\mathbf{x}}$ (*for every* $\mathbf{x} \in \mathrm{H}$) *for the map* $t \mapsto \mathbf{f}(t, \mathbf{x})$ *of* I *into* E*, the following conditions are satisfied:* 1° $\mathbf{f}_{\mathbf{x}}$ *is regulated on* I *for every* $\mathbf{x} \in \mathrm{H}$; 2° *the map* $\mathbf{x} \mapsto \mathbf{f}_{\mathbf{x}}$ *of* H *into the set* $\mathcal{F}(\mathrm{I}, \mathrm{E})$ *of maps from* I *into* E *is continuous when one endows* $\mathcal{F}(\mathrm{I}, \mathrm{E})$ *with the topology of compact convergence* (*Gen. Top.*, X, p. 278). *Under these conditions:*

1° *For every continuous map* $\mathbf{u}$ *of* I *into* H *the function* $t \mapsto \mathbf{f}(t, \mathbf{u}(t))$ *is regulated on* I*; more precisely, the right* (resp. *left*) *limit of this function at a point* $t_0 \in \mathrm{I}$ *is equal to the right* (resp. *left*) *limit of the function* $t \mapsto \mathbf{f}(t, \mathbf{u}(t_0))$ *at the point* t_0.

2° *If* $(\mathbf{u}_n)$ *is a sequence of maps of* I *into* H *which converges uniformly to a continuous function* $\mathbf{u}$ *of* I *into* H *on every compact subset of* I*, then the sequence of functions* $t \mapsto \mathbf{f}(t, \mathbf{u}_n(t))$ *converges uniformly to* $\mathbf{f}(t, \mathbf{u}(t))$ *on every compact subset of* I.

1° Let $\mathbf{c}$ be the right limit of $\mathbf{f}(t, \mathbf{u}(t_0))$ at the point t_0; for every $\varepsilon > 0$ there is a compact neighbourhood V of t_0 in I such that $\|\mathbf{f}(t, \mathbf{u}(t_0)) - \mathbf{c}\| \leqslant \varepsilon$ for $t \in \mathrm{V}$ and $t > t_0$. On the other hand, there exists $\delta > 0$ such that the relations

$$\mathbf{x} \in \mathrm{H}, \qquad \|\mathbf{x} - \mathbf{u}(t_0)\| \leqslant \delta$$

imply $\|\mathbf{f}(s, \mathbf{x}) - \mathbf{f}(s, \mathbf{u}(t_0))\| \leqslant \varepsilon$ for all $s \in \mathrm{V}$; if $\mathrm{W} \subset \mathrm{V}$ is a neighbourhood of t_0 in I such that $\|\mathbf{u}(t) - \mathbf{u}(t_0)\| \leqslant \delta$ for every $t \in \mathrm{W}$, then $\|\mathbf{f}(t, \mathbf{u}(t)) - \mathbf{c}\| \leqslant 2\varepsilon$ for $t \in \mathrm{W}$ and $t > t_0$, which proves that $\mathbf{c}$ is the right limit of $\mathbf{f}(t, \mathbf{u}(t))$ at the point t_0.

2° Let K be a compact subset of I; since $\mathbf{u}$ is continuous on I, $\mathbf{u}(\mathrm{K})$ is a compact subset of H; for every $\varepsilon > 0$ and $\mathbf{x} \in \mathbf{u}(\mathrm{K})$ there exists a number $\delta_{\mathbf{x}}$ such that, for every $\mathbf{y} \in \mathrm{H}$, $\|\mathbf{y} - \mathbf{x}\| \leqslant \delta_{\mathbf{x}}$ and every $t \in \mathrm{K}$, one has $\|\mathbf{f}(t, \mathbf{y}) - \mathbf{f}(t, \mathbf{x})\| \leqslant \varepsilon$. There is a finite number of points $\mathbf{x}_i \in \mathbf{u}(\mathrm{K})$ such that the closed balls with centre $\mathbf{x}_i$ and radius $\frac{1}{2}\delta_{\mathbf{x}_i}$ form a cover of $\mathbf{u}(\mathrm{K})$; let $\delta = \mathrm{Min}(\delta_{\mathbf{x}_i})$. By hypothesis there exists an integer n_0 such that for $n \geqslant n_0$ one has $\|\mathbf{u}_n(t) - \mathbf{u}(t)\| \leqslant \frac{1}{2}\delta$ for every $t \in \mathrm{K}$. Now, for every $t \in \mathrm{K}$ there exists an index i such that

$$\|\mathbf{u}(t) - \mathbf{x}_i\| \leqslant \tfrac{1}{2}\delta_{\mathbf{x}_i} ;$$

consequently one has $\|\mathbf{u}_n(t) - \mathbf{x}_i\| \leqslant \delta_{\mathbf{x}}$, whence $\|\mathbf{f}(t, \mathbf{u}_n(t)) - \mathbf{f}(t, \mathbf{u}(t))\| \leqslant 2\varepsilon$ for every $t \in \mathrm{K}$ and every $n \geqslant n_0$.

For the rest of this section I *will denote an interval contained in* $\mathbf{R}$*, not reducing to a single point,* H *an open nonempty set contained in the normed space* E*, and* $\mathbf{f}$ *a map from* $\mathrm{I} \times \mathrm{H}$ *into* E *satisfying the hypotheses of lemma 1.*

PROPOSITION 1. *Let* t_0 *be a point of* I *and* $\mathbf{x}_0$ *a point of* H*; for a continuous function* $\mathbf{u}$ *to be a solution of the equation* (1) *on* I *and take the value* $\mathbf{x}_0$ *at the point* t_0*, it is necessary and sufficient that it satisfies the relation*

$$\mathbf{u}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{u}(s))\, ds \tag{6}$$

for every $t \in \mathrm{I}$.

Indeed, by lemma 1, if $\mathbf{u}$ is a solution of (1) on I, then $\mathbf{f}(t, \mathbf{u}(t))$ is regulated, so the right-hand side of (6) is defined and equal to $\mathbf{u}(t)$ for every $t \in \mathrm{I}$. Conversely, if $\mathbf{u}$ is a continuous function that satisfies (6) then $\mathbf{f}(t, \mathbf{u}(t))$ is regulated, by lemma 1, so $\mathbf{u}$ has derivative equal to $\mathbf{f}(t, \mathbf{u}(t))$ except at the points of a countable subset of I.

COROLLARY. *At every point of* I *distinct from the left* (resp. *right*) *endpoint of this interval, every solution* $\mathbf{u}$ *of* (1) *on* I *admits a left* (resp. *right*) *derivative equal to the left* (*right*) *limit of* $\mathbf{f}(t, \mathbf{u}(t))$ *at this point.*

PROPOSITION 2. *If* $\mathbf{f}$ *is a continuous map from* $\mathrm{I} \times \mathrm{H}$ *into* E*, then every solution of* (1) *on* I *is a strict solution.*

Indeed, such a solution $\mathbf{u}$ is a primitive of the continuous function $\mathbf{f}(t, \mathbf{u}(t))$ (II, p. 66, prop. 3).

Furthermore, we note that a continuous function $\mathbf{f}$ on $\mathrm{I} \times \mathrm{H}$ satisfies the conditions of lemma 1 (*Gen. Top.*, X, p. 286, cor. 3).

In the sequel we shall choose $t_0 \in \mathrm{I}$ and $\mathbf{x}_0 \in \mathrm{H}$ arbitrarily and investigate whether there exist solutions of (1) on I (or on a neighbourhood of t_0 in I) taking the value $\mathbf{x}_0$ at the point t_0 (or, what comes to the same, solutions of (6)).

3. EXISTENCE OF APPROXIMATE SOLUTIONS

Given a number $\varepsilon > 0$ we shall say that a continuous map $\mathbf{u}$ of I into H is an *approximate solution to within* ε of the differential equation

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \tag{1}$$

if, at all the points of the complement of a *countable* subset of I, the function $\mathbf{u}$ admits a derivative which satisfies the condition

$$\left\|\mathbf{u}'(t) - \mathbf{f}(t, \mathbf{u}(t))\right\| \leqslant \varepsilon. \tag{7}$$

Let $(t_0, \mathbf{x}_0)$ be a point of $\mathrm{I} \times \mathrm{H}$; since $\mathbf{f}$ satisfies the hypotheses of lemma 1 (IV, p. 165) there exist a compact neighbourhood J of t_0 in I such that $\mathbf{f}(t, \mathbf{x}_0)$ is bounded on J, and an open ball S with centre $\mathbf{x}_0$ contained in H, such that $\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{x}_0)$ is bounded on $\mathrm{J} \times \mathrm{S}$; it follows that $\mathbf{f}(t, \mathbf{x})$ is *bounded* on $\mathrm{J} \times \mathrm{S}$. *Throughout this subsection* J *will denote a compact interval which is a neighbourhood of* t_0 *in* I*,* S *will be an open ball with centre* $\mathbf{x}_0$ *and radius* r *contained in* H*, with* J *and* S *such that* $\mathbf{f}$ *is bounded on* $\mathrm{J} \times \mathrm{S}$*; and* M *will denote the supremum of* $\|\mathbf{f}(t, \mathbf{x})\|$ *over* $\mathrm{J} \times \mathrm{S}$.

PROPOSITION 3. *On every compact interval with left* (*or right*) *endpoint* t_0 *contained in* J*, with length less than* $r/(\mathrm{M} + \varepsilon)$*, there exists an approximate solution to within* ε *of equation* (1)*, with values in* S*, and equal to* $\mathbf{x}_0$ *at* t_0.

We suppose that t_0 is not the right-hand endpoint of J, and prove the proposition for intervals with left-hand endpoint t_0. Let $\mathfrak{M}$ be the set of solutions of (1) to within ε, each of which takes values in S, is equal to $\mathbf{x}_0$ at t_0, and is defined on a half open interval $[t_0, b[$ contained in J (the interval depending on the approximate solution under consideration). First we show that $\mathfrak{M}$ is not empty. Let $\mathbf{c}$ be right limit of $\mathbf{f}(t, \mathbf{x}_0)$ at t_0; by lemma 1 (IV, p. 165) the function $\mathbf{f}\big(t, \mathbf{x}_0 + \mathbf{c}(t - t_0)\big)$ has a right limit equal to $\mathbf{c}$ at t_0, so the restriction of the function $\mathbf{x}_0 + \mathbf{c}(t - t_0)$ to a sufficiently small half open interval $[t_0, b[$ will belong to $\mathfrak{M}$.

We order the set $\mathfrak{M}$ by the relation "$\mathbf{u}$ is a restriction of $\mathbf{v}$", and show that $\mathfrak{M}$ is *inductive* (*Set Theory*, III, p. 154). Let $(\mathbf{u}_\alpha)$ be a totally ordered subset of $\mathfrak{M}$ and $[t_0, b_\alpha[$ the interval where $\mathbf{u}_\alpha$ is defined: for $b_\alpha \leqslant b_\beta$ the function $\mathbf{u}_\beta$ is thus an extension of $\mathbf{u}_\alpha$. The union of the intervals $[t_0, b_\alpha[$ is an interval $[t_0, b[$ contained in J, and there exists one and only one function $\mathbf{u}$ defined on $[t_0, b[$ that coincides with $\mathbf{u}_\alpha$ on $[t_0, b_\alpha[$ for each α; among the b_α there is an increasing sequence (b_{α_n}) tending to b; since $\mathbf{u}$ agrees with $\mathbf{u}_{\alpha_n}$ on $[t_0, b_{\alpha_n}[$ the function $\mathbf{u}$ admits a derivative satisfying (7) at all the points of the complement of a countable subset of $[t_0, b[$, and so is the supremum of the set $(\mathbf{u}_\alpha)$ in $\mathfrak{M}$.

By Zorn's lemma (*Set Theory*, III, p. 154, th. 2), $\mathfrak{M}$ admits a *maximal* element $\mathbf{u}_0$; we shall show that if $[t_0, t_1[$ is the interval where $\mathbf{u}_0$ is defined, then either t_1 is the right-hand endpoint of J, or else $t_1 - t_0 \geqslant r/(\mathrm{M} + \varepsilon)$. We argue by contradiction, supposing that neither of these conditions holds; first we show that one can extend $\mathbf{u}_0$ by continuity at the point t_1; in fact, for any s and t in $[t_0, t_1[$,

$$\|\mathbf{u}_0(s) - \mathbf{u}_0(t)\| \leqslant (\mathrm{M} + \varepsilon)\,|s - t|$$

by the mean value theorem; Cauchy's criterion shows that $\mathbf{u}_0$ admits a left limit $\mathbf{x}_1 \in \mathrm{S}$ at the point t_1. Now let $\mathbf{c}_1$ be the right limit at t_1 of the function $\mathbf{f}(t, \mathbf{x}_1)$; one has $\|\mathbf{c}_1\| \leqslant \mathrm{M}$; the same argument as that at the beginning of the proof shows that one can extend $\mathbf{u}_0$ to a half open interval with left-hand endpoint t_1 by the function $\mathbf{x}_1 + \mathbf{c}_1(t - t_1)$, so that the extended function belongs to $\mathfrak{M}$, which is absurd. This proves the proposition.

When $\mathbf{f}$ is *uniformly continuous* on $\mathrm{J} \times \mathrm{S}$ one can prove prop. 3 without using Zorn's lemma (IV, p. 199, exerc. 1*a*)).

PROPOSITION 4. *The set of approximate solutions of* (1) *to within* ε, *defined on the same interval* $\mathrm{K} \subset \mathrm{J}$ *and taking values in* S, *is uniformly equicontinuous.*

Indeed, if $\mathbf{u}$ is any function in this set and s and t are two points of K, then by the mean value theorem

$$\|\mathbf{u}(s) - \mathbf{u}(t)\| \leqslant (\mathrm{M} + \varepsilon)\,|s - t|\,.$$

COROLLARY (Peano's theorem). *If* E *is of finite dimension over* **R** *then there exists a solution of* (1) *with values in* S *and equal to* $\mathbf{x}_0$ *at* t_0, *on every compact interval* K *with left* (*or right*) *endpoint* t_0 *contained in* J *and having length* $< r/\mathrm{M}$.

Indeed, by prop. 3, once n is large enough there is an approximate solution $\mathbf{u}_n$ of (1) to within $1/n$, defined on K, with values in S, and equal to $\mathbf{x}_0$ at t_0. Further, from some value of n on, $\mathbf{u}_n(\mathrm{K})$ is contained in a *closed* ball with centre $\mathbf{x}_0$ and radius $< r$, independent of n. The set of $\mathbf{u}_n$ is equicontinuous (prop. 4), and since E is finite dimensional the set S is relatively compact in E; so for every $t \in \mathrm{K}$ the set of $\mathbf{u}_n(t)$ is relatively compact in E. By Ascoli's theorem (*Gen. Top.*, X, p. 290, th. 2) the set of $\mathbf{u}_n$ is relatively compact in the space $\mathcal{F}(\mathrm{K};\mathrm{E})$ of maps from K into E endowed with the uniform norm. Thus there is a sequence extracted from $\left(\mathbf{u}_{n_k}\right)$ of $\left(\mathbf{u}_n\right)$ which converges uniformly on K to a continuous function $\mathbf{u}$. One has $\mathbf{u}(\mathrm{K}) \subset \mathrm{S}$, so $t \mapsto \mathbf{f}(t, \mathbf{u}(t))$ is defined on K; by lemma 1 (IV, p. 165), $\mathbf{f}(t, \mathbf{u}_{n_k}(t))$ converges uniformly to $\mathbf{f}(t, \mathbf{u}(t))$ on K; by (IV, p. 4, formula (7)), $\mathbf{u}_{n_k}$ is a primitive of a function which tends uniformly to $\mathbf{f}(t, \mathbf{u}(t))$ on K, so (II, p. 52, th. 1) $\mathbf{u}$ is a solution of (1) on K, and equal to $\mathbf{x}_0$ at the point t_0.

Remarks. 1) There can be *infinitely many* integrals of a differential equation (1) taking the same value at a given point. For example, the scalar differential equation $x' = 2\sqrt{|x|}$ has all the functions defined by

$$\begin{array}{lll} u(t) = 0 & \text{for} & -\beta \leqslant t \leqslant \alpha \\ u(t) = -(t+\beta)^2 & \text{for} & t \leqslant -\beta \\ u(t) = (t-\alpha)^2 & \text{for} & t \geqslant \alpha \end{array}$$

as integrals taking the value 0 at the point $t = 0$, for any positive numbers α and β.

2) Peano's theorem no longer holds when E is an arbitrary complete normed space of *infinite* dimension (IV, p. 204, exerc. 18).

4. COMPARISON OF APPROXIMATE SOLUTIONS

In what follows, I and H denote, as above, an interval contained in $\mathbf{R}$ and an open set in the normed space E, respectively; t_0 is a point of I.

DEFINITION 1. *Given a positive real function $t \mapsto k(t)$ defined on* I, *one says that a map* $\mathbf{f}$ *from* $\mathrm{I} \times \mathrm{H}$ *into* E *is Lipschitz with respect to the function $k(t)$ if, for every* $\mathbf{x} \in \mathrm{H}$, *the function $t \mapsto \mathbf{f}(t, \mathbf{x})$ is regulated on* I, *and if, for every $t \in \mathrm{I}$ and every pair of points* $\mathbf{x}_1$, $\mathbf{x}_2$ *of* H, *one has* (*the* "Lipschitz condition")

$$\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leqslant k(t)\, \|\mathbf{x}_1 - \mathbf{x}_2\|\,. \tag{8}$$

We shall say that $\mathbf{f}$ is *Lipschitz* (without being more specific) on $\mathrm{I} \times \mathrm{H}$ if it is Lipschitz on this set for some *constant* $k \geqslant 0$. It is immediate that a Lipschitz function on $\mathrm{I} \times \mathrm{H}$ satisfies the hypotheses of lemma 1 of IV, p. 165 (the converse being false); when $\mathbf{f}$ is Lipschitz (on $\mathrm{I} \times \mathrm{H}$) one says that the differential equation

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \tag{1}$$

is *Lipschitz* (on $\mathrm{I} \times \mathrm{H}$).

Example. When $\mathrm{E} = \mathbf{R}$ and H is an interval in $\mathbf{R}$, if the function $f(t, x)$ admits a *partial derivative* f'_x (II, p. 74) at every point (t, x) of $\mathrm{I} \times \mathrm{H}$, such that $\left|f'_x(t, x)\right| \leqslant k(t)$ on $\mathrm{I} \times \mathrm{H}$, then condition (8) is satisfied, by the mean value theorem; we shall see later how this example generalizes to the case where E is an arbitrary normed space.

If $\mathbf{f}$ is Lipschitz on $\mathrm{I} \times \mathrm{H}$ then $\mathbf{f}$ is *bounded* on $\mathrm{J} \times \mathrm{S}$ for every compact subinterval $\mathrm{J} \subset \mathrm{I}$ and every open ball $\mathrm{S} \subset \mathrm{H}$. Thus prop. 3 (IV, p. 166) can be applied, and demonstrates the existence of approximate solutions of equation (1). But we also have the following proposition, which allows us to *compare* two approximate solutions:

PROPOSITION 5. *Let $k(t)$ be a real regulated function and > 0 on* I, *and let* $\mathbf{f}(t, \mathbf{x})$ *be a function that is defined and Lipschitz with respect to $k(t)$ on $\mathrm{I} \times \mathrm{H}$. If* $\mathbf{u}$ *and* $\mathbf{v}$ *are two approximate solutions of* (1), *to within ε_1 and ε_2 respectively, defined on* I *with values in* H, *then, for all $t \in \mathrm{I}$ such that $t \geqslant t_0$,*

$$\|\mathbf{u}(t) - \mathbf{v}(t)\| \leqslant \|\mathbf{u}(t_0) - \mathbf{v}(t_0)\|\, \Phi(t) + (\varepsilon_1 + \varepsilon_2)\Psi(t) \tag{9}$$

where

$$\begin{cases} \Phi(t) = 1 + \displaystyle\int_{t_0}^{t} k(s) \exp\left(\int_s^t k(\tau)\, d\tau\right) ds \\ \Psi(t) = t - t_0 + \displaystyle\int_{t_0}^{t} (s - t_0) k(s) \exp\left(\int_s^t k(\tau)\, d\tau\right) ds. \end{cases} \tag{10}$$

From the relation $\left\|\mathbf{u}'(t) - \mathbf{f}(t, \mathbf{u}(t))\right\| \leqslant \varepsilon_1$, valid on the complement of a countable set, one deduces, from the mean value theorem, that

$$\left\|\mathbf{u}(t) - \mathbf{u}(t_0) - \int_{t_0}^{t} \mathbf{f}(s, \mathbf{u}(s))\, ds\right\| \leqslant \varepsilon_1(t - t_0)$$

and similarly

$$\left\|\mathbf{v}(t) - \mathbf{v}(t_0) - \int_{t_0}^{t} \mathbf{f}(s, \mathbf{v}(s))\, ds\right\| \leqslant \varepsilon_2(t - t_0)$$

whence

$$\begin{aligned} \|\mathbf{u}(t) - \mathbf{v}(t)\| \leqslant{} & \|\mathbf{u}(t_0) - \mathbf{v}(t_0)\| \\ & + \left\|\int_{t_0}^{t} (\mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \mathbf{v}(s)))\, ds\right\| + (\varepsilon_1 + \varepsilon_2)(t - t_0). \end{aligned}$$

By the Lipschitz condition (8) one has

$$\begin{aligned} \left\|\int_{t_0}^{t} (\mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \mathbf{v}(s)))\, ds\right\| &\leqslant \int_{t_0}^{t} \|\mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \mathbf{v}(s))\|\, ds \\ &\leqslant \int_{t_0}^{t} k(s)\, \|\mathbf{u}(s) - \mathbf{v}(s)\|\, ds \end{aligned}$$

whence, putting $w(t) = \|\mathbf{u}(t) - \mathbf{v}(t)\|$,

$$w(t) \leqslant w(t_0) + (\varepsilon_1 + \varepsilon_2)(t - t_0) + \int_{t_0}^{t} k(s) w(s)\, ds. \tag{11}$$

The proposition is thus a consequence of the following lemma:

Lemma 2. If w is a continuous real function on the interval $[t_0, t_1]$ *and satisfies the inequality*

$$w(t) \leqslant \varphi(t) + \int_{t_0}^{t} k(s)w(s)\,ds \tag{12}$$

where φ is a regulated function $\geqslant 0$ on $[t_0, t_1]$, then, for $t_0 \leqslant t \leqslant t_1$,

$$w(t) \leqslant \varphi(t) + \int_{t_0}^{t} \varphi(s)k(s)\exp\left(\int_{s}^{t} k(\tau)\,d\tau\right) ds. \tag{13}$$

Put $y(t) = \int_{t_0}^{t} k(s)w(s)\,ds$; the relation (12) implies that

$$y'(t) - k(t)y(t) \leqslant \varphi(t)k(t) \tag{14}$$

on the complement of a countable set.

Put $z(t) = y(t)\exp\left(-\int_{t_0}^{t} k(s)\,ds\right)$; then (14) is equivalent to

$$z'(t) \leqslant \varphi(t)k(t)\exp\left(-\int_{t_0}^{t} k(s)\,ds\right).$$

On applying the mean value theorem (I, p. 15, th. 2) to this inequality, and noting that $z(t_0) = 0$, we obtain

$$z(t) \leqslant \int_{t_0}^{t} \varphi(s)k(s)\exp\left(-\int_{t_0}^{s} k(\tau)\,d\tau\right) ds$$

whence

$$y(t) \leqslant \int_{t_0}^{t} \varphi(s)k(s)\exp\left(\int_{s}^{t} k(\tau)\,d\tau\right) ds$$

and since $w(t) \leqslant \varphi(t) + y(t)$ one thus obtains (13).

COROLLARY. *Let* $\mathbf{f}$ *be a Lipschitz function for the constant $k > 0$, defined on* $\mathrm{I}\times\mathrm{H}$. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are two approximate solutions of* (1) *to within ε_1 and ε_2 respectively, defined on* I *and taking their values in* H, *then, for all $t \in \mathrm{I}$,*

$$\|\mathbf{u}(t) - \mathbf{v}(t)\| \leqslant \|\mathbf{u}(t_0) - \mathbf{v}(t_0)\|\, e^{k|t-t_0|} + (\varepsilon_1 + \varepsilon_2)\frac{e^{k|t-t_0|} - 1}{k}. \tag{15}$$

This inequality is in fact an immediate consequence of (9) when $t \geqslant t_0$; to prove it for $t \leqslant t_0$ it suffices to apply it to the equation

$$\frac{d\mathbf{x}}{ds} = -\mathbf{f}(-s, \mathbf{x})$$

obtained from (1) by the change of variable $t = -s$.

Remarks. 1) When $k = 0$ the inequality (15) is replaced by the inequality

$$\|\mathbf{u}(t) - \mathbf{v}(t)\| \leqslant \|\mathbf{u}(t_0) - \mathbf{v}(t_0)\| + (\varepsilon_1 + \varepsilon_2)\,|t - t_0|$$

whose proof is immediate.

2) When E is of *finite* dimension, and $\mathbf{f}$ is Lipschitz on $\mathrm{I} \times \mathrm{H}$, one can show the existence of approximate solutions of (1) (IV, p. 166, prop. 3) without using the axiom of choice (IV, p. 199, exerc. 1 *b*)).

PROPOSITION 6. *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be two functions defined on* $\mathrm{I} \times \mathrm{H}$*, satisfying the hypotheses of lemma 1 of* IV, p. 165*, and such that, on* $\mathrm{I} \times \mathrm{H}$*,*

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{g}(t, \mathbf{x})\| \leqslant \alpha. \tag{16}$$

Suppose further that $\mathbf{g}$ *is Lipschitz for the constant* $k > 0$ *on* $\mathrm{I} \times \mathrm{H}$*. In these circumstances, if* $\mathbf{u}$ *is an approximate solution of* $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ *to within* ε_1*, defined on* I*, with values in* H*, and* $\mathbf{v}$ *is an approximate solution of* $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$ *to within* ε_2*, defined on* I*, with values in* H*, then, for all* $t \in \mathrm{I}$

$$\|\mathbf{u}(t) - \mathbf{v}(t)\| \leqslant \|\mathbf{u}(t_0) - \mathbf{v}(t_0)\|\, e^{k|t-t_0|} + (\alpha + \varepsilon_1 + \varepsilon_2)\frac{e^{k|t-t_0|} - 1}{k}\,. \tag{17}$$

Indeed

$$\left\|\mathbf{u}'(t) - \mathbf{g}(t, \mathbf{u}(t))\right\| \leqslant \alpha + \varepsilon_1$$

for all t in the complement in I of a countable subset of I; in other words, $\mathbf{u}$ is an approximate solution of $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$ to within $\alpha + \varepsilon_1$, so the inequality (17) follows on applying prop. 5 of IV, p. 169.

5. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF LIPSCHITZ AND LOCALLY LIPSCHITZ EQUATIONS

THEOREM 1 (Cauchy). *Let* $\mathbf{f}$ *be a Lipschitz function on* $\mathrm{I} \times \mathrm{H}$*, let* J *be a compact subinterval of* I*, not reducing to a single point,* t_0 *a point of* J*,* S *an open ball with centre* $\mathbf{x}_0$ *and radius* r*, contained in* H*, and* M *the least upper bound of* $\|\mathbf{f}(t, \mathbf{x})\|$ *on* $\mathrm{J} \times \mathrm{S}$*. In these circumstances, for every compact interval* K *that does not reduce to a single point and is contained in the intersection of* J *with* $]t_0 - r/M,\ t_0 + r/M[$*, and contains* t_0*, there exists one and only one solution of the differential equation* $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ *defined on* K*, with values in* S*, and equal to* $\mathbf{x}_0$ *at the point* t_0*.*

Indeed, for $\varepsilon > 0$ sufficiently small, the set F_ε of approximate solutions of (1) to within ε, defined on K, with values in S, and equal to $\mathbf{x}_0$ at the point t_0, is not empty (IV, p. 166, prop. 3); further, if $\mathbf{u}$ and $\mathbf{v}$ belong to F_ε then, by (15) (IV, p. 170)

$$\|\mathbf{u}(t) - \mathbf{v}(t)\| \leqslant 2\varepsilon\frac{e^{k|t-t_0|} - 1}{k}$$

for all $t \in \mathrm{K}$, so the sets F_ε form a filter base $\mathfrak{G}$ which converges *uniformly* on K to a continuous function $\mathbf{w}$, equal to $\mathbf{x}_0$ at t_0; also $\mathbf{w}$ takes values in S, since, for ε small enough, the functions $\mathbf{u} \in \mathrm{F}_\varepsilon$ take their values in a closed ball contained in S. Since $\mathbf{f}(t, \mathbf{u}(t))$ tends uniformly on K to $\mathbf{f}(t, \mathbf{w}(t))$ along $\mathfrak{G}$, $\mathbf{w}$ satisfies equation (6) of IV, p. 165, so is a solution of (1). The uniqueness of the solution follows immediately from inequality (15) of IV, p. 170 where one takes $\varepsilon_1 = \varepsilon_2 = 0$ and $\mathbf{u}(t_0) = \mathbf{v}(t_0)$.

We shall say that a function $\mathbf{f}$ defined on $\mathrm{I} \times \mathrm{H}$ is *locally Lipschitz* if, for every point $(t, \mathbf{x})$ of $\mathrm{I} \times \mathrm{H}$, there exists a neighbourhood V of t (with respect to I) and a neighbourhood S of $\mathbf{x}$ such that $\mathbf{f}$ is Lipschitz on $\mathrm{V} \times \mathrm{S}$ (for a constant k *depending on* V *and* S). By the Borel-Lebesgue theorem, for every compact interval $\mathrm{J} \subset \mathrm{I}$ and every point $\mathbf{x}_0 \in \mathrm{H}$ there exists an open ball S with centre $\mathbf{x}_0$, contained in H, such that $\mathbf{f}$ is Lipschitz on $\mathrm{J} \times \mathrm{S}$; thus $\mathbf{f}$ satisfies the hypotheses of lemma 1 of IV, p. 3. When $\mathbf{f}$ is locally Lipschitz on $\mathrm{I} \times \mathrm{H}$ we shall say that the equation $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ is *locally Lipschitz* on $\mathrm{I} \times \mathrm{H}$.

We shall generalize and clarify th. 1 of IV, p. 10 for locally Lipschitz equations; we restrict ourselves to the case where t_0 is the *left endpoint* of the interval I; one can pass easily to the case where t_0 is an arbitrary point of I (*cf.* IV, §IV ??, p. 9, corollary).

THEOREM 2. *Let* $\mathrm{I} \subset \mathbf{R}$ *be an interval* (*not reducing to a single point*) *with left-hand endpoint* $t_0 \in \mathrm{I}$, *let* H *be a nonempty open set in* E, *and* $\mathbf{f}$ *a locally Lipschitz function on* $\mathrm{I} \times \mathrm{H}$.

1° *For* $\mathbf{x}_0 \in \mathrm{H}$ *there exists a largest interval* $\mathrm{J} \subset \mathrm{I}$, *with left-hand endpoint* $t_0 \in \mathrm{J}$, *on which there exists an integral* $\mathbf{u}$ *of the equation* $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ *taking values in* H *and equal to* $\mathbf{x}_0$ *at the point* t_0*; this integral is unique.*

2° *If* $\mathrm{J} \neq \mathrm{I}$ *then* J *is a half-open interval* $[t_0, \beta[$ *of finite length; further, for every compact subset* $\mathrm{K} \subset \mathrm{H}$ *the set* $\overset{-1}{\mathbf{u}}(\mathrm{K})$ *is a compact subset of* $\mathbf{R}$.

3° *If* J *is bounded, and if* $\mathbf{f}(t, \mathbf{u}(t))$ *is bounded on* J, *then* $\mathbf{u}(t)$ *has a left limit* $\mathbf{c}$ *at the right-hand endpoint of* J*; further, if* $\mathrm{J} \neq \mathrm{I}$ *then* $\mathbf{c}$ *is a boundary point of* H *in* E.

1° Let $\mathfrak{M}$ be the set of intervals L (not reducing to a single point) with left-hand endpoint $t_0 \in \mathrm{L}$ which are contained in I and are such that on L there is a solution of (1) (IV, p. 163) with values in H and equal to $\mathbf{x}_0$ at t_0; by th. 1 (IV, p. 171) the set $\mathfrak{M}$ is not empty. Let L and L′ be two intervals belonging to $\mathfrak{M}$, and suppose, for example, that $\mathrm{L} \subset \mathrm{L}'$; if $\mathbf{u}$ and $\mathbf{v}$ are two integrals of (1) defined respectively on L and L′, with values in H, and equal to $\mathbf{x}_0$ at t_0, we shall see that $\mathbf{v}$ is an *extension* of $\mathbf{u}$. Indeed, let t_1 be the supremum of the set of $t \in \mathrm{L}$ such that $\mathbf{u}(s) = \mathbf{v}(s)$ for $t_0 \leqslant s \leqslant t$; we shall show that t_1 is the right-hand endpoint of L. If this were not so, we would have $\mathbf{u}(t_1) = \mathbf{v}(t_1)$ by continuity, and $\mathbf{x}_1 = \mathbf{u}(t_1)$ would belong to H; since $\mathbf{f}$ is locally Lipschitz, th. 1 shows that there can exist only one integral of (1) defined on a neighbourhood of t_1 with values in H and equal to $\mathbf{x}_1$ at t_1; it is therefore a contradiction to suppose that t_1 is not the right-hand endpoint of L. We now see that if J is the *union* of the intervals $\mathrm{L} \in \mathfrak{M}$ there exists one and only one integral $\mathbf{u}$ of (1), defined on J, with values in H and equal to $\mathbf{x}_0$ at t_0.

2° Suppose that $\mathrm{J} \neq \mathrm{I}$ and let β be the right endpoint of J; if β is the right-hand endpoint of I then $\beta \in \mathrm{I}$ (so β is finite) and $\mathrm{J} = [t_0, \beta[$ by hypothesis. Suppose then that β is not the right-hand endpoint of I; if $\beta \in \mathrm{J}$ then $\mathbf{u}(\beta) = \mathbf{c}$ belongs to H; by th. 1 there exists an integral of (1) with values in H, defined on an interval

$$[\beta, \beta_1[\subset \mathrm{I}$$

and equal to $\mathbf{c}$ at β; then J would not be the largest of the intervals in $\mathfrak{M}$, which is absurd; so $\mathrm{J} = [t_0, \beta[$.

If K is a compact subset of H then $\overset{-1}{\mathbf{u}}(\mathrm{K})$ is closed in J; we shall see that there exists a $\gamma \in \mathrm{J}$ such that $\overset{-1}{\mathbf{u}}(\mathrm{K})$ is contained in $[t_0, \gamma]$, which will prove that $\overset{-1}{\mathbf{u}}(\mathrm{K})$ is compact. If not, there would be a point $\mathbf{c} \in \mathrm{K}$ such that $(\beta, \mathbf{c})$ is a cluster point of the set of points $(t, \mathbf{u}(t))$ such that $t < \beta$ and $\mathbf{u}(t) \in \mathrm{K}$. Since $\beta \in \mathrm{I}$ and $\mathbf{c} \in \mathrm{H}$ there exists a neighbourhood V of β in I, and an open ball S with centre $\mathbf{c}$ and radius r contained in H, such that $\mathbf{f}$ is Lipschitz and bounded on $\mathrm{V} \times \mathrm{S}$; let M be the supremum of $\|\mathbf{f}(t, \mathbf{x})\|$ over this set. By hypothesis there exists a $t_1 \in \mathrm{J}$ such that $\beta - t_1 < r/2\mathrm{M}$, $t_1 \in \mathrm{V}$ and $\|\mathbf{u}(t_1) - \mathbf{c}\| \leqslant r/2$; th. 1 shows that there exists one and only one integral of (1), with values in H, defined on an interval $[t_1, t_2]$ *containing* β, and equal to $\mathbf{u}(t_1)$ at t_1; since this interval coincides with $\mathbf{u}$ on the interval $[t_1, \beta[$ it follows that $\mathrm{J} = [t_0, \beta[$ is not the largest interval in $\mathfrak{M}$, which is absurd.

3° Suppose that J is bounded and that $\|\mathbf{f}(t, \mathbf{u}(t))\| \leqslant M$ on J; then $\left\|\mathbf{u}'(t)\right\| \leqslant \mathrm{M}$ on the complement of a countable subset of J; then, by the mean value theorem, $\|\mathbf{u}(s) - \mathbf{u}(t)\| \leqslant \mathrm{M}\,|s - t|$ for any s and t in J; by the Cauchy criterion, $\mathbf{u}$ has a left limit $\mathbf{c}$ at the right endpoint β of J. If $\mathrm{J} \neq \mathrm{I}$ then $\mathbf{c}$ cannot belong to H, for on extending $\mathbf{u}$ by continuity at β, $\mathbf{u}$ would be an integral of (1) *with values in* H, defined on an interval $[t_0, \beta]$ and equal to $\mathbf{x}_0$ at t_0; then one would have $\mathrm{J} = [t_0, \beta]$, contradicting what we have seen in 2°.

COROLLARY 1. *If* $\mathrm{H} = \mathrm{E}$ *and* $\mathrm{J} \neq \mathrm{I}$ *then* $\mathbf{f}(t, \mathbf{u}(t))$ *is not bounded on* J*; if, further,* E *is finite dimensional, then* $\|\mathbf{u}(t)\|$ *has left limit* $+\infty$ *at the right-hand endpoint of* J.

The first part is an immediate consequence of the third part of th. 2. If E is finite dimensional then every closed ball $\mathrm{S} \subset \mathrm{E}$ is compact, so the second part of th. 2 shows that there exists a $\gamma \in \mathrm{J}$ such that $\mathbf{u}(t) \notin \mathrm{S}$ for $t > \gamma$.

If E is finite dimensional it can happen that $\mathrm{J} \neq \mathrm{I}$ but that $\|\mathbf{u}(t)\|$ remains *bounded* as t tends to the right-hand endpoint of J (IV, p. 200, exerc. 5).

COROLLARY 2. *If, on* $\mathrm{I} \times \mathrm{H}$*, the function* $\mathbf{f}$ *is Lipschitz with respect to a regulated function* $k(t)$*, and if the right-hand endpoint* β *of* J *belongs to* I*, then* $\mathbf{u}$ *has a left limit at* β*; if* $\mathrm{H} = \mathrm{E}$ *and if* $\mathbf{f}$ *is Lipschitz with respect to a regulated function* $k(t)$ *on* $\mathrm{I} \times \mathrm{E}$*, then* $\mathrm{J} = \mathrm{I}$.

Indeed, if $\beta \in \mathrm{I}$, there exists a compact neighbourhood V of β in I, such that $\mathbf{f}(t, \mathbf{x}_0)$ and $k(t)$ are bounded on V; then $\|\mathbf{f}(t, \mathbf{x})\| \leqslant m\,\|x\| + h$ (m and h constant) on $V \times H$, whence $\left\|\mathbf{u}'(t)\right\| \leqslant m\,\|\mathbf{u}(t)\| + h$ on the complement of a countable subset of $V \cap J$, so that $\|\mathbf{u}(t)\| \leqslant m\int_{t_0}^{t} \|\mathbf{u}(s)\|\,ds + q$ (q constant) on $V \cap J$; lemma 2 (IV, p. 170) shows that $\|\mathbf{u}(t)\| \leqslant c\,e^{mt} + d$ (c and d constant) on $V \cap J$, and thus $\mathbf{f}(t, \mathbf{u}(t))$ remains *bounded* on J, and the corollary results from th. 2 of IV, p. 172.

Examples. 1) For a differential equation of the form $\mathbf{x}' = \mathbf{g}(t)$, where $\mathbf{g}$ is regulated on I, every integral $\mathbf{u}$ is clearly defined on all of I. One should note that $\mathbf{u}$ can be bounded on I without $\mathbf{g}(t)$ being so.

2) For the scalar equation $x' = \sqrt{1 - x^2}$ one has $I = \mathbf{R}$, $H = \,]-1, +1[$. If one takes $t_0 = x_0 = 0$ the corresponding integral is $\sin t$ *on the largest interval containing* 0 *where the derivative of* $\sin t$ *is positive,* that is to say, on $]-\pi/2, +\pi/2[$; at the endpoints of this interval the integral tends to an endpoint of H.

3) For the scalar equation $x' = 1 + x^2$ one has $I = H = \mathbf{R}$; the integral that vanishes at $t = 0$ is $\tan t$, and the largest interval containing 0 where this function is continuous is $J = \,]-\pi/2, +\pi/2[$; and $|\tan t|$ tends to $+\infty$ at the endpoints of J (*cf.* IV, p. 173, cor. 1).

4) For the scalar equation $x' = \sin tx$ one has $I = H = \mathbf{R}$ and the right-hand side is bounded on $I \times H$, so (IV, p. 173, cor. 1) every integral is defined on all of $\mathbf{R}$.

6. CONTINUITY OF INTEGRALS AS FUNCTIONS OF A PARAMETER

Prop. 6 (IV, p. 171) shows that when a differential equation

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$$

is "close" to a Lipschitz equation $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$ and when one supposes that *both* equations have an approximate solution on the same interval, then these approximate solutions are "close"; we shall clarify this result by showing that the existence of solutions of the Lipschitz equation $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$ on an interval *implies* that of approximate solutions of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ on the same interval, so long as, on the latter, the values of the solution of $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$ are not "too close" to the *boundary* of H.

PROPOSITION 7. *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be two functions defined on* $I \times H$, *satisfying the hypotheses of lemma 1 of* IV. p. 165, *and such that, on* $I \times H$

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{g}(t, \mathbf{x})\| \leqslant \alpha\,. \tag{16}$$

Suppose further that $\mathbf{g}$ *is Lipschitz with respect to a constant* $k > 0$ *on* $I \times H$ *and that* $\mathbf{f}$ *is locally Lipschitz on* $I \times H$, *or that* E *is finite dimensional. Let* $(t_0, \mathbf{x}_0)$ *be a point of* $I \times H$, μ *a number* > 0, *and*

$$\varphi(t) = \mu\,e^{k(t-t_0)} + \alpha\,\frac{e^{k(t-t_0)} - 1}{k}\,.$$

Let $\mathbf{u}$ *be an integral of the equation* $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$ *defined on an interval* $K = [t_0, b[$ *contained in* I, *equal to* $\mathbf{x}_0$ *at the point* t_0, *and such that for all* $t \in K$ *the closed ball with centre* $\mathbf{u}(t)$ *and radius* $\varphi(t)$ *is contained in* H. *Under these conditions, for every* $\mathbf{y} \in H$ *such that* $\|\mathbf{y} - \mathbf{x}_0\| \leqslant \mu$ *there exists an integral* $\mathbf{v}$ *of* $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$, *defined on* K, *with values in* H, *and equal to* $\mathbf{y}$ *at the point* t_0; *moreover,* $\|\mathbf{u}(t) - \mathbf{v}(t)\| \leqslant \varphi(t)$ *on* K.

Let $\mathfrak{M}$ be the family of integrals of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ each of which takes its values in H, is equal to $\mathbf{y}$ at t_0, and is defined on a half-open interval $[t_0, \tau[$ contained in I (depending on the interval considered). By th. 1 of IV, p. 171 (when $\mathbf{f}$ is locally Lipschitz) or IV, p. 167 corollary (when E is finite dimensional), $\mathfrak{M}$ is not empty, and the same reasoning as in prop. 3 of IV, p. 166 shows that $\mathfrak{M}$ is *inductive* for the order "$\mathbf{v}$ is a restriction of $\mathbf{w}$". Let $\mathbf{v}_0$ be a maximal element of $\mathfrak{M}$ and $[t_0, t_1[$ the interval of definition of $\mathbf{v}_0$; by prop. 6 of IV, p. 171, it all comes down to proving that $t_1 \geqslant b$. If not, one would have

$$\|\mathbf{u}(t) - \mathbf{v}_0(t)\| \leqslant \varphi(t)$$

on the interval $[t_0, t_1[$ by prop. 6; now on the compact interval $[t_0, t_1]$ the regulated function $\mathbf{g}(t, \mathbf{u}(t))$ is bounded, so the function $\mathbf{g}(t, \mathbf{v}_0(t))$ is bounded on the interval $[t_0, t_1[$, for $\|\mathbf{g}(t, \mathbf{v}_0(t))\| \leqslant \|\mathbf{g}(t, \mathbf{u}(t))\| + k\varphi(t)$ on this interval. Since $\mathbf{v}_0$ is an approximate solution of $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$ to within α on $[t_0, t_1[$ there exists a number $\mathrm{M} > 0$ such that $\left\|\mathbf{v}_0'(t)\right\| \leqslant \mathrm{M}$ on this interval, except at the points of a countable set; the mean value theorem now shows that $\|\mathbf{v}_0(s) - \mathbf{v}_0(t)\| \leqslant \mathrm{M}\,|s - t|$ for every pair of points $s,\ t$ of $[t_0, t_1[$, so (by Cauchy's criterion) $\mathbf{v}_0(t)$ has a left limit $\mathbf{c}$ at the point t_1, and, by continuity, one has $\|\mathbf{c} - \mathbf{u}(t_1)\| \leqslant \varphi(t_1)$, and so $\mathbf{c} \in \mathrm{H}$. Now one sees, from IV, p. 171, th. 1 or IV, p. 167, corollary, that there exists an integral of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ defined on an interval $[t_1, t_2[$ and equal to $\mathbf{c}$ at t_1, which contradicts the definition of $\mathbf{v}_0$.

THEOREM 3. *Let* F *be a topological space and let* $\mathbf{f}$ *be a map from* $\mathrm{I} \times \mathrm{H} \times \mathrm{F}$ *into* E *such that, for every* $\xi \in \mathrm{F}$, *the function* $(t, \mathbf{x}) \mapsto \mathbf{f}(t, \mathbf{x}, \xi)$ *is Lipschitz on* $\mathrm{I} \times \mathrm{H}$, *and such that, when* ξ *tends to* ξ_0, $\mathbf{f}(t, \mathbf{x}, \xi)$ *tends uniformly to* $\mathbf{f}(t, \mathbf{x}, \xi_0)$ *on* $\mathrm{I} \times \mathrm{H}$. *Let* $\mathbf{u}_0(t)$ *be an integral of* $\mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \xi_0)$ *defined on an interval* $\mathrm{J} = [t_0, b[$ *contained in* I, *with values in* H, *and equal to* $\mathbf{x}_0$ *at* t_0. *For every compact interval* $[t_0, t_1]$ *contained in* J *there exists a neighbourhood* V *of* ξ_0 *in* F *such that, for every* $\xi \in \mathrm{V}$, *the equation* $\mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \xi)$ *has an integral* (*and only one*) $\mathbf{u}(t, \xi)$ *defined on* $[t_0, t_1]$, *with values in* H *and equal to* $\mathbf{x}_0$ *at* t_0; *moreover, when* ξ *tends to* ξ_0 *the solution* $\mathbf{u}(t, \xi)$ *tends uniformly to* $\mathbf{u}_0(t)$ *on* $[t_0, t_1]$.

Indeed, let $r > 0$ be such that for $t_0 \leqslant t \leqslant t_1$ the closed ball with centre $\mathbf{u}_0(t)$ and radius r is contained in H; if $\mathbf{f}(t, \mathbf{x}, \xi)$ is Lipschitz with respect to the constant $k > 0$ on $\mathrm{I} \times \mathrm{H}$ we take α small enough that $\alpha\,\dfrac{e^{k(t_1 - t_0)} - 1}{k} < r$; taking V such that, for every $\xi \in \mathrm{V}$, one has $\|\mathbf{f}(t, \mathbf{x}, \xi) - \mathbf{f}(t, \mathbf{x}, \xi_0)\| \leqslant \alpha$ on $\mathrm{I} \times \mathrm{H}$, one can apply prop. 7 of IV, p. 174; moreover,

$$\|\mathbf{u}(t, \xi) - \mathbf{u}_0(t)\| \leqslant \alpha\,\frac{e^{k(t_1 - t_0)} - 1}{k}$$

on $[t_0, t_1]$, which completes the proof of the theorem.

Remark. When $\mathrm{H} = \mathrm{E}$ and the condition (16) of IV, p. 174 is satisfied on $\mathrm{I} \times \mathrm{E}$, prop. 7 of IV, p. 174 applies to *every* solution $\mathbf{u}$ of $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$ on *any* interval where this solution is defined; one may even assume that $\mathbf{g}(t,\mathbf{x})$ is Lipschitz with respect to a regulated function $k(t)$ though not necessarily bounded on this interval.

7. DEPENDENCE ON INITIAL CONDITIONS

Let $\mathbf{x}' = \mathbf{f}(t,\mathbf{x})$ be a locally Lipschitz equation on $\mathrm{I}\times\mathrm{H}$; by th. 2 (IV, p. 172), for every point $(t_0,\mathbf{x}_0)$ of $\mathrm{I}\times\mathrm{H}$ there exists a *largest interval* $\mathrm{J}(t_0,\mathbf{x}_0)\subset\mathrm{I}$, not reducing to a single point, containing t_0, and on which there exists an integral (and only one) of this equation, equal to $\mathbf{x}_0$ at t_0; we shall clarify the manner in which this integral, and the interval $J(t_0,\mathbf{x}_0)$ where it is defined, depend on the point $(t_0,\mathbf{x}_0)$.

THEOREM 4. *Let* $\mathbf{f}$ *be a locally Lipschitz function on* $\mathrm{I}\times\mathrm{H}$ *and* $(a,\mathbf{b})$ *an arbitrary point of* $\mathrm{I}\times\mathrm{H}$.

1° *There exist an interval* $\mathrm{K}\subset\mathrm{I}$, *a neighbourhood of* a *in* I, *and a neighbourhood* V *of* $\mathbf{b}$ *in* H *such that, for every point* $(t_0,\mathbf{x}_0)$ *of* $\mathrm{K}\times\mathrm{V}$, *there exists an integral* (*and only one*) $\mathbf{u}(t,t_0,\mathbf{x}_0)$ defined on K, *with values in* H *and equal to* $\mathbf{x}_0$ *at the point* t_0 (in other words, $J(t_0,\mathbf{x}_0)\supset\mathrm{K}$ for all $(t_0,\mathbf{x}_0)\in\mathrm{K}\times\mathrm{V}$).

2° *The map* $(t,t_0,\mathbf{x}_0)\mapsto\mathbf{u}(t,t_0,\mathbf{x}_0)$ *of* $\mathrm{K}\times\mathrm{K}\times\mathrm{V}$ *into* H *is uniformly continuous.*

3° *There exists a neighbourhood* $\mathrm{W}\subset\mathrm{V}$ *of* $\mathbf{b}$ *in* H *such that, for every point*

$$(t,t_0,\mathbf{x}_0)\in\mathrm{K}\times\mathrm{K}\times\mathrm{W},$$

the equation $\mathbf{x}_0=\mathbf{u}(t_0,t,\mathbf{x})$ *has a unique solution* $\mathbf{x}$ *on* V *equal to* $\mathbf{u}(t,t_0,\mathbf{x}_0)$ ("resolution of the integral with respect to the constant of integration").

1° Let S be a ball with centre $\mathbf{b}$ and radius r contained in H, and J_0 an interval contained in I, a neighbourhood of a in I, such that $\mathbf{f}$ is bounded and Lipschitz (with respect to some constant k) on $\mathrm{J}_0\times\mathrm{S}$; denote by M the supremum of $\|\mathbf{f}(t,\mathbf{x})\|$ on $\mathrm{J}_0\times\mathrm{S}$. Then there exist (IV, p. 171, th. 1) an interval $\mathrm{J}\subset\mathrm{J}_0$, a neighbourhood of a in I, and an integral $\mathbf{v}$ of $\mathbf{x}'=\mathbf{f}(t,\mathbf{x})$ defined on J, with values in S and equal to $\mathbf{b}$ at a. We shall see that the open ball V with centre $\mathbf{b}$ and radius $r/2$, and the intersection K of J with an interval $]a-l,a+l[$, where l is *small enough,* are as required. Indeed, prop. 7 of IV, p. 174 (applied to the set $\mathrm{J}_0\times\mathrm{S}$ and the case where $\alpha=0$) shows that there exists an integral of $\mathbf{x}'=\mathbf{f}(t,\mathbf{x})$ *defined on* K, with values in S, and equal to $\mathbf{x}_0$ at a point $t_0\in\mathrm{K}$, provided that

$$\|\mathbf{v}(t)-\mathbf{b}\|+\|\mathbf{v}(t_0)-\mathbf{x}_0\|\,e^{k|t-t_0|}<r \tag{18}$$

for every $t\in\mathrm{K}$. Now, by the mean value theorem, one has

$$\|\mathbf{v}(t)-\mathbf{b}\|\leqslant\mathrm{M}\,|t-a|\leqslant\mathrm{M}l$$

for every $t\in\mathrm{K}$; since $\|\mathbf{x}_0-\mathbf{b}\|<r/2$ one sees that it suffices to take l such that

$$\mathrm{M}l+(\mathrm{M}l+r/2)e^{2kl}<r \tag{19}$$

for the relation (18) to be satisfied for every $(t,t_0,\mathbf{x}_0)$ of $\mathrm{K}\times\mathrm{K}\times\mathrm{V}$.

2° By the mean value theorem we have

$$\|\mathbf{u}(t_1, t_0, \mathbf{x}_0) - \mathbf{u}(t_2, t_0, \mathbf{x}_0)\| \leqslant \mathrm{M}\,|t_2 - t_1| \tag{20}$$

for all t_0, t_1, t_2 in K and $\mathbf{x}_0$ in V. Now prop. 5 (IV, p. 169) shows that

$$\|\mathbf{u}(t, t_0, \mathbf{x}_1) - \mathbf{u}(t, t_0, \mathbf{x}_2)\| \leqslant e^{2kt}\ \|\mathbf{x}_2 - \mathbf{x}_1\| \tag{21}$$

for every t and t_0 in K, and $\mathbf{x}_1$ and $\mathbf{x}_2$ in V. Finally, if t_1 and t_2 are any two points in K,

$$\|\mathbf{u}(t_1, t_2, \mathbf{x}_0) - \mathbf{u}(t_2, t_2, \mathbf{x}_0)\| \leqslant \mathrm{M}\,|t_2 - t_1|$$

by the mean value theorem, that is to say

$$\|\mathbf{u}(t_1, t_2, \mathbf{x}_0) - \mathbf{x}_0\| \leqslant \mathrm{M}\,|t_2 - t_1|\ ;$$

since $\mathbf{u}(t, t_2, \mathbf{x}_0)$ is identical to the integral which takes the value $\mathbf{u}(t_1, t_2, \mathbf{x}_0)$ at the point t_1, prop. 5 (IV, p. 169) shows that

$$\|\mathbf{u}(t, t_1, \mathbf{x}_0) - \mathbf{u}(t, t_2, \mathbf{x}_0)\| \leqslant \mathrm{M}e^{2kl}\ |t_2 - t_1| \tag{22}$$

for all t, t_1, t_2 in K and $\mathbf{x}_0$ in V. The three inequalities (20), (21) and (22) thus prove the uniform continuity of the map $(t, t_0, \mathbf{x}_0) \mapsto \mathbf{u}(t, t_0, \mathbf{x}_0)$ on $\mathrm{K} \times \mathrm{K} \times \mathrm{V}$.

3° By (20), we have $\|\mathbf{u}(t, t_0, \mathbf{x}_0) - \mathbf{x}_0\| \leqslant \mathrm{M}\,|t - t_0| \leqslant 2\mathrm{M}l$ on

$$\mathrm{K} \times \mathrm{K} \times \mathrm{V}.$$

If l is taken small enough, so that $2\mathrm{M}l < r/4$, one then sees that if $\mathbf{x}_0$ is any point of the open ball W with centre $\mathbf{b}$ and radius $r/4$, that $\mathbf{u}(t, t_0, \mathbf{x}_0) \in \mathrm{V}$ for any t and t_0 in K. If $\mathbf{x} = \mathbf{u}(t, t_0, \mathbf{x}_0)$, the function $s \mapsto \mathbf{u}(s, t, \mathbf{x})$ is then defined on K and is equal to the integral of (1) which takes the value $\mathbf{x}$ at the point t, that is, to $\mathbf{u}(s, t_0, \mathbf{x}_0)$; in particular

$$\mathbf{x}_0 = \mathbf{u}(t_0, t_0, \mathbf{x}_0) = \mathbf{u}(t_0, t, \mathbf{x}).$$

Moreover, if $\mathbf{y} \in \mathrm{V}$ is such that $\mathbf{x}_0 = \mathbf{u}(t_0, t, \mathbf{y})$, then the integral $s \mapsto \mathbf{u}(s, t, \mathbf{y})$ takes the value $\mathbf{x}_0$ at t_0 so is identical to $s \mapsto \mathbf{u}(s, t_0, \mathbf{x}_0)$, which consequently takes the value $\mathbf{x}$ at t, which shows that $\mathbf{y} = \mathbf{x}$ and concludes the proof.

§ 2. LINEAR DIFFERENTIAL EQUATIONS

1. EXISTENCE OF INTEGRALS OF A LINEAR DIFFERENTIAL EQUATION

Let E be a complete normed space over the field $\mathbf{R}$, and J an interval in $\mathbf{R}$, not reducing to a point. One says that the differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}) \tag{1}$$

where $\mathbf{f}$ is defined on $\mathrm{J} \times \mathrm{E}$, is a *linear* equation if, for every $t \in \mathrm{J}$, the map $\mathbf{x} \mapsto \mathbf{f}(t, \mathbf{x})$ is a *continuous affine linear map* [1] from E into itself; if one puts $\mathbf{b}(t) = \mathbf{f}(t, 0)$ the map $\mathbf{x} \mapsto \mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, 0) = \mathbf{f}(t, \mathbf{x}) - \mathbf{b}(t)$ is then a continuous linear map from E to itself; from now on we shall denote this map by $A(t)$ and write $A(t).\mathbf{x}$, (or simply $A(t)\mathbf{x}$) for its value at the point $\mathbf{x} \in \mathrm{E}$; thus the linear differential equation (1) may be written

$$\frac{d\mathbf{x}}{dt} = A(t).\mathbf{x} + \mathbf{b}(t) \tag{2}$$

where $\mathbf{b}$ is a map from J into E; when $\mathbf{b} = 0$ one says that the linear differential equation (2) is *homogeneous*.

Examples. 1) When E is of finite dimension n over $\mathbf{R}$ one can identify the endomorphism $A(t)$ with its *matrix* $\left(a_{ij}(t)\right)$ *with respect to* any *basis of* E (*Alg.*, II, p. 343); when one identifies a vector $\mathbf{x} \in E$ with the column matrix $\left(x_j\right)$ of its components with respect to the basis of E under consideration the expression $A(t).\mathbf{x}$ conforms to the general conventions of Algebra (*Alg.*, II, p. 343, prop. 2). In this case, equation (2) is equivalent to the system of scalar differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t) \qquad (1 \leqslant i \leqslant n). \tag{3}$$

2) Let G be a *complete normed algebra* over $\mathbf{R}$, and $\mathbf{a}(t)$, $\mathbf{b}(t)$ and $\mathbf{c}(t)$ three maps from J into G; the equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{a}(t)\mathbf{x} + \mathbf{x}\,\mathbf{b}(t) + \mathbf{c}(t)$$

is a linear differential equation; here $A(t)$ is the linear map $\mathbf{x} \mapsto \mathbf{a}(t)\mathbf{x} + \mathbf{x}\,\mathbf{b}(t)$ of G to itself.

For every $t \in \mathrm{J}$, $A(t)$ is an element of the set $\mathscr{L}(\mathrm{E})$ of continuous linear maps from E to itself (continuous endomorphisms of E); one knows (*Gen. Top.*, X, p. 298) that $\mathscr{L}(\mathrm{E})$, endowed with the *norm* $\|\mathrm{U}\| = \sup_{\|\mathbf{x}\| \leqslant 1} \|\mathrm{U}\mathbf{x}\|$ is a *complete normed algebra* over the field $\mathbf{R}$, and that $\|\mathrm{UV}\| \leqslant \|\mathrm{U}\|\ \|\mathrm{V}\|$.

Throughout this section we shall assume that the following conditions are satisfied:

a) *The map* $t \mapsto A(t)$ *of* J *into* $\mathscr{L}(\mathrm{E})$ *is regulated.*

b) *The map* $t \mapsto \mathbf{b}(t)$ *of* J *into* E *is regulated.*

When E has dimension n, $\mathscr{L}(\mathrm{E})$ is isomorphic to $\mathbf{R}^{n^2}$ (as a topological vector space) and condition *a*) means that each of the elements $a_{ij}(t)$ of the matrix $A(t)$ is a *regulated* function on J.

[1] Recall that if E is of finite dimension then every affine linear map from E into itself is continuous (*Gen. Top.*, VI, p. 33 and 37).

Since $\|A(t')\mathbf{x} - A(t)\mathbf{x}\| \leqslant \|A(t') - A(t)\| \, \|\mathbf{x}\|$, the map

$$t \mapsto A(t).\mathbf{x} + \mathbf{b}(t)$$

is *regulated* for every $\mathbf{x} \in \mathrm{E}$; further,

$$\|A(t)\mathbf{x}_1 - A(t)\mathbf{x}_2\| = \|A(t)(\mathbf{x}_1 - \mathbf{x}_2)\| \leqslant \|A(t)\| \; \|\mathbf{x}_1 - \mathbf{x}_2\|$$

for any $t \in \mathrm{J}$ and $\mathbf{x}_1$, $\mathbf{x}_2$ in E; in other words, the right-hand side of (2) satisfies the conditions of lemma 1 of IV, p. 165 and is *Lipschitz* with respect to the *regulated* function $\|A(t)\|$ on $\mathrm{J} \times \mathrm{E}$. In consequence (IV, p. 173, cor. 2):

THEOREM 1. *Let $t \mapsto A(t)$ be a regulated map of* J *into* $\mathscr{L}(\mathrm{E})$, *and $t \mapsto \mathbf{b}(t)$ be a regulated map of* J *into* E. *For every point $(t_0, \mathbf{x}_0)$ of* $\mathrm{J} \times \mathrm{E}$ *the linear equation* (2) *admits one and only one solution defined on all of* J *and equal to $\mathbf{x}_0$ at the point t_0.*

2. LINEARITY OF THE INTEGRALS OF A LINEAR DIFFERENTIAL EQUATION

Solving a linear differential equation (2) is a linear problem (*Alg.*, II, p. 240); the homogeneous linear equation

$$\frac{d\mathbf{x}}{dt} = A(t).\mathbf{x} \tag{4}$$

is said to be *associated* with the inhomogeneous equation (2); and one knows (*Alg.*, II, p. 241, prop. 14) that if $\mathbf{u}_1$ is an integral of the inhomogeneous equation (2) then every integral of this equation is of the form $\mathbf{u} + \mathbf{u}_1$ where $\mathbf{u}_1$ is a solution of the associated homogeneous equation (4), and conversely. We shall first study in this subsection the integrals of a *homogeneous* equation (4).

PROPOSITION 1. *The set $\mathcal{I}$ of integrals of the homogeneous linear equation* (4), *defined on* J, *is a vector subspace of the space $\mathcal{C}(\mathrm{J}; \mathrm{E})$ of continuous maps from* J *into* E.

The proof is immediate.

THEOREM 2. *For every point $(t_0, \mathbf{x}_0)$ of* $\mathrm{J} \times \mathrm{E}$ *let $\mathbf{u}(t, t_0, \mathbf{x}_0)$ be the integral of the homogeneous equation* (4) *defined on* J *and equal to $\mathbf{x}_0$ at t_0.*

1° *For every point $t \in \mathrm{J}$ the map $\mathbf{x}_0 \mapsto \mathbf{u}(t, t_0, \mathbf{x}_0)$ is a bijective bicontinuous linear map $C(t, t_0)$ of* E *to itself.*

2° *The map $t \mapsto C(t, t_0)$ of* J *into* $\mathscr{L}(\mathrm{E})$ *is identical to the integral of the homogeneous linear differential equation*

$$\frac{dU}{dt} = A(t)\,U \tag{5}$$

which takes the value I (the identity map of E to itself) *at the point t_0.*

3° *For any points s, t, u of* J

$$C(s, u) = C(s, t)C(t, u), \qquad\qquad C(s, t) = (C(t, s))^{-1}. \tag{6}$$

By prop. 1, $\mathbf{u}(t, t_0, \mathbf{x}_1) + \mathbf{u}(t, t_0, \mathbf{x}_2)$ (resp. $\lambda\mathbf{u}(t, t_0, \mathbf{x}_0)$) is an integral of (4) and takes the value $\mathbf{x}_1 + \mathbf{x}_2$ (resp. $\lambda\mathbf{x}_0$) at t_0, so, by th. 1 of IV, p. 179, is identical to $\mathbf{u}(t, t_0, \mathbf{x}_1 + \mathbf{x}_2)$ (resp. $\mathbf{u}(t, t_0, \lambda\mathbf{x}_0)$); the map $\mathbf{x}_0 \mapsto \mathbf{u}(t, t_0, \mathbf{x}_0)$ is thus a linear map $C(t, t_0)$ of E into itself, and one can write $\mathbf{u}(t, t_0, \mathbf{x}_0) = C(t, t_0).\mathbf{x}_0$.

Since the map $(X, Y) \mapsto XY$ of $\mathscr{L}(\mathrm{E}) \times \mathscr{L}(\mathrm{E})$ into $\mathscr{L}(\mathrm{E})$ is *continuous* (*Gen. Top.*, X, p. 298, prop. 8), the map $t \mapsto A(t)U$ of J into $\mathscr{L}(\mathrm{E})$ is regulated for all $U \in \mathscr{L}(\mathrm{E})$; further (*Gen. Top.*, X, p. 296)

$$\|A(t)X - A(t)Y\| = \|A(t)(X - Y)\| \leqslant \|A(t)\| \; \|X - Y\| ,$$

so one can apply th. 1 of IV, p. 179 to the homogeneous linear equation (5); let $V(t)$ be the integral of this equation defined on J and equal to I at t_0. One has (I, p. 6, prop. 3)

$$\frac{d}{dt}\big(V(t)\mathbf{x}_0\big) = \frac{dV(t)}{dt}\mathbf{x}_0 = A(t)\big(V(t)\mathbf{x}_0\big)$$

and for $t = t_0$ we have $V(t)\mathbf{x}_0 = I\mathbf{x}_0 = \mathbf{x}_0$; by th. 1 of IV, p. 179 one must have $V(t).\mathbf{x}_0 = C(t, t_0)\mathbf{x}_0$ for all $\mathbf{x}_0 \in E$, that is, $V(t) = C(t, t_0)$; this proves that $C(t, t_0)$ belongs to $\mathscr{L}(\mathrm{E})$, in other words, that $\mathbf{x}_0 \mapsto C(t, t_0).\mathbf{x}_0$ is continuous on E, and that the map $t \mapsto C(t, t_0)$ is the integral of (5) which is equal to I at t_0.

Finally, the integral $s \mapsto C(s, u).\mathbf{x}_0$ of (4) is equal to $C(t, u).\mathbf{x}_0$ at the point t, so, by definition,

$$C(s, u).\mathbf{x}_0 = C(s, t)\big(C(t, u).\mathbf{x}_0\big) = \big(C(s, t)\, C(t, u)\big).\mathbf{x}_0$$

for any $\mathbf{x}_0 \in \mathrm{E}$, whence the first relation (6); since $C(s, s) = I$ one has $C(s,t)C(t,s) = I$, for any s and t in J; this proves (*Set Theory*, II, p. 86, corollary) that $C(t, t_0)$ is a bijective map of E onto itself, with inverse map $C(t_0, t)$. This completes the proof of the theorem.

One says that $C(t, t_0)$ is the *resolvent* of equation (2) of IV, p. 178.

COROLLARY 1. *The map which to every point* $\mathbf{x}_0 \in \mathrm{E}$ *associates the continuous function* $t \mapsto C(t, t_0).\mathbf{x}_0$, *defined on* J, *is an isomorphism of the normed space* E *onto the vector space* $\mathcal{I}$ *of integrals of* (4), *endowed with the topology of compact convergence.*

It is certainly a bijective linear map of E onto $\mathcal{I}$; now $C(t, t_0)$ is bounded on a compact set $\mathrm{K} \subset \mathrm{J}$, so $\|C(t, t_0).\mathbf{x}_0\| \leqslant \mathrm{M}\,\|\mathbf{x}_0\|$ for any $t \in \mathrm{K}$ and $\mathbf{x}_0 \in \mathrm{E}$, which shows that this map is continuous; and since

$$C(t_0, t_0).\mathbf{x}_0 = \mathbf{x}_0 ,$$

it is clear that the inverse map is also continuous.

COROLLARY 2. *The map* $(s, t) \mapsto C(s, t)$ *of* $\mathrm{J} \times \mathrm{J}$ *into* $\mathscr{L}(\mathrm{E})$ *is continuous.*

By (6) we have $C(s,t) = C(s,t_0)\left(C(t,t_0)\right)^{-1}$; now, the map $(X,Y) \mapsto XY$ of $\mathscr{L}(\mathrm{E}) \times \mathscr{L}(\mathrm{E})$ into $\mathscr{L}(\mathrm{E})$ is continuous, as is the map $X \mapsto X^{-1}$ of the (open) group of invertible elements of $\mathscr{L}(\mathrm{E})$ onto itself (TG, IX, p. 40, prop. 14).

One may note that the map

$$t \mapsto C(t_0,t) = (C(t,t_0))^{-1}$$

admits a derivative equal to $-(C(t,t_0))^{-1}(dC(t,t_0)/dt)(C(t,t_0))^{-1}$ (on the complement of a countable set) (I, p. 8, prop. 4), that is to say (by IV, p. 179, formula (5)) equal to $-C(t_0,t)\,A(t)$.

COROLLARY 3. *Let* K *be a compact interval contained in* J, *and let* $k = \sup_{t\in \mathrm{K}} \|A(t)\|$. *For all* t *and* t_0 *in* K

$$\|C(t,t_0) - I\| \leqslant e^{k|t-t_0|} - 1\,. \tag{7}$$

Indeed, $\|A(t)\mathbf{x}_0\| \leqslant k\,\|\mathbf{x}_0\|$ for all $t \in \mathrm{K}$; on K the constant function equal to $\mathbf{x}_0$ is thus an approximate integral to within $k\,\|\mathbf{x}_0\|$ by equation (4) of IV, p. 18; by formula (15) of IV, p. 170, one thus has

$$\|C(t,t_0)\mathbf{x}_0 - \mathbf{x}_0\| \leqslant \|\mathbf{x}_0\|\left(e^{k|t-t_0|} - 1\right)$$

for any t and t_0 in K, and $\mathbf{x}_0$ in E, which is equivalent to the inequality (7) by the definition of the norm on $\mathscr{L}(\mathrm{E})$.

PROPOSITION 2. *Let* B *be a continuous endomorphism of* E, *independent of* t, *and commuting with* $A(t)$ *for all* $t \in \mathrm{J}$*; then* B *commutes with* $C(t,t_0)$ *for all* t *and* t_0 *in* J.

Indeed, by (5)

$$\frac{d}{dt}\big(BC\big) = BAC = ABC \qquad \text{and} \qquad \frac{d}{dt}\big(CB\big) = ACB,$$

so $\dfrac{d}{dt}\big(BC - CB\big) = A(BC - CB)$; but $BC(t_0,t_0) - C(t_0,t_0)B = 0$, so (IV, p. 179, th. 1) $BC(t,t_0) - C(t,t_0)B = 0$ for all $t \in \mathrm{J}$.

An important instance of prop. 2 is that where E is endowed with the structure of a normed vector space with respect to the *field of complex numbers* $\mathbf{C}$, and where, for every $t \in \mathrm{J}$, $A(t)$ is an endomorphism of E for this vector space structure; this means that $A(t)$ commutes with the continuous endomorphism $\mathbf{x} \mapsto i\mathbf{x}$ of E (for the vector space structure *over* $\mathbf{R}$); then $C(t,t_0)$ commutes with this endomorphism, which means that for any t and t_0 in J, the map $C(t,t_0)$ is a continuous endomorphism for the normed vector space structure of E over $\mathbf{C}$.

3. INTEGRATING THE INHOMOGENEOUS LINEAR EQUATION

Integrating the inhomogeneous linear equation

$$\frac{d\mathbf{x}}{dt} = A(t).\mathbf{x} + \mathbf{b}(t) \tag{2}$$

reduces to integrating the associated homogeneous equation

$$\frac{d\mathbf{x}}{dt} = A(t).\mathbf{x} \tag{4}$$

and evaluating a primitive. With the notation of th. 2 of IV, p. 179, let us put $\mathbf{x} = C(t, t_0).\mathbf{z}$, whence, from the second formula (6) of IV, p. 179, $\mathbf{z} = C(t_0, t).\mathbf{x}$; if $\mathbf{x}$ is an integral of (2) then $\mathbf{z}$ is an integral of the equation $\frac{d}{dt}\big(C(t, t_0).\mathbf{z}\big) = A(t)C(t, t_0).\mathbf{z} + \mathbf{b}(t)$; since the bilinear map

$$(U, \mathbf{y}) \mapsto U.\mathbf{y}$$

of $\mathscr{L}(\mathrm{E}) \times \mathrm{E}$ into E is continuous (*Gen. Top.*, X, p. 297, prop. 6), $\mathbf{z}$ admits a derivative (except on a countable subset of J) and one has, by the formula for differentiating a bilinear function (I, p. 6, prop. 3)

$$\frac{d}{dt}\big(C(t, t_0).\mathbf{z}\big) = \frac{dC(t, t_0)}{dt}.\mathbf{z} + C(t, t_0).\frac{d\mathbf{z}}{dt} = A(t)C(t, t_0).\mathbf{z} + C(t, t_0).\frac{d\mathbf{z}}{dt}$$

(replacing $dC(t, t_0)/dt$ by $A(t)C(t, t_0)$ according to (5) (IV, p. 179)). The equation for $\mathbf{z}$ then reduces to $C(t, t_0).d\mathbf{z}/dt = \mathbf{b}(t)$, or again to

$$\frac{d\mathbf{z}}{dt} = C(t_0, t).\mathbf{b}(t) \tag{8}$$

by the second formula (6) of IV, p. 179. Now the right-hand side of equation (8) is a regulated function on J, having been obtained by substituting the regulated functions U and $\mathbf{y}$ in the continuous bilinear function $U.\mathbf{y}$ (*cf.* II, p. 55, cor. 2); equation (8) thus has one and only one integral taking the value $\mathbf{x}_0$ at t_0, given by the formula

$$\mathbf{z}(t) = \mathbf{x}_0 + \int_{t_0}^{t} C(t_0, s).\mathbf{b}(s)\, ds. \tag{9}$$

Since one has $C(t, t_0). \int_{t_0}^{t} C(t_0, s).\mathbf{b}(s)\, ds = \int_{t_0}^{t} C(t, t_0)C(t_0, s).\mathbf{b}(s)\, ds$ (II, p. 59, formula (9)), one obtains (taking account of the first formula (6) of IV, p. 179) the following result:

PROPOSITION 3. *With the notation of th.* 2 (IV, p. 179), *for every point* $(t_0, \mathbf{x}_0)$ *of* $\mathrm{J} \times \mathrm{E}$ *the integral of the linear equation* (2) *defined on* J *and equal to* $\mathbf{x}_0$ *at* t_0 *is given by the formula*

$$\mathbf{u}(t) = C(t, t_0).\mathbf{x}_0 + \int_{t_0}^{t} C(t, s).\mathbf{b}(s)\, ds. \tag{10}$$

The method which leads to formula (10), which consists of taking the function $\mathbf{z}$ as a new unknown function, is often called the "method of variation of parameters".

4. FUNDAMENTAL SYSTEMS OF INTEGRALS OF A LINEAR SYSTEM OF SCALAR DIFFERENTIAL EQUATIONS

In this subsection and the next we shall consider the case where E is a vector space *of finite dimension n over the field* **C** of complex numbers (so of dimension $2n$ over **R**), and where, for every $t \in \mathrm{J}$, $A(t)$ is an endomorphism of E *with respect to the vector space structure over* **C**. One can then identify $A(t)$ with its matrix $\big(a_{ij}(t)\big)$ with respect to a basis of E (over the field **C**), the a_{ij} this time being n^2 *complex* functions defined and regulated on J; letting x_j $(1 \leqslant j \leqslant n)$ denote the (complex) components of a vector $\mathbf{x} \in \mathrm{E}$ with respect to the chosen basis, the linear equation

$$\frac{d\mathbf{x}}{dt} = A(t).\mathbf{x} + \mathbf{b}(t) \tag{2}$$

is again equivalent to the system

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t) \qquad (1 \leqslant i \leqslant n). \tag{3}$$

Theorems 1 (IV, p. 179) and 2 (IV, p. 179) and prop. 2 (IV, p. 181) then show that for every $\mathbf{x}_0 = (x_{k0})_{1\leqslant k\leqslant n}$ in E there exists one and only one integral $\mathbf{u} = (u_k)_{1\leqslant k\leqslant n}$ of the equation

$$\frac{d\mathbf{x}}{dt} = A(t).\mathbf{x} \tag{4}$$

defined on E and equal to $\mathbf{x}_0$ at the point t_0; this integral can be written

$$\mathbf{u}(t, t_0, \mathbf{x}_0) = C(t, t_0).\mathbf{x}_0,$$

$C(t, t_0)$ being an *invertible* square matrix $(c_{ij}(t, t_0))$ of order n whose entries are continuous complex functions on $\mathrm{J} \times \mathrm{J}$ and such that $t \mapsto c_{ij}(t, t_0)$ is a primitive of a regulated function on J.

In the particular case where $n = 1$ the system (3) reduces to a single scalar equation

$$\frac{dx}{dt} = a(t)x + b(t) \tag{11}$$

($a(t)$ and $b(t)$ being complex regulated functions on J); one verifies immediately that the (one element) matrix $C(t, t_0)$ is equal to $\exp\left(\int_{t_0}^{t} a(s)\,ds\right)$; the integral of (11) equal to x_0 at the point t_0 is thus given explicitly by the formula

$$u(t) = x_0 \exp\left(\int_{t_0}^{t} a(s)\,ds\right) + \int_{t_0}^{t} b(s)\exp\left(\int_{t_0}^{t} a(\tau)\,d\tau\right) ds. \tag{12}$$

In the space $\mathcal{C}(\mathrm{J};\mathrm{E})$ of continuous maps from J into E, endowed with the topology of compact convergence, the set $\mathcal{I}$ of integrals of equation (4) is a vector subspace (over **C**) *isomorphic* to E, therefore to $\mathbf{C}^n$ (IV, p. 180, cor. 1, and IV, p. 181, prop. 2).

A *basis* $(\mathbf{u}_j)_{1\leqslant j\leqslant n}$ of this space (over the field $\mathbf{C}$) is called a *fundamental system* of integrals of (4).

PROPOSITION 4. *For the n integrals $\mathbf{u}_j$ $(1 \leqslant j \leqslant n)$ of equation* (4) *to form a fundamental system it is necessary and sufficient that their values $\mathbf{u}_j(t_0)$ at a point $t_0 \in \mathrm{J}$ be linearly independent vectors in* E.

Indeed, the map which to every $\mathbf{x}_0 \in \mathrm{E}$ associates the integral $t \mapsto C(t, t_0).\mathbf{x}_0$ is an isomorphism of E onto $\mathcal{I}$ (IV, p. 180, cor. 1 and IV, p. 181, prop. 2).

If $\left(\mathbf{e}_j\right)_{1\leqslant j\leqslant n}$ is any basis of E over $\mathbf{C}$, the n integrals

$$\mathbf{u}_j(t) = C(t, t_0).\mathbf{e}_j \qquad (1 \leqslant j \leqslant n)$$

thus form a fundamental system; if one identifies $C(t, t_0)$ with its matrix with respect to the basis $\left(\mathbf{e}_j\right)$ the integrals $\mathbf{u}_j$ are precisely the *columns* of the matrix $C(t, t_0)$. The integral of (4) that takes the value $\mathbf{x}_0 = \sum_{j=1}^{n} \lambda_j \mathbf{e}_j$ at the point t_0 is then $C(t, t_0).\mathbf{x}_0 = \sum_{k=1}^{n} \lambda_k \mathbf{u}_k(t)$.

Given *any* n integrals $\mathbf{u}_j$ $(1 \leqslant j \leqslant n)$ of (4) one terms the *determinant* of these n integrals at a point $t \in \mathrm{J}$ with respect to a basis $\left(\mathbf{e}_j\right)_{1\leqslant j\leqslant n}$ of E, the determinant

$$\Delta(t) = \left(\mathbf{u}_1(t),\ \mathbf{u}_2(t),\ \ldots,\ \mathbf{u}_n(t)\right) \tag{13}$$

of the n vectors $\mathbf{u}_j(t)$ with respect to the basis $\left(\mathbf{e}_j\right)$ (*Alg*., III, p. 522). One has (*Alg*., III, p. 523, prop. 2)

$$\Delta(t) = \Delta(t_0) \det\left(C(t, t_0)\right). \tag{14}$$

By prop. 4 of IV, p. 184, for $\left(\mathbf{u}_j\right)_{1\leqslant j\leqslant n}$ to be a fundamental system of integrals of (4) it is necessary and sufficient that the determinant $\Delta(t)$ of the $\mathbf{u}_j$ be $\neq 0$ at some one point t_0 of J; the formula (14) then shows that $\Delta(t) \neq 0$ at every point of J, in other words, that the vectors $\mathbf{u}_j(t)$ $(1 \leqslant j \leqslant n)$ are always linearly independent.

PROPOSITION 5. *The determinant of the matrix $C(t, t_0)$ is given by the formula*

$$\det\left(C(t, t_0)\right) = \exp\left(\int_{t_0}^{t} \operatorname{Tr}(A(s))\, ds\right). \tag{15}$$

Indeed, if one puts $\delta(t) = \det\left(C(t, t_0)\right)$ one has, by the formula for the derivative of a determinant (I, p. 8, formula (3))

$$\frac{d\delta}{dt} = \operatorname{Tr}\left(\frac{dC(t, t_0)}{dt}\left(C(t, t_0)\right)^{-1}\right)\delta(t)$$

that is, by the differential equation (5) of IV, p. 179 satisfied by $C(t, t_0)$,

$$\frac{d\delta}{dt} = \mathrm{Tr}\big(A(t)\big)\,\delta(t).$$

Since $\delta(t_0) = 1$ the formula (15) follows from the expression (12) (IV, p. 183) for the integral of a scalar linear equation.

Specifying n linearly independent integrals of (4) determines *all* the integrals of this equation, as we have just seen. We shall now show that for $1 \leqslant p \leqslant n$ the knowledge of *p linearly independent integrals* $\mathbf{u}_j$ $(1 \leqslant j \leqslant p)$ of equation (4) reduces the integration of this equation to that of a *homogeneous linear system of $n - p$ scalar equations*. Suppose that on an interval $\mathrm{K} \subset \mathrm{J}$ there are $n - p$ maps

$$\mathbf{u}_{p+k} \qquad (1 \leqslant k \leqslant n - p)$$

of K into E, which are primitives of regulated functions on K, and such that, for every $t \in \mathrm{K}$, the n vectors $\mathbf{u}_j(t)$ $(1 \leqslant j \leqslant n)$ form a *basis* of E.

For every point $t_1 \in \mathrm{J}$ there always exists an interval K, a neighbourhood of t_1 in J, on which there are defined $n - p$ functions $\mathbf{u}_{p+k}$ $(1 \leqslant k \leqslant n - p)$ having the preceding properties. For, let $(\mathbf{e}_i)_{1 \leqslant i \leqslant n}$ be a basis of E; there exist $n - p$ vectors of this basis which form with the $\mathbf{u}_j(t_1)$ $(1 \leqslant j \leqslant p)$ a basis of E (*Alg.*, II, p. 292, th. 2); suppose for example that they are $\mathbf{e}_{p+1}, \ldots, \mathbf{e}_n$; since the determinant $\det(\mathbf{u}_1(t), \ldots, \mathbf{u}_p(t), \mathbf{e}_{p+1}, \ldots, \mathbf{e}_n)$ (with respect to the basis $(\mathbf{e}_i)$) is a continuous function of t and does not vanish for $t = t_1$ there exists a neighbourhood K of t_1 on which it does not vanish; one can then take $\mathbf{u}_{p+k}(t) = \mathbf{e}_{p+k}$ $(1 \leqslant k \leqslant n - p)$ for $t \in \mathrm{K}$.

There exists an invertible matrix $B(t)$ of order n, whose elements are primitives of regulated functions on K, such that $B(t).\mathbf{e}_j = \mathbf{u}_j(t)$ for $1 \leqslant j \leqslant n$. Let us put $\mathbf{x} = B(t).\mathbf{y}$; then $\mathbf{y}$ satisfies the equation $\dfrac{dB}{dt}.\mathbf{y} + B(t).\dfrac{d\mathbf{y}}{dt} = A(t)B(t).\mathbf{y}$, which can also be written

$$\frac{d\mathbf{y}}{dt} = \big(B(t)\big)^{-1}\left(A(t)B(t) - \frac{dB}{dt}\right).\mathbf{y} = H(t).\mathbf{y}$$

where $H(t) = \big(h_{jk}(t)\big)$ is a matrix with regulated entries on K. By the definition of $B(t)$ this linear equation admits the p constant vectors $\mathbf{e}_j$ $(1 \leqslant j \leqslant p)$ as integrals; one concludes immediately that necessarily $h_{jk}(t) = 0$ for $1 \leqslant k \leqslant p$; thus the components y_k of $\mathbf{y}$ (with respect to the basis $(\mathbf{e}_i)$) with index $k \geqslant p + 1$ satisfy a homogeneous linear system of $n - p$ equations; once the solutions of this system are determined, the dy_j/dt for indices $j \leqslant p$ are linear functions of the y_k with $k \geqslant p + 1$, so are known, and the primitives of these functions will give the y_j for indices $j \leqslant p$.

In particular, when one knows $n - 1$ linearly independent integrals of equation (4) of IV, p. 183, integrating this equation reduces to that of a single homogeneous scalar equation, and then the evaluation of n primitives.

Remarks. 1) All the above applies also to the case where E is of dimension n *over the field* $\mathbf{R}$ and $A(t)$ is an endomorphism of E for every $t \in \mathrm{J}$: one has only to replace $\mathbf{C}$ by $\mathbf{R}$ throughout.

2) Let $A(t) = (a_{ij}(t))$ be a square matrix of order n whose entries are real (resp. complex) regulated functions of t on J, and let $C(t, t_0) = (c_{ij}(t, t_0))$ be the resolvent matrix of the corresponding linear system (3) (IV, p. 22). Let F be an arbitrary complete normed space over $\mathbf{R}$ (resp. $\mathbf{C}$) and let us consider the system of linear differential equations

$$\frac{d\mathbf{y}_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)\,\mathbf{y}_j$$

where the unknown functions $\mathbf{y}_j$ *take their values in* F. It is immediate that the solution $(\mathbf{u}_j)_{1 \leqslant j \leqslant n}$ of this system such that $\mathbf{u}_j(t_0) = \mathbf{d}_j$ for $1 \leqslant j \leqslant n$ ($\mathbf{d}_j$ being arbitrary in F) is given by the formulae

$$\mathbf{u}_i(t) = \sum_{j=1}^{n} c_{ij}(t, t_0)\,\mathbf{d}_j \qquad (1 \leqslant i \leqslant n).$$

We consider in particular the case where $A(t)$ is an endomorphism of a vector space E of finite dimension n *over* $\mathbf{C}$, such that there exists a basis of E with respect to which the matrix of $A(t)$ has *real* elements for every $t \in \mathrm{J}$. Then the above shows (by th. 1 of IV, p. 179) that the resolvent matrix $C(t, t_0)$ with respect to the same basis also has *real* elements: it suffices to consider the vector space E_0 *over* $\mathbf{R}$ generated by the basis of E under consideration, and to remark that the restriction of $A(t)$ to E_0 is an endomorphism of this vector space.

5. ADJOINT EQUATION

Assuming always that the space E is of *finite* dimension n over $\mathbf{C}$, let E^* be its *dual* (A, II, p. 40), which is a space of dimension n over $\mathbf{C}$ (*Alg*., II, p. 299, th. 4); the canonical bilinear form $\langle \mathbf{x}, \mathbf{x}^* \rangle$ defined on $\mathrm{E} \times \mathrm{E}^*$ (*Alg*., II, p. 234) is *continuous* on this product (being a polynomial in the components of $\mathbf{x} \in \mathrm{E}$ and $\mathbf{x}^* \in \mathrm{E}^*$).

Given a homogeneous linear equation (4) (IV, p. 183), where $t \mapsto A(t)$ is a regulated map of J into $\mathscr{L}(\mathrm{E})$, let us see if there exists a map $t \mapsto \mathbf{v}(t)$ of J into E^*, a primitive of a regulated function on J, and such that the scalar function $t \mapsto \langle \mathbf{u}(t), \mathbf{v}(t) \rangle$ is *constant* on J when $\mathbf{u}$ is an *arbitrary* solution of (4); it comes to the same to write that the derivative of this function should be zero at every point where $\mathbf{u}$ and $\mathbf{v}$ are differentiable, that is, one must have

$$\left\langle \frac{d\mathbf{u}}{dt}, \mathbf{v}(t) \right\rangle + \left\langle \mathbf{u}(t), \frac{d\mathbf{v}}{dt} \right\rangle = 0$$

at such points.

Now, by (4), $\left\langle \dfrac{d\mathbf{u}}{dt}, \mathbf{v}(t) \right\rangle = \langle A(t).\mathbf{u}(t), \mathbf{v}(t) \rangle = -\langle \mathbf{u}(t), B(t).\mathbf{v}(t) \rangle$ where $-B(t)$ is the *transpose* of $A(t)$ (*Alg*., II, p. 234). The relation that $\mathbf{v}$ must satisfy can thus be written

$$\left\langle \mathbf{u}(t)\,,\ \frac{d\mathbf{v}}{dt} - B(t).\mathbf{v}(t) \right\rangle = 0$$

at all points where $A(t)$ is continuous and $\mathbf{v}(t)$ is differentiable. Now for such a point t and an *arbitrary* point $\mathbf{x}_0 \in \mathrm{E}$ there exists, by th. 1 of IV, p. 179, a solution $\mathbf{u}$ of (4) such that $\mathbf{u}(t) = \mathbf{x}_0$; one thus must have $\left\langle \mathbf{x}_0\,,\ \frac{d\mathbf{v}}{dt} - B(t).\mathbf{v}(t) \right\rangle = 0$ for *all* $\mathbf{x}_0 \in \mathrm{E}$, which entails $\frac{d\mathbf{v}}{dt} - B(t).\mathbf{v}(t) = 0$. Consequently:

PROPOSITION 6. *The map* $t \mapsto \mathbf{v}(t)$ *of* J *into* E^*, *a primitive of a regulated function on* J, *is such that* $\langle \mathbf{u}(t)\,,\ \mathbf{v}(t)\rangle$ *is constant on* J *for every solution* $\mathbf{u}$ *of the equation* (4) *of* IV, p. 183 *if and only if* $\mathbf{v}$ *is a solution of the homogeneous linear equation*

$$\frac{d\mathbf{x}}{dt} = B(t).\mathbf{x} \tag{16}$$

where $-B(t)$ *is the transpose of* $A(t)$.

Equation (16) is called the *adjoint* of (4); clearly (4) is the adjoint of (16). The elements of the matrix $B(t)$ being regulated functions of t on J, the results obtained above on linear equations apply to (16). In particular, the integral of (16) taking the value $\mathbf{x}_0^*$ at the point t_0 can be written as $H(t, t_0).\mathbf{x}_0^*$, where $H(t, t_0)$ is a bijective linear map of E^* onto itself, identical to the integral of the equation

$$\frac{dV}{dt} = B(t)V \tag{17}$$

which takes the value I at the point t_0. As a result one has (with the notation of IV, p. 179)

$$\langle C(t, t_0).\mathbf{x}_0\,,\ H(t, t_0).\mathbf{x}_0^* \rangle = \langle \mathbf{x}_0\,,\ \mathbf{x}_0^* \rangle$$

for any $\mathbf{x}_0 \in \mathrm{E}$ and $\mathbf{x}_0^* \in \mathrm{E}^*$, which shows that

$$H(t, t_0) = \check{C}(t, t_0) \tag{18}$$

(the *contragradient* of $C(t, t_0)$). In particular, if one knows a fundamental system of integrals of the adjoint equation (16) the matrix $H(t, t_0)$ is determined, so is $C(t, t_0)$ also, and as a result, so are *all* the integrals of equation (4).

Remark. Let E and F be two complete normed spaces over $\mathbf{R}$ (or over $\mathbf{C}$), and $(\mathbf{x}, \mathbf{y}) \mapsto \langle \mathbf{x}\,,\ \mathbf{y}\rangle$ a *continuous* bilinear form on $\mathrm{E} \times \mathrm{F}$, such that the relation "$\langle \mathbf{x}\,,\ \mathbf{y}\rangle = 0$ for all $\mathbf{y} \in \mathrm{F}$" (resp. $\langle \mathbf{x}\,,\ \mathbf{y}\rangle = 0$ for all $\mathbf{x} \in \mathrm{E}$") implies $\mathbf{x} = 0$ (resp. $\mathbf{y} = 0$). Suppose further that for every $t \in \mathrm{J}$ there is a continuous linear map $B(t)$ of F into itself, such that $\langle A(t).\mathbf{x}\,,\ \mathbf{y}\rangle + \langle \mathbf{x}\,,\ B(t).\mathbf{y}\rangle = 0$ for every $(\mathbf{x}, \mathbf{y}) \in \mathrm{E} \times \mathrm{F}$. In these circumstances one sees as before that for a map $t \mapsto \mathbf{v}(t)$ of J into F, a primitive of a regulated function, to be such that $\langle \mathbf{u}(t)\,,\ \mathbf{v}(t)\rangle$ is *constant* for *every* integral $\mathbf{u}$ of (4), it is necessary and sufficient that $\mathbf{v}$ should be an integral of (16), which again we call the *adjoint* of (4).

6. LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Suppose again that E is an *arbitrary* complete normed space over $\mathbf{R}$; let A be a continuous endomorphism of A, *independent of* t, and consider the homogeneous linear equation

$$\frac{d\mathbf{x}}{dt} = A.\mathbf{x}. \tag{19}$$

When E is of *finite* dimension the equation (19) is equivalent to a homogeneous system (3) (IV, p. 183) of scalar differential equations, where the coefficients a_{ij} are *constant*.

By th. 1 (IV, p. 179), every integral of (19) is defined on *all of* $\mathbf{R}$; by th. 2 (IV, p. 179) the integral of (19) taking the value $\mathbf{x}_0$ at a point $t_0 \in \mathbf{R}$ can be written as $C(t, t_0)\mathbf{x}_0$, where $C(t, t_0)$ is a bijective bicontinuous linear map of E onto itself that satisfies the equation

$$\frac{dU}{dt} = AU \tag{20}$$

and is such that $C(t_0, t_0) = I$. Moreover, we have the identity

$$C(t + \tau, t_0 + \tau) = C(t, t_0) \tag{21}$$

for any $\tau \in \mathbf{R}$: indeed, one has $dC(s, t_0 + \tau)/ds = AC(s, t_0 + \tau)$ by (20), and since A is constant one also has

$$\frac{dC(t + \tau, t_0 + \tau)}{dt} = AC(t + \tau, t_0 + \tau);$$

moreover

$$C(t_0 + \tau, t_0 + \tau) = I = C(t_0, t_0),$$

whence we have the identity (21), since the integral of (20) equal to I at the point t_0 is unique.

If one puts $C_0(t) = C(t, 0)$ then $C(t, t_0) = C_0(t - t_0)$; moreover, for every $\lambda \in \mathbf{R}$, $C_0(\lambda t)$ is identical to the integral of the equation

$$\frac{dU}{dt} = \lambda AU \tag{22}$$

which takes the value I at the point 0. We make the following definition:

DEFINITION 1. *Given a continuous endomorphism* A *of* E *we denote by* e^A *or* $\exp A$ *the automorphism of* E *equal to the value at the point* $t = 1$ *of the integral of the equation* (20) *which takes the value* I *at the point* $t = 0$.

With this notation the remarks preceding def. 1 show that

$$C(t, t_0) = \exp\big(A(t - t_0)\big). \tag{23}$$

The exponential notation just introduced is justified by the following properties, which are entirely analogous to those of the function $\exp z$, for z real or complex (*cf.* III, p. 98 and 106):

PROPOSITION 7. 1° *The map* $X \mapsto e^X$ *is a continuous map of* $\mathscr{L}(\mathrm{E})$ *into the group of automorphisms of* E (invertible elements of $\mathscr{L}(\mathrm{E})$).

2° *The map* $t \mapsto e^{Xt}$ *of* $\mathbf{R}$ *into* $\mathscr{L}(\mathrm{E})$ *is differentiable, and*

$$\frac{d}{dt}\left(e^{Xt}\right) = Xe^{Xt} = e^{Xt}X. \tag{24}$$

3° *For any* $X \in \mathscr{L}(\mathrm{E})$ *one has*

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!} \tag{25}$$

the right-hand side being absolutely and uniformly convergent on every bounded subset of $\mathscr{L}(\mathrm{E})$*; in particular,* $e^{It} = e^t I$ *for* $t \in \mathbf{R}$.

4° *If* X *and* Y *commute then* Y *and* e^Y *commute with* e^X*, and*

$$e^{X+Y} = e^X e^Y. \tag{26}$$

The relation (24) follows from the expression (23) for $C(t, 0)$ and from the fact that this function is an integral of (20); by recursion on n one deduces from (24) that $t \mapsto e^{Xt}$ is indefinitely differentiable on $\mathbf{R}$ and that

$$\mathrm{D}^n\left(e^{Xt}\right) = X^n e^{Xt}.$$

By Taylor's formula one thus can write

$$e^X = I + \frac{X}{1!} + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!} + X^{n+1}\int_0^1 \frac{(1-t)^n}{n!}\, e^{Xt}\, dt. \tag{27}$$

On the other hand, cor. 3 of IV, p. 181 shows that $\left\|e^{Xt}\right\| \leqslant \exp\left(\|X\|\,|t|\right)$. Thus the remainder $r_n(X) = X^{n+1}\int_0^1 \frac{(1-t)^n}{n!}\, e^{Xt} dt$ in formula (27) satisfies the inequality

$$\|r_n(X)\| \leqslant \frac{\|X\|^{n+1}}{(n+1)!}\, e^{\|X\|}$$

whence one deduces the formula (25), the series on the right-hand side being absolutely and uniformly convergent on every bounded subset of $\mathscr{L}(\mathrm{E})$. For every pair of elements X, T of $\mathscr{L}(\mathrm{E})$ one thus has

$$e^{X+T} - e^X = \sum_{n=1}^{\infty} \frac{1}{n!}\left((X+T)^n - X^n\right).$$

Now, one can write $(X+T)^n - X^n = \sum_{(V_i)} V_1 V_2 \ldots V_n$, where the sum is taken over the $2^n - 1$ sequences (V_i) of elements of $\mathscr{L}(\mathrm{E})$ such that $V_i = X$ or $V_i = T$ for $1 \leqslant i \leqslant n$, and at least one of the V_i is equal to T; the inequality

$$\left\|(X+T)^n - X^n\right\| \leqslant \big(\,\|X\| + \|T\|\,\big)^n - \|X\|^n$$

follows immediately, whence

$$\|\exp(X+T) - \exp X\| \leqslant \exp\big(\,\|X\| + \|T\|\,\big) - \exp\|X\| \tag{28}$$

which establishes the continuity of the map $X \mapsto \exp X$.

Finally, if X and Y commute, then Y commutes with e^{Xt} (IV, p. 181, prop. 2), so

$$\frac{d}{dt}\big(e^{Xt}\,e^{Yt}\big) = Xe^{Xt}e^{Yt} + e^{Xt}(Ye^{Yt}) = (X+Y)e^{Xt}e^{Yt}.$$

Since, on the other hand, $e^{Xt}\,e^{Yt}$ is equal to I for $t = 0$, one has $e^{Xt}e^{Yt} = e^{(X+Y)t}$, whence formula (26). From this latter, one deduces in particular that for arbitrary real s and t one has

$$e^{X(s+t)} = e^{Xs}\,e^{Xt} \tag{29}$$

and also that

$$e^{-X} = \big(e^X\big)^{-1}. \tag{30}$$

On the contrary one notes that (26) need not remain valid if X and Y are no longer assumed to commute: indeed it would imply that $\exp X$ and $\exp Y$ always commute, which is not the case, as is shown by simple examples (IV, p. 204, exerc. 3).

Now let us assume that E is a vector space *of finite dimension over the field* $\mathbf{C}$, and A an endomorphism of E (for its vector space structure over $\mathbf{C}$) which one can identify with its matrix with respect to a basis of E; then, for all $t \in \mathbf{R}$, e^{At} is an automorphism of E for the same structure (IV, p. 181, prop. 2). Let r_k $(1 \leqslant k \leqslant q)$ be the distinct roots (in $\mathbf{C}$) of the *characteristic polynomial* $\varphi(r) = \det(A - rI)$ of the endomorphism A (the "characteristic roots" of A); if n_k is the order of multiplicity of r_k then $\sum_{k=1}^{q} n_k = n$. One knows that (*Alg.*, VII, 31, n° 3) to each root r_k there corresponds a subspace E_k of E, of dimension n_k, such that E_k is *stable* under A, and that E is the *direct sum* of the E_k: E_k can be defined as the subspace of vectors $\mathbf{x}$ such that

$$(A - r_k I)^{n_k}.\mathbf{x} = 0.$$

Let $\mathbf{a}$ be any vector in E; one can write $\mathbf{a} = \sum_{k=1}^{q} \mathbf{a}_k$, where $\mathbf{a}_k \in \mathrm{E}_k$; the integral of the equation (19) of IV, p. 188, taking the value $\mathbf{a}$ at the point $t = 0$ is thus given by

$$\mathbf{u}(t) = e^{At}.\mathbf{a} = \sum_{k=1}^{q} e^{At}.\mathbf{a}_k = \sum_{k=1}^{q} e^{r_k t} e^{(A - r_k I)t}.\mathbf{a}_k.$$

But since $\mathbf{a}_k \in \mathrm{E}_k$ one has

$$e^{(A-r_kI)t}.\mathbf{a}_k = \mathbf{a}_k + \frac{t}{1!}(A-r_kI).\mathbf{a}_k + \frac{t^2}{2!}(A-r_kI)^2.\mathbf{a}_k + \cdots + \frac{t^{n_k-1}}{(n_k-1)!}(A-r_kI)^{n_k-1}.\mathbf{a}_k. \tag{31}$$

Thus every integral of the equation (19) of IV, p. 188, can be written as

$$\mathbf{u}(t) = \sum_{k=1}^{q} e^{r_kt}\mathbf{p}_k(t) \tag{32}$$

where $\mathbf{p}_k(t)$ is a polynomial in t, with coefficients in the vector space E_k, and of degree $\leqslant n_k - 1$. In particular, if all the roots of the characteristic equation of A are *simple,* the spaces E_k $(1 \leqslant k \leqslant n)$ are all of dimension 1 over the field $\mathbf{C}$, so there exist n vectors $\mathbf{c}_k$ such that the n functions $e^{r_kt}\mathbf{c}_k$ $(1 \leqslant k \leqslant n)$ form a fundamental system of integrals of the equation (19) of IV, p. 188.

The characteristic roots of the endomorphism A are also called *characteristic roots of the linear equation* (19) of IV, p. 188. One may observe that one obtains the characteristic equation of A by writing that the function $\mathbf{c}e^{rt}$ is an integral of (19) for a vector $\mathbf{c} \neq 0$.

When one has determined the roots r_k $(1 \leqslant k \leqslant q)$ explicitly, and thus the order of multiplicity n_k of r_k, in practice one obtains the integrals of (19) by writing that this equation is satisfied by the expression (32) of IV, p. 191, where $\mathbf{p}_k$ is an arbitrary polynomial of degree $\leqslant n_k - 1$, with coefficients *in* E; on identifying the coefficients of e^{r_kt} (for $1 \leqslant k \leqslant q$) in the two sides of the equation so obtained one obtains linear equations for the coefficients of the polynomials $\mathbf{p}_k$: one establishes easily that these equations determine the terms of degree > 0 of $\mathbf{p}_k$ as functions of the constant term, and that the latter is a solution of the equation $(A - r_kI)^{n_k}.\mathbf{x} = 0$, which defines the subspace E_k (method of "undetermined coefficients").

Remark. When there exists a basis of E such that the matrix of A with respect to this basis has *real* elements (*cf.* IV, p. 186, *Remark* 2), the characteristic equation of A has real coefficients. For every $\mathbf{x} = (\xi_k)_{1\leqslant k\leqslant n}$ of E, expressed in the basis under consideration, let $\overline{\mathbf{x}} = (\overline{\xi}_k)_{1\leqslant k\leqslant n}$; the map $\mathbf{x} \mapsto \overline{\mathbf{x}}$ is an antilinear involution of E. One knows (*Alg.*, VII) that, if r_k is a non-real root of the characteristic equation, and E_k is the corresponding stable subspace, then $\overline{r}_k$ is a characteristic root having the same multiplicity n_k as r_k, and the image E'_k of E_k under the map $\mathbf{x} \mapsto \overline{\mathbf{x}}$ is the stable subspace corresponding to $\overline{r}_k$. One deduces from this that if $\mathbf{u}_j$ $(1 \leqslant j \leqslant n_k)$ are n_k linearly independent integrals with values in E_k, then the $2n_k$ integrals $\mathbf{u}_j + \overline{\mathbf{u}}_j$, $i(\mathbf{u}_j - \overline{\mathbf{u}}_j)$ are linearly independent, and have, with respect to the chosen basis of E, components which are *real* functions of E. If r_k is a real characteristic root *Remark* 2 of IV, p. 186, shows that (with the same notation) there are n_k linearly independent integrals $\mathbf{v}_j$ $(1 \leqslant j \leqslant n_k)$ with values in E_k whose components are real. That way one obtains a fundamental system of integrals of (19) whose components are all *real*.

7. LINEAR EQUATIONS OF ORDER *n*

One calls a *linear differential equation of order n* an equation of the form

$$\mathrm{D}^n x - a_1(t)\mathrm{D}^{n-1}x - \cdots - a_{n-1}(t)\mathrm{D}x - a_n(t)x = b(t) \tag{33}$$

where the a_k $(1 \leqslant k \leqslant n)$ and b are real (complex) functions of the real variable t, defined on an interval J of $\mathbf{R}$. The general procedure of IV, p. 164 shows that this equation is equivalent to the linear system of n first order equations

$$\begin{cases} \dfrac{dx_k}{dt} = x_{k+1} & (1 \leqslant k \leqslant n-1) \\ \dfrac{dx_n}{dt} = a_1(t)x_n + a_2(t)x_{n-1} + \cdots + a_n(t)x_1 + b(t) \end{cases} \tag{34}$$

that is to say, to the linear equation

$$\frac{d\mathbf{x}}{dt} = A(t).\mathbf{x} + \mathbf{b}(t) \tag{35}$$

where we have put $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{C}^n$, $\mathbf{b}(t) = (0, 0, \ldots, 0, b(t))$, and where the matrix $A(t)$ is defined by

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ a_n(t) & a_{n-1}(t) & a_{n-2}(t) & \ldots & a_1(t) \end{pmatrix}.$$

The study of the linear equation of order n thus consists of applying the general results above to the particular linear equation (35). For every interval J where the functions a_j $(1 \leqslant j \leqslant n)$ and b are *regulated* there exists one and only one function u, defined on J, having a continuous derivative of order $n-1$, and a regulated derivative of order n on this interval (except at the points of a countable set), satisfying (33) on the complement of a countable subset of J, and such that

$$u(t_0) = x_0, \qquad \mathrm{D}u(t_0) = x_0', \ \ldots, \ D^{n-1}u(t_0) = x_0^{(n-1)} \tag{36}$$

where t_0 is an arbitrary point of J, and $x_0, x_0', \ldots, x_0^{(n-1)}$ are n arbitrary complex numbers.

For the p integrals u_j $(1 \leqslant j \leqslant p)$ of the homogeneous equation

$$\mathrm{D}^n x - a_1(t)\mathrm{D}^{n-1}x - \cdots - a_{n-1}(t)\mathrm{D}x - a_n(t)x = 0 \tag{37}$$

associated with (33) to be linearly independent (in the space $\mathcal{C}(\mathrm{J};\mathbf{C})$ of continuous maps from J into $\mathbf{C}$, considered as a vector space over $\mathbf{C}$), it is necessary and sufficient that the corresponding p integrals $\mathbf{u}_j = (u_j, \mathrm{D}u_j, \ldots, \mathrm{D}^{n-1}u_j)$ of the homogeneous equation $d\mathbf{x}/dt = A(t).\mathbf{x}$ should be linearly independent (in the space $\mathcal{C}(\mathrm{J};\mathbf{C}^n)$ of continuous maps from J into $\mathbf{C}^n$). It is clear that this condition is necessary.

Conversely, if there are n complex constants λ_j, not all zero, such that $\sum_{j=1}^{n} \lambda_j u_j(t) = 0$ identically on J, one deduces that $\sum_{j=1}^{n} \lambda_j \mathrm{D}^k u_j(t) = 0$ on J for every integer k such that $1 \leqslant k \leqslant n-1$, which means that $\sum_{j=1}^{n} \lambda_j \mathbf{u}_j(t) = 0$ on J.

Consequently (IV, p. 180, cor. 1)

PROPOSITION 8. *The set of integrals of the homogeneous linear equation* (37), *defined on* J, *is a vector space of dimension* n *over the field* $\mathbf{C}$.

Given any n integrals u_j $(1 \leqslant j \leqslant n)$ of equation (37) one calls the *Wronskian* of this system of integrals the determinant (with respect to the canonical basis of $\mathbf{C}^n$) of the corresponding n integrals $\mathbf{u}_j$ of the equation $d\mathbf{x}/dt = A(t).\mathbf{x}$, that is to say, the function

$$\mathrm{W}(t) = \begin{vmatrix} u_1(t) & u_2(t) & \dots & u_n(t) \\ \mathrm{D}u_1(t) & \mathrm{D}u_2(t) & \dots & \mathrm{D}u_n(t) \\ \vdots & \vdots & \dots & \vdots \\ \mathrm{D}^{n-1}u_1(t) & \mathrm{D}^{n-1}u_2(t) & \dots & \mathrm{D}^{n-1}u_n(t) \end{vmatrix}.$$

For the n integrals u_j to be linearly independent it is necessary and sufficient that $\mathrm{W}(t) \neq 0$ on J; moreover, it suffices for this that $\mathrm{W}(t_0) \neq 0$ at *only one* t_0 of J (IV, p. 184, prop. 4); further, one has (IV, p. 184, prop. 5)

$$\mathrm{W}(t) = \mathrm{W}(t_0)\ \exp\left(\int_{t_0}^{t} a_1(s)\,ds\right). \tag{38}$$

We identify the resolvent $C(t, t_0)$ of equation (35) with its matrix with respect to the canonical basis of $\mathbf{C}^n$; the columns $\mathbf{v}_j(t, t_0)$ $(1 \leqslant j \leqslant n)$ of this matrix are then n linearly independent integrals

$$\mathbf{v}_j(t, t_0) = (v_j(t, t_0), \mathrm{D}v_j(t, t_0), \dots, \mathrm{D}^{n-1}v_j(t, t_0))$$

of the homogeneous equation $d\mathbf{x}/dt = A(t).\mathbf{x}$ which correspond to n linearly independent integrals $v_j(t, t_0)$ of equation (37) such that

$$\mathrm{D}^{k-1}v_j(t_0, t_0) = \delta_{jk}$$

(Kronecker delta) for $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant n$ (on agreeing to put $\mathrm{D}^0 v_j = v_j$). It results in particular that the method of variation of parameters (IV, p. 182) applied to equation (35) here gives as a particular integral of (33), equal to 0 together with its first $n-1$ derivatives at the point t_0, the function

$$w(t) = \int_{t_0}^{t} v_n(t, s) b(s)\,ds. \tag{39}$$

In the particular case of the equation $D^n x = b(t)$ the formula (39) again gives the formula expressing the n^{th} primitive of the regulated function $b(t)$ which vanishes with its first $n-1$ derivatives at the point t_0, namely

$$w(t) = \int_{t_0}^{t} b(s) \frac{(t-s)^{n-1}}{(n-1)!} ds$$

(II, p. 62, formula (19)): the integral of $D^n x = 0$ which vanishes together with its first $n-2$ derivatives at the point t_0, and whose $n-1^{th}$ derivative is equal to 1 there, is actually the polynomial $(t-t_0)^{n-1}/(n-1)!$.

8. LINEAR EQUATIONS OF ORDER *n* WITH CONSTANT COEFFICIENTS

If the coefficients a_j in equation (33) are *constant,* the corresponding matrix A is constant; the characteristic equation is obtained by writing that e^{rt} is a solution, which gives

$$r^n - a_1 r^{n-1} - \ldots - a_{n-1} r - a_n = 0. \tag{40}$$

Let r_j $(1 \leqslant j \leqslant q)$ be the distinct roots of this equation, and n_j $(1 \leqslant j \leqslant q)$ the multiplicity of the root r_j $\left(\sum_{j=1}^{q} n_j = n\right)$. By the results of IV, p. 188 to 194, to each root r_j there corresponds, for the homogeneous equation

$$D^n x - a_1 D^{n-1} x - \cdots - a_{n-1} D x - a_n x = 0 \tag{41}$$

a system of n_j *linearly independent* integrals

$$u_{jk}(t) = e^{r_j t} p_{jk}(t),$$

where p_{jk} is a polynomial (with complex coefficients) of degree $\leqslant n_j - 1$ $(1 \leqslant k \leqslant n_j)$; further, the n integrals u_{jk} $(1 \leqslant j \leqslant q,\ 1 \leqslant k \leqslant n_j)$ so obtained are *linearly independent*. It follows that the n_j polynomials p_{jk} $(1 \leqslant k \leqslant n_j)$ are linearly independent in the space of polynomials in t of degree $\leqslant n_j - 1$, so form a *basis* (over $\mathbf{C}$) of this space, since the latter is of dimension n_j. In other words:

PROPOSITION 9. *Let r_j $(1 \leqslant j \leqslant q)$ be the distinct roots of the characteristic equation* (40), *and let n_j be the multiplicity of the root r_j $(1 \leqslant j \leqslant q)$. Then the n functions $t^k e^{r_j t}$ $(1 \leqslant k \leqslant n_j,\ 1 \leqslant j \leqslant q)$ are linearly independent integrals of the homogeneous equation* (41).

One can prove this result directly in the following way. It follows from equation (41) that the n^{th} derivative of every integral of this equation is differentiable on $\mathbf{R}$, from which one deduces immediately, by induction on the integer $m > n$, that every integral of (41) admits a derivative of order m, that is, is *indefinitely differentiable* on $\mathbf{R}$. Let $\mathcal{D}$ be the (non-topological) vector space over $\mathbf{C}$ of indefinitely differentiable complex-valued functions on $\mathbf{R}$; the map $x \mapsto Dx$ is an endomorphism of this space, and equation (41) can be written

$$f(\mathrm{D})x = 0 \tag{42}$$

where $f(\mathrm{D}) = \mathrm{D}^n - a_1\mathrm{D}^{n-1} - \cdots - a_{n-1}\mathrm{D} - a_n$ (*Alg.*, IV, p. 4).

PROPOSITION 10. *Let g and h be two relatively prime polynomials such that $f = gh$. The subspace of solutions of* (42) *is the direct sum of the subspaces of solutions of the two equations*

$$g(\mathrm{D})x = 0, \qquad h(\mathrm{D})x = 0.$$

Now, by the Bezout identity (*Alg.*, VII. 2, th. 1) there exist two polynomials $p(\mathrm{D})$ and $q(\mathrm{D})$ such that $p(\mathrm{D})g(\mathrm{D}) + q(\mathrm{D})h(\mathrm{D}) = 1$. For every solution x of (42) one can write $x = y + z$, where $y = p(\mathrm{D})g(\mathrm{D})x$ and $z = q(\mathrm{D})h(\mathrm{D})x$; and then $h(\mathrm{D})y = p(\mathrm{D})(f(\mathrm{D})x) = 0$ and $g(\mathrm{D})z = q(\mathrm{D})(f(\mathrm{D})x) = 0$. On the other hand, if both $g(\mathrm{D})x = 0$ and $h(\mathrm{D})x = 0$, one deduces that

$$x = p(\mathrm{D})(g(\mathrm{D})x) + q(\mathrm{D})(h(\mathrm{D})x) = 0,$$

which completes the proof.

With the preceding notation one can write

$$f(\mathrm{D}) = \prod_{j=1}^{q} (\mathrm{D} - r_j)^{n_j}$$

and prop. 10, by induction on q, shows that the subspace of solutions of (42) is the direct sum of the subspaces of solutions of the q equations

$$(\mathrm{D} - r_j)^{n_j} x = 0 \qquad (1 \leqslant j \leqslant q). \tag{43}$$

Now, for every complex number r one has

$$\mathrm{D}(e^{rt}x\big) = e^{rt}\big(\mathrm{D} + r)x \tag{44}$$

so (43) is equivalent to

$$\mathrm{D}^{n_j}\left(e^{-r_j t}x\right) = 0$$

and thus has as solutions the functions $e^{r_j t}p_j(t)$, where p_j runs through the set of polynomials of degree $\leqslant n_j - 1$; thus one recovers prop. 9 of IV, p. 194.

Assuming that the homogeneous equation (41) has been solved (that is, assuming that the characteristic roots have been found), one knows that the method of variation of parameters allows one to find the solutions of the inhomogeneous equation

$$\mathrm{D}^n x - a_1\mathrm{D}^{n-1}x - \cdots - a_{n-1}\mathrm{D}x - a_n x = b(t) \tag{45}$$

where $b(t)$ is an *arbitrary* regulated function (IV, p. 182); we note that if $b(t)$ is *indefinitely differentiable* on an interval J then all the integrals of (45) are indefinitely differentiable on J. In the particular case $b(t) = e^{\alpha t} p(t)$, where p is a *polynomial* (with complex coefficients) and α is an arbitrary complex number, one obtains an integral of (45) more simply by the following method. Put $x = e^{\alpha t} y$; the equation

$$f(\mathrm{D})x = e^{\alpha t} p(t)$$

can, by (44), be written

$$f(\alpha + \mathrm{D})y = p(t)$$

or, by Taylor's formula applied to the polynomial $f(\mathrm{D})$,

$$\frac{f^{(n)}(\alpha)}{n!}\mathrm{D}^n y + \frac{f^{(n-1)}(\alpha)}{(n-1)!}\mathrm{D}^{n-1} y + \cdots + \frac{f'(\alpha)}{1!}\mathrm{D}y + f(\alpha)y = p(t). \tag{46}$$

Let m be the degree of the polynomial $p(t) = \sum_{k=0}^{m} \lambda_k t^{m-k}$; if $f(\alpha) \neq 0$ (that is, if α is not a characteristic root), there exists one and only one polynomial $u(t) = \sum_{k=0}^{m} c_k t^{m-k}$ *of degree* m which is a solution of (46), for the coefficients c_k are determined by the system of linear equations

$$f(\alpha)c_k + \binom{m-k+1}{1} f'(\alpha)c_{k-1} + \binom{m-k+2}{2} f''(\alpha)c_{k-2} + \cdots + \binom{m}{k} f^{(k)}(\alpha)c_0 = \lambda_k \qquad (0 \leqslant k \leqslant m)$$

which clearly admits one and only one solution. If, on the other hand, α is a characteristic root, and h is its multiplicity, the preceding calculation shows that there is one and only one polynomial of degree m such that every solution of $\mathrm{D}^h y = v(t)$ is an integral; in other words, every polynomial solution of (46) is then of degree $m + h$ ("resonance").

9. SYSTEMS OF LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

With the notation of n° 8, let us consider more generally a system of m differential equations of the form

$$\sum_{k=1}^{n} p_{jk}(\mathrm{D})\, x_k = b_j(t) \qquad (1 \leqslant j \leqslant m) \tag{47}$$

where the unknowns x_k $(1 \leqslant k \leqslant n)$ and the right-hand sides b_j $(1 \leqslant j \leqslant m)$ are complex functions of the real variable t, and where the $p_{jk}(\mathrm{D})$ are polynomials (of any degree) with *constant* (complex) coefficients in the differentiation operator D $(1 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant n)$.

Such systems are not of the same type as those considered in IV, p. 1164 (formula (5)), as the following example shows:

$$\begin{cases} \mathrm{D}x_1 = a(t) \\ \mathrm{D}^2x_1 + \mathrm{D}x_2 + x_2 = b(t). \end{cases} \tag{48}$$

We shall confine ourselves to the case where the functions $b_j(t)$ are *indefinitely differentiable* on the interval J, and we shall look only for solutions $\big(x_k\big)_{1\leqslant k\leqslant n}$ which are indefinitely differentiable on J. On putting $\mathbf{b}(t) = \big(b_1(t), \dots, b_m(t)\big)$, (a map from J into $\mathbf{C}^m$), and $\mathbf{x} = \big(x_1, x_2, \dots, x_n\big)$, the system (47) can be written as

$$P(\mathrm{D})\,\mathbf{x} = \mathbf{b}(t) \tag{49}$$

where $P(\mathrm{D})$ is the matrix $\big(p_{jk}(\mathrm{D})\big)$ with m rows and n columns, whose coefficients belong to the ring $\mathbf{C}[\mathrm{D}]$ of polynomials in D, with coefficients in $\mathbf{C}$. Let $f_j(\mathrm{D})$ $(1 \leqslant j \leqslant r \leqslant \mathrm{Min}(m, n))$ be the nonzero *similarity invariants* of the matrix $P(\mathrm{D})$; one knows (*Alg.*, VII, p. 32) that these are well-determined monic polynomials, such that f_j divides f_{j+1} for $1 \leqslant j \leqslant r-1$ (r being the *rank* of $P(\mathrm{D})$); further, there are two square matrices $U(\mathrm{D})$ and $V(\mathrm{D})$ of order m and n respectively, *invertible* (in the rings of square matrices of order m and n respectively, *with coefficients in the ring* $\mathbf{C}[\mathrm{D}]$ *of polynomials in* D *with complex coefficients*), and such that all the entries of the matrix $Q(\mathrm{D}) = \big(q_{jk}(\mathrm{D})\big) = U(\mathrm{D})\,P(\mathrm{D})\,V(\mathrm{D})$ are zero, apart from the diagonal terms $q_{jj}(\mathrm{D}) = f_j(\mathrm{D})$ for $1 \leqslant j \leqslant r$. Now we put $\mathbf{y} = V^{-1}(\mathrm{D})\,\mathbf{x}$; the equation (49) is equivalent to the equation

$$U(\mathrm{D})\big(P(\mathrm{D})\big(V(\mathrm{D})\,\mathbf{y}\big)\big) = U(\mathrm{D})\,\mathbf{b}$$

that is, to

$$Q(\mathrm{D})\,\mathbf{y} = U(\mathrm{D})\,\mathbf{b} \tag{50}$$

since $U(\mathrm{D})$ is invertible. Now, if $\mathbf{y} = (y_1, y_2, \dots, y_n)$, and if

$$U(\mathrm{D})\,\mathbf{b}(t) = \big(c_1(t), \dots, c_m(t)\big),$$

then equation (50) can be written

$$f_j(\mathrm{D})\,y_j = c_j(t) \qquad \text{for } 1 \leqslant j \leqslant r \tag{51}$$

$$0 = c_j(t) \qquad \text{for } r+1 \leqslant j \leqslant m. \tag{52}$$

The system thus does not admit any indefinitely differentiable solutions unless the conditions (52) are satisfied; the determination of the y_j for indices $j \leqslant r$ then reduces to integrating the r linear differential equations with constant coefficients (51); the y_j for indices $> r$ are arbitrary indefinitely differentiable functions. Once the solutions $\mathbf{y}$ of (50) have thus been determined, one deduces the solutions of (47) from the formula $\mathbf{x} = V(\mathrm{D})\,\mathbf{y}$.

Remarks. 1) Some of the polynomials $f_j(\mathrm{D})$ may reduce to nonzero constants; the corresponding y_j are then completely determined.

2) When the b_j are all zero, that is, if the system (47) is *homogeneous,* the conditions (52) are always satisfied; if, further, $r = n$, one sees that the set of solutions of (47) is a vector space over $\mathbf{C}$, of dimension equal to the *sum of the degrees* of the $f_j(\mathrm{D})$, that is, to the *degree* of $\det(P(\mathrm{D}))$.

3) Given the polynomials $p_{jk}(\mathrm{D})$, a system (47) which admits solutions when the right-hand sides are indefinitely differentiable (or differentiable of a certain order) may not admit them when the right hand sides are arbitrary regulated functions: this is shown by the example (48), which has no solution when $a(t)$ is not a primitive. We shall not undertake here to seek for supplementary possibility conditions which enter when the right-hand sides are arbitrary regulated functions.

EXERCISES

§ 1.

1) *a*) With the notation of IV, p. 165, suppose that $\mathbf{f}$ is uniformly continuous on $\mathrm{J} \times \mathrm{S}$. Prove prop. 3 of IV, p. 166 without using the axiom of choice. (Let δ be such that the relations $|t_2 - t_1| \leqslant \delta$, $\|\mathbf{x}_2 - \mathbf{x}_1\| \leqslant \delta$ imply $\|\mathbf{f}(t_2, \mathbf{x}_2) - \mathbf{f}(t_1, \mathbf{x}_1)\| \leqslant \varepsilon$; consider a subdivision of J into intervals of length $\leqslant \inf(\delta, \delta/M)$ and define the approximate solution by induction.)

b) When E is finite dimensional and $\mathbf{f}$ is Lipschitz on $\mathrm{I} \times \mathrm{H}$ prove prop. 3 of IV, p. 166 without using the axiom of choice. (Remark that for every $\delta > 0$ there exists a finite number of points $\mathbf{x}_i$ of S such that every point of S is at a distance $\leqslant \delta$ from one of the $\mathbf{x}_i$; then proceed as in *a*), considering the regulated functions $t \mapsto \mathbf{f}(t, \mathbf{x}_i)$, of which there are only a finite number.)

2) Given two numbers $b > 0$, $M > 0$ and an arbitrary number $\varepsilon > 0$, give an example of a scalar differential equation $x' = g(x)$ such that $|g(x)| \leqslant M$ for $|x| \leqslant b$ and which admits an integral $x = u(t)$ continuous on the interval $]-b/M - \varepsilon, b/M + \varepsilon[$, but which does not have a finite limit at the point $x = \dfrac{b}{M} + \varepsilon$ (define g so that the integral considered has a continuous derivative on $]-b/M - \varepsilon, b/M + \varepsilon[$, this derivative being equal to the constant M on $]-b/M, b/M[$).

3) Let $\mathrm{S} \subset \mathrm{H}$ be an open ball with centre $\mathbf{x}_0$ and radius r, and $\mathbf{f}$ a Lipschitz function for the constant $k > 0$ on $\mathrm{I} \times \mathrm{S}$; suppose further that $t \mapsto \mathbf{f}(t, \mathbf{x}_0)$ is bounded on I, and denote by M_0 the supremum of $\|\mathbf{f}(t, \mathbf{x}_0)\|$ over $t \in \mathrm{I}$. Show that for all $t_0 \in \mathrm{I}$ there is an integral $\mathbf{u}$ of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$, with values in S, equal to $\mathbf{x}_0$ at the point t_0 and defined on the intersection of I and the interval $]t_0 - \lambda, t_0 + \lambda[$, where

$$\lambda = \frac{1}{k} \log\left(1 + \frac{kz}{M_0}\right)$$

(note that one has $\|\mathbf{u}(t) - \mathbf{x}_0\| \leqslant \mathrm{M}(t - t_0) + k \displaystyle\int_{t_0}^{t} \|\mathbf{u}(s) - \mathbf{x}_0\| \, ds$ for $t > t_0$).

¶ 4) *a*) Let $\mathrm{I} =]t_0 - a, t_0 + a[$ be an open interval in $\mathbf{R}$, let S be an open ball with centre $\mathbf{x}_0$ and radius r in E, and $\mathbf{f}$ a locally Lipschitz function on $\mathrm{I} \times \mathrm{S}$. Let $(s, z) \mapsto h(s, z)$ be a function $\geqslant 0$ defined and continuous for $0 \leqslant s \leqslant a$ and $0 \leqslant z \leqslant r$, such that for every $s \in [0, a]$ the map $z \mapsto h(s, z)$ is increasing; suppose that $\|\mathbf{f}(t, \mathbf{x})\| \leqslant h(|t - t_0|, \|\mathbf{x} - \mathbf{x}_0\|)$ on $\mathrm{I} \times \mathrm{S}$. Let φ be a primitive of a regulated function on an interval $[0, \alpha]$ (with $\alpha < a$), with values in $[0, r[$, such that $\varphi(0) = 0$ and $\varphi'(s) > h(s, \varphi(s))$ on $[0, \alpha]$ except on a countable set. Show that the integral $\mathbf{u}$ of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$, equal to $\mathbf{x}_0$ at the point t_0, is defined on $]t_0 - \alpha, t_0 + \alpha[$, and that $\|\mathbf{u}(t) - \mathbf{x}_0\| \leqslant \varphi(|t - t_0|)$ on this interval.

b) Deduce from *a*) that if $\mathbf{f}$ is defined on $\mathrm{I} \times \mathrm{E}$, and there is a function h defined, continuous, increasing and > 0 on $[0, +\infty[$, such that $\int_0^{+\infty} dz/h(z) = +\infty$, and that $\|\mathbf{f}(t, \mathbf{x})\| \leqslant h(\|x\|)$, then every integral of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ is defined on all of I.

¶ 5) Let E be the complete normed space of sequences $\mathbf{x} = (x_n)$ of real numbers such that $\lim_{n\to\infty} x_n = 0$, normed by $\|\mathbf{x}\| = \sup_n |x_n|$. For every integer $n \geqslant 0$ let $\mathbf{e}_n$ be the sequence every term of which is zero, apart from the term with index n, which is equal to 1; the space E is the direct sum of the subspace V_n of dimension 2 generated by $\mathbf{e}_n$ and $\mathbf{e}_{n+1}$, and the closed subspace W_n generated by the $\mathbf{e}_k$ with indices different from n and $n+1$. Let $\mathbf{f}_n$ be a continuous Lipschitz function on E, with values in E, constant on every equivalence class modulo W_n, equal to $\mathbf{e}_{n+1} - \mathbf{e}_n$ on the line joining $\mathbf{e}_n$ to $\mathbf{e}_{n+1}$, and equal to 0 on the intersection of V_n and the ball $\|\mathbf{x}\| \leqslant \frac{1}{4}$. On the other hand, for every integer $n > 0$ let φ_n be a real function defined and continuous on the interval $\left[\dfrac{1}{n+1}, \dfrac{1}{n}\right]$, equal to 0 at the endpoints of this interval, and such that $\int_{1/n+1}^{1/n} \varphi_n(t)\, dt = 1$. Let I be the interval $[0, 1]$ in $\mathbf{R}$; consider on $\mathrm{I} \times \mathrm{E}$ the function $\mathbf{f}$ with values in E, defined as follows: $\mathbf{f}(0, \mathbf{x}) = 0$ for every $\mathbf{x} \in \mathrm{E}$; for $\dfrac{1}{n+1} \leqslant t \leqslant \dfrac{1}{n}$, let $\mathbf{f}(t, \mathbf{x}) = \varphi_n(t)\mathbf{f}_n(\mathbf{x})$. Show that $\mathbf{f}$ is continuous and locally Lipschitz on $\mathrm{I} \times \mathrm{E}$, but that there exists an integral $\mathbf{u}$ of the equation $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$, defined and bounded on $]0, 1]$, equal to $\mathbf{e}_n$ for $t = 1/n$, and consequently having no limit as t tends to 0.

¶ 6) *a*) Let $\mathrm{I} = [0, +\infty[$; if $\mathbf{f}$ is Lipschitz on $\mathrm{I} \times \mathrm{E}$ for a regulated function $k(t) > 0$ such that the integral $\int_0^{+\infty} k(t)\, dt$ converges, show that every integral of the equation $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ is defined on all of I.

b) If further the integral $\int_0^{+\infty} \|\mathbf{f}(t, \mathbf{x}_0)\|\, dt$ converges (for a certain point $\mathbf{x}_0 \in \mathrm{E}$), show that every integral of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ tends to a finite limit as t tends to $+\infty$ (first show that every integral is bounded as t tends to $+\infty$).

7) Consider the system of scalar differential equations

$$\frac{dx_i}{dt} = \sum_{j,k=1}^{n} c_{ijk} x_j x_k \qquad (1 \leqslant i \leqslant n)$$

where the c_{ijk} are constants such that $c_{kji} = -c_{ijk}$. Show that the integrals of this system are defined for all values of t (note that $\sum_{i=1}^{n} x_i^2$ is constant for every integral $\mathbf{x} = (x_i)$).

8) Let $\mathbf{f}$ be a function defined on $\mathrm{I} \times \mathrm{H}$ satisfying the conditions of lemma 1 of IV, p. 165, and such that for a constant k with $0 < k < 1$, and a point $t_0 \in \mathrm{I}$, one has, for all $t \in \mathrm{I}$ and $\mathbf{x}_1,\ \mathbf{x}_2 \in \mathrm{H}$,

$$\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leqslant \frac{k}{|t - t_0|}\ \|\mathbf{x}_1 - \mathbf{x}_2\|\,.$$

Show that if $\mathbf{u}$ and $\mathbf{v}$ are two approximate solutions of the equation $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$, to within ε_1 and ε_2 respectively, defined on I, with values in H, and having the same value at the point t_0, then, for every $t \in \mathrm{I}$,

$$\|\mathbf{u}(t) - \mathbf{v}(t)\| \leqslant \frac{\varepsilon_1 - \varepsilon_2}{1 - k}\ |t - t_0|\,.$$

Deduce from this that th. 1 (IV, p. 171) and 2 (IV, p. 172) remain valid for the equation $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ under the indicated conditions.

¶ 9) Let I be an interval in $\mathbf{R}$, and t_0 a point of I, let S be a ball with radius r and centre $\mathbf{x}_0$ in E, let G be the normed space of bounded maps from I × S into E, the norm of such a map $\mathbf{f}$ being the supremum $\|\mathbf{f}\|$ of $\|\mathbf{f}(t, \mathbf{x})\|$ over I × S. For every $M > 0$ let G_M be the ball $\|\mathbf{f}\| \leqslant M$ in G. Let L be the subset of G formed by the Lipschitz maps of I × S into E; for every function $\mathbf{f} \in L \cap G_M$ let $\mathbf{u} = U(\mathbf{f})$ be the integral of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ such that $\mathbf{u}(t_0) = \mathbf{x}_0$, with values in S, and defined on the intersection J_M of I and the interval $\left]t_0 - \frac{r}{M}, t_0 + \frac{r}{M}\right[$ (th. 1).

a) Let $(\mathbf{f}_n)$ be a sequence of functions belonging to $L \cap G_M$; if $\mathbf{f}_n$ converges uniformly on I × S to a function $\mathbf{f}$, then every cluster point (for the topology of compact convergence) of the sequence of $\mathbf{u}_n = U(\mathbf{f}_n)$ in the space F of bounded maps from J_M into E, is an integral of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ taking the value $\mathbf{x}_0$ at the point t_0. Conversely, every integral $\mathbf{v}$ of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ defined on J_M and such that $\mathbf{v}(t_0) = \mathbf{x}_0$, is also an integral of an equation $\mathbf{x}' = \mathbf{g}(t, \mathbf{x})$, where $\mathbf{g}$ is Lipschitz and arbitrarily close to $\mathbf{f}$ in G (consider the equation

$$\mathbf{x}' = \mathbf{f}_n(t, \mathbf{x}) + \mathbf{v}'(t) - \mathbf{f}_n(t, \mathbf{v}(t))).$$

b) Suppose further that E is of *finite* dimension. Show that if $\mathbf{f} \in G_M$ satisfies the hypotheses of lemma 1, then for every $t \in J_M$ the set $H(t)$ of values at the point t of the integrals of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ which take the value $\mathbf{x}_0$ at the point t_0 is a *compact* and *connected* set (to see that $H(t)$ is closed, use Ascoli's theorem; to see that it is connected, use *a*): if $\mathbf{x}_1, \mathbf{x}_2$ are two points of $H(t)$ and $\varepsilon > 0$ is arbitrary, show that there exists a connected set Φ of functions $\mathbf{g}$ belonging to $L \cap G_M$ such that $\|\mathbf{f} - \mathbf{g}\| \leqslant \varepsilon$ for every function $\mathbf{g} \in \Phi$, and that the set of values of the functions $U(\mathbf{g})$ at the point t is connected and contains $\mathbf{x}_1$ and $\mathbf{x}_2$. Conclude by passing to the limit along an ultrafilter finer than the neighbourhood filter of 0 in $\mathbf{R}_+$).

10) Let f be a continuous bounded real function defined on the box $P : |t - t_0| < a$, $|x - x_0| < b$ of $\mathbf{R}^2$. Let M be the supremum of $|f(t, x)|$ over P, and let $I =]t_0 - \alpha, t_0 + \alpha[$, where $\alpha = \inf(a, b/M)$. Show that the upper and lower envelopes of the set Φ of integrals of $x' = f(t, x)$ defined on I and taking the value x_0 at the point t_0, are again integrals of $x' = f(t, x)$, which one calls the *maximal* and *minimal* integral of this equation corresponding to the point (t_0, x_0) (remark that the set Φ is equicontinuous and closed for the topology of uniform convergence on I).

For every $\tau \in I$ let ξ be the value of the minimal integral (corresponding to (t_0, x_0)) at the point τ. Show that the minimal integral corresponding to the point (τ, ξ) is identical to the minimal integral corresponding to the point (t_0, x_0) on an interval of the form $[\tau, \tau + h[$ if $\tau > t_0$, and of the form $]\tau - h, \tau]$ if $\tau < t_0$.

Deduce from this that there exists a largest open interval $]t_1, t_2[$ containing t_0 and contained in $]t_0 - a, t_0 + a[$, such that the minimal integral u corresponding to (t_0, x_0) can be extended by continuity to $]t_1, t_2[$, so that at every point t of $]t_1, t_2[$, the value $u(t)$ belongs to $]x_0 - b, x_0 + b[$, and that u is identical with the minimal integral corresponding to the point $(t, u(t))$ on an interval of the form $[t, t + h[$ if $t > t_0$ and of the form $]t - h, t]$ if $t < t_0$; show further that either $t_1 = t_0 - a$ (resp. $t_2 = t_0 + a$) or that $\lim_{t \to t_1} u(t) = x_0 \pm b$ (resp. $\lim_{t \to t_2} u(t) = x_0 \pm b$).

11) *a*) On the box P : $|t - t_0| < a$, $|x - x_0| < b$ let g and h be two continuous real functions such that $g(t, x) < h(t, x)$ on P. Let u (resp. v) be an integral of $x' = g(t, x)$ (resp. $x' = h(t, x)$) such that $u(t_0) = x_0$ (resp. $v(t_0) = x_0$) defined on an interval $[t_0, t_0 + c[$; show that for $t_0 < t < t_0 + c$ one has $u(t) < v(t)$ (consider the supremum τ of the t for which this inequality holds).

b) Let u be the maximal integral of $x' = g(t, x)$ corresponding to the point (t_0, x_0) (exerc. 10), defined on an interval $[t_0, t_0 + c[$, with values in $|x - x_0| < b$. Show that in every compact interval $[t_0, t_0 + d]$ contained in $[t_0, t_0 + c[$ the minimal and maximal integrals of the equation $x' = g(t, x) + \varepsilon$ corresponding to the point (t_0, x_0) are defined once $\varepsilon > 0$ is small enough, and converge uniformly to u when ε tends to 0 through values > 0.

c) Let g and h be two real continuous functions defined on P and such that $g(t, x) \leqslant h(t, x)$ on P. Let $[t_0, t_0 + c[$ be an interval on which are defined an integral u of $x' = g(t, x)$ such that $u(t_0) = x_0$, and the maximal integral v of $x' = h(t, x)$ corresponding to the point (t_0, x_0). Show that $u(t) \leqslant v(t)$ on this interval (reduce to case *a*) using *b*)).

12) Let u be the integral of the equation $x' = \lambda + \dfrac{x^2}{1 + t^2}$ equal to 0 for $t = 0$, and let J be the largest interval with left-hand endpoint 0 on which u is continuous.

a) Show that if $\lambda \leqslant \frac{1}{4}$ one has $J = [0, +\infty[$ (use exerc. 4 of IV, p. 199).

b) Show that if $\lambda > \frac{1}{4}$ one has $J = [0, a[$ where

$$\sinh \frac{\pi}{2\sqrt{\lambda}} < a < \sinh \frac{\pi}{\sqrt{4\lambda - 1}}$$

(put $x = y\sqrt{1 + t^2}$ and use exerc. 11).

¶ 13) *a*) Let $I = [t_0, t_0 + c]$ and let ω be a continuous real function, $\geqslant 0$, defined on $I \times \mathbf{R}$. Let S be a ball with centre $\mathbf{x}_0$ in a complete normed space E, and let $\mathbf{f}$ be a continuous map from $I \times S$ into E such that for any $t \in I$, $\mathbf{x}_1 \in S$ and $\mathbf{x}_2 \in S$ one has $\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leqslant \omega(t, \|\mathbf{x}_1 - \mathbf{x}_2\|)$. Let $\mathbf{u}$ and $\mathbf{v}$ be two integrals of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ defined on I, with values in S, such that $\mathbf{u}(t_0) = \mathbf{x}_1$, $\mathbf{v}(t_0) = \mathbf{x}_2$; let w be the *maximal* integral (exerc. 10) of $z' = \omega(t, z)$ corresponding to the point $(t_0, \|\mathbf{x}_1 - \mathbf{x}_2\|)$, assumed to be defined on I; show that, on I, one has $\|\mathbf{u}(t) - \mathbf{v}(t)\| \leqslant w(t)$. (Let $w(t, \varepsilon)$ be the maximal integral of $z' = \omega(t, z) + \varepsilon$ corresponding to the point $(t_0, \|\mathbf{x}_1 - \mathbf{x}_2\|)$, where $\varepsilon > 0$ is sufficiently small. Show that $\|\mathbf{u}(t) - \mathbf{v}(t)\| \leqslant w(t, \varepsilon)$ for every $\varepsilon > 0$, arguing by contradiction.)

b) In the statement of the hypotheses of *a*), replace I by the interval $I' =]t_0 - c, t_0]$. Show that if w is the *minimal* integral on this interval of $z' = \omega(t, z)$ corresponding to the point $(t_0, \|\mathbf{x}_1 - \mathbf{x}_2\|)$, then $\|\mathbf{u}(t) - \mathbf{v}(t)\| \geqslant w(t)$ on I′ (same method).

¶ 14) *a*) Let $\omega(t, z)$ be a continuous real function, $\geqslant 0$, defined for $0 < t \leqslant a$ and $z \geqslant 0$. Suppose that $z = 0$ is the only integral of $z' = \omega(t, z)$ defined for $0 < t \leqslant a$ and such that $\lim_{t \to 0} z(t) = 0$ and $\lim_{t \to 0} z(t)/t = 0$. Let $I = [t_0, t_0 + a[$, let S be a ball with centre $\mathbf{x}_0$ in E, and $\mathbf{f}$ a map from $I \times S$ into E, such that for any $t \in I$, $\mathbf{x}_1 \in S$ and $\mathbf{x}_2 \in S$ one has $\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leqslant \omega(|t - t_0|, \|\mathbf{x}_1 - \mathbf{x}_2\|)$. Show that, on an interval with left-hand endpoint t_0 contained in I, the equation $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ can have only one solution $\mathbf{u}$ such that $\mathbf{u}(t_0) = \mathbf{x}_0$. (Argue by contradiction: if $\mathbf{v}$ is a second integral of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$, bound $\|\mathbf{u}(t) - \mathbf{v}(t)\|$ below on an interval with left-hand endpoint t_0, with the help of exerc. 13 *b*), and thus obtain a contradiction.)

Apply to the case where $\omega(t,z) = k(z/t)$ with $0 \leqslant k < 1$ (*cf.* IV, p. 200, exerc. 8).

b) The result of *a*) applies for $\omega(t,z) = z/t$; but show by an example in this case that if $\mathbf{u}$, $\mathbf{v}$ are two integrals to within ε of $\mathbf{x}' = \mathbf{f}(t,\mathbf{x})$, equal to $\mathbf{x}_0$ at the point t_0, it is not possible to bound $\|\mathbf{u}(t) - \mathbf{v}(t)\|$ above by a number depending only on t (and *not* on the function $\mathbf{f}$). (Take for $\mathbf{f}$ a continuous map of $\mathbf{R}_+ \times \mathbf{R}$ into $\mathbf{R}$, equal to x/t for $t \geqslant \alpha$ and for $0 < t < \alpha$ and $|x| \leqslant t^2/(\alpha - t)$, and independent of x for the other points (t,x).)

c) Let θ be a continuous real function, $\geqslant 0$ on the interval $[0,a]$. Show that if the integral $\int_0^a \frac{\theta(t)}{t}\,dt$ converges then the result of *a*) applies for $\omega(t,z) = \frac{1+\theta(t)}{t}z$; on the other hand, if $\int_0^a \frac{\theta(t)}{t}\,dt$ is infinite, give an example of a continuous function $\mathbf{f}$ such that $\|\mathbf{f}(t,\mathbf{x}_1) - \mathbf{f}(t,\mathbf{x}_2)\| \leqslant \frac{1+\theta(|t-t_0|)}{|t-t_0|}\|\mathbf{x}_1 - \mathbf{x}_2\|$, but such that the equation $\mathbf{x}' = \mathbf{f}(t,\mathbf{x})$ has infinitely many integrals equal to $\mathbf{x}_0$ at the point t_0 (method similar to that of *b*)).

15) Let f be a real function, defined and continuous for $|t| \leqslant a$, $|x| \leqslant b$, such that $f(t,x) < 0$ for $tx > 0$ and $f(t,x) > 0$ for $tx < 0$; show that $x = 0$ is the only integral of the equation $x' = f(t,x)$ which takes the value 0 at the point $t = 0$ (argue by contradiction).

16) Let E be a vector space of *finite* dimension, and $\mathbf{f}$ a bounded function on $\mathrm{I}\times\mathrm{H}$ satisfying the conditions of IV, p. 165, lemma 1, such that the equation $\mathbf{x}' = \mathbf{f}(t,\mathbf{x})$ has *a unique* solution $\mathbf{u}$ defined on I, with values in H, equal to $\mathbf{x}_0$ at the point t_0. Suppose further that for sufficiently large n there is an approximate integral $\mathbf{u}_n$ of $\mathbf{x}' = \mathbf{f}(t,\mathbf{x})$ to within $1/n$, defined on I, with values in H, equal to $\mathbf{x}_0$ at the point t_0. Show that the sequence $(\mathbf{u}_n)$ converges uniformly to $\mathbf{u}$ on every compact interval contained in I (use the fact that the sequence $(\mathbf{u}_n)$ is equicontinuous on I).

17) To study the equation $x' = \sin tx$ (IV, p. 174, *Example* 4) one puts $u = xy$ and considers the solutions of the corresponding equation $u' = \frac{u}{t} + t\sin u = \mathrm{F}(t,u)$ which are > 0 for $t > 0$ (one has $u(0) = 0$ for every solution of this kind). Denote by Γ_k the curve defined by the relations $(2k-1)\pi < u < 2k\pi$, $t\sin u + \frac{u}{t} = 0$ for each integer $k \geqslant 1$; by D_k the open set defined by $(2k-1)\pi < u < 2k\pi$, $t\sin u + \frac{u}{t} < 0$; and by E the complement of the union of the $\overline{\mathrm{D}}_k$ in the set of (t,u) such that $t > 0$ and $u > 0$.

a) Show that every integral curve which cuts a line $u = 2k\pi$ also cuts the line $u = (2k+1)\pi$.

b) If an integral curve C cuts a curve Γ_k at a point (t_0,u_0), then the function u is increasing for $0 < t < t_0$, decreasing for $t > t_0$, and, when t tends to $+\infty$, $u(t)$ tends to $(2k-1)\pi$, the curve C remaining in D_k for $t > t_0$.

c) Show that there is no integral curve C contained in E and such that $u(t)$ tends to $+\infty$ when t tends to $+\infty$. (Form the differential equation $dx/du = \mathrm{G}(u,x)$ between x and u along C. If C lies entirely in E, then $x^2 > u$ for $u = (2k - \frac{1}{2})\pi$; on the other hand, estimate $x^2(u)$ by using the preceding differential equation, and obtain a contradiction.)

d) Show that for every integer k there exists one and only one integral curve C contained in E and such that $u(t)$ tends to $2k\pi$ when t tends to $+\infty$. (On putting $v = 2k\pi - u$ one obtains the equation $v' = \dfrac{v - 2k\pi}{t} + t \sin v$; compare this equation to $v' = \dfrac{v - 2k\pi}{t} + tv$ with t tending to $+\infty$ and v to 0.)

18) Let E be the space of sequences $\mathbf{x} = (x_n)_{n \in \mathbf{N}}$ of real numbers such that $\lim\limits_{n \to \infty} x_n = 0$, endowed with the norm $\|\mathbf{x}\| = \sup\limits_n |x_n|$, which is in fact a complete normed space. For each $\mathbf{x} = (x_n) \in \mathrm{E}$ let $\mathbf{y} = \mathbf{f}(\mathbf{x})$ be the element (y_n) of E defined by $y_n = |x_n|^{\frac{1}{2}} + \dfrac{1}{n+1}$; the function $\mathbf{f}$ is continuous on E. Show that there is no solution of the differential equation $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ defined on a neighbourhood of 0 in $\mathbf{R}$, with values in E, and equal to 0 at the point $t = 0$ (*cf.* IV, p. 202, exerc. 11).

§ 2.

1) Let E be a complete normed space over $\mathbf{R}$, F a topological space, J an interval of $\mathbf{R}$ not reducing to a single point; let $(t, \xi) \mapsto A(t, \xi)$ be a map from $\mathrm{J} \times \mathrm{F}$ into $\mathscr{L}(\mathrm{E})$, such that when ξ tends to ξ_0, $A(t, \xi)$ tends uniformly to $A(t, \xi_0)$ in J. If $C(t, t_0, \xi)$ is the resolvent of the linear equation $d\mathbf{x}/dt = A(t, \xi).\mathbf{x}$ show that, for every compact interval $\mathrm{K} \subset \mathrm{J}$, $C(t, t_0, \xi)$ tends uniformly to $C(t, t_0, \xi_0)$ on $\mathrm{K} \times \mathrm{K}$ as ξ tends to ξ_0.

2) Let $t \mapsto A(t)$ be a regulated map from J into $\mathscr{L}(\mathrm{E})$ such that, for any two points s, t of J, $A(s)$ and $A(t)$ *commute*. Put $B(t) = \displaystyle\int_{t_0}^{t} A(s)\, ds$. Show that the resolvent $C(t, t_0)$ of the equation $d\mathbf{x}/dt = A(t).\mathbf{x}$ is equal to $\exp(B(t))$.

If $t \mapsto A_1(t)$, $t \mapsto A_2(t)$ are two regulated functions from J into $\mathscr{L}(\mathrm{E})$ such that, for any two points s, t of J, $A_1(s)$, $A_1(t)$, $A_2(s)$, $A_2(t)$ commute pairwise, show that

$$\exp\left(\int_{t_0}^{t} \left(A_1(s) + A_2(s)\right) ds\right) = \exp\left(\int_{t_0}^{t} A_1(s)\, ds\right) \exp\left(\int_{t_0}^{t} A_2(s)\, ds\right).$$

3) Given the two matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

show that $\exp(A + B) \neq \exp(A) \exp(B)$.

4) If P is any automorphism in $\mathscr{L}(\mathrm{E})$ show that $\exp(PAP^{-1}) = P \exp(A) P^{-1}$ for every $A \in \mathscr{L}(\mathrm{E})$.

5) Show by an example that a linear equation $d\mathbf{x}/dt = A.\mathbf{x} + e^{\alpha t}\mathbf{p}(t)$, where $A \in \mathscr{L}(\mathrm{E})$ is independent of t and $\mathbf{p}$ is a polynomial (with coefficients in E) can have an integral which is equal to a polynomial of the *same degree* as $\mathbf{p}$, even when α is a characteristic root of A.

6) Let $f(\mathrm{X})$ be a polynomial of degree n with complex coefficients, and let

$$\frac{1}{f(\mathrm{X})} = \sum_{j=1}^{q} \sum_{h=1}^{n_j} \frac{\lambda_{jh}}{(\mathrm{X} - r_j)^h}$$

be the decomposition of the rational fraction $1/f(\mathrm{X})$ into simple elements (*Alg.*, VII. 7, n° 2). Show that, for any regulated function b, the function

$$t \mapsto \sum_{j=1}^{q} \sum_{h=1}^{n_j} \lambda_{jh} \int_{t_0}^{t} \frac{(t-s)^{h-1}}{(h-1)!} e^{r_j(t-s)} b(s)\, ds$$

is an integral of the equation $f(\mathrm{D})\, x = b(t)$.

7) *a*) Let $t \mapsto A(t)$ be a map from an interval $\mathrm{J} \subset \mathbf{R}$ into $\mathscr{L}(\mathrm{E})$, such that, for every $\mathbf{x} \in \mathrm{E}$, the map $t \mapsto A(t).\mathbf{x}$ of J into E is continuous. Show that $t \mapsto \|A(t)\|$ is *bounded* on every compact interval $\mathrm{K} \subset \mathrm{J}$ (use *Gen. Top.*, IX, p. 194, th. 2). Under these conditions show that, for every point $(t_0, \mathbf{x}_0) \in \mathrm{J} \times \mathrm{E}$ the equation $d\mathbf{x}/dt = A(t).\mathbf{x} + \mathbf{b}(t)$ has one and only one solution defined on J and equal to $\mathbf{x}_0$ at the point t_0. Further, if one denotes this solution by $\mathbf{u}(t, t_0, \mathbf{x}_0)$, the map $\mathbf{x}_0 \mapsto \mathbf{u}(t, t_0, \mathbf{x}_0)$ is a bijective bicontinuous linear map $C(t, t_0)$ of E onto itself, which satisfies the relations (6) (IV, p. 179) and (7) (IV, p. 181); further, the map $(s, t) \mapsto C(s, t)$ is continuous from $\mathrm{J} \times \mathrm{J}$ into $\mathscr{L}(\mathrm{E})$.

b) Take for E the space of sequences $\mathbf{x} = (x_n)_{n \in \mathbf{N}}$ of real numbers such that $\lim_{n \to \infty} x_n = 0$, endowed with the norm $\|\mathbf{x}\| = \sup_n |x_n|$, under which E is complete. For each $t \in \mathrm{J} = [0, 1]$, let $A(t)$ be the linear map of E into itself such that

$$A(t).\mathbf{x} = \left(\frac{1}{1 + nt} x_n \right)_{n \in \mathbf{N}}.$$

Show that $A(t)$ satisfies the conditions of *a*) but that the map $t \mapsto A(t)$ of J into $\mathscr{L}(\mathrm{E})$ is not continuous at the point $t = 0$, and that the resolvent $C(t, t_0)$ of the equation $d\mathbf{x}/dt = A(t).\mathbf{x}$ is such that the map $t \mapsto C(t, t_0)$ of J into $\mathscr{L}(\mathrm{E})$ is not differentiable at the point $t = 0$.

¶ 8) Let G be a complete normed algebra with unit $\mathbf{e}$ over the field $\mathbf{R}$.

a) Let $t \mapsto \mathbf{a}(t)$ be a regulated function on an interval $\mathrm{J} \subset \mathbf{R}$ with values in G. Show that the integral $\mathbf{u}$ of the linear equation $d\mathbf{x}/dt = \mathbf{a}(t)\mathbf{x}$ which takes the value $\mathbf{e}$ at a point $t_0 \in \mathrm{J}$ is invertible on J, and that its inverse is the solution of the equation $d\mathbf{x}/dt = -\mathbf{x}\mathbf{a}(t)$ (if $\mathbf{v}$ is the integral of this last equation which takes the value $\mathbf{e}$ at the point t_0, consider the linear equations satisfied by $\mathbf{uv}$ and $\mathbf{vu}$). Deduce that, for every $\mathbf{x}_0 \in \mathrm{G}$, the integral of $d\mathbf{x}/dt = \mathbf{a}(t)\mathbf{x}$ which takes the value $\mathbf{x}_0$ at the point t_0 is equal to $\mathbf{u}(t)\mathbf{x}_0$.

b) Let $\mathbf{a}(t)$, $\mathbf{b}(t)$ be two regulated functions on J, and $\mathbf{u}$ and $\mathbf{v}$ the integrals of the equations $d\mathbf{x}/dt = \mathbf{a}(t)\mathbf{x}$, $d\mathbf{x}/dt = \mathbf{x}\mathbf{b}(t)$ taking the value $\mathbf{e}$ at the point t_0. Show that the integral of the equation $d\mathbf{x}/dt = \mathbf{a}(t)\mathbf{x} + \mathbf{x}\mathbf{b}(t)$which takes the value $\mathbf{x}_0$ at the point t_0 is equal to $\mathbf{u}(t)\mathbf{x}_0\mathbf{v}(t)$.

c) Let $\mathbf{a}(t)$, $\mathbf{b}(t)$, $\mathbf{c}(t)$, $\mathbf{d}(t)$ be four regulated functions on J, and $(\mathbf{u}, \mathbf{v})$ a solution of the system of two linear equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{a}(t)\mathbf{x} + \mathbf{b}(t)\mathbf{y}, \qquad \frac{d\mathbf{y}}{dt} = \mathbf{c}(t)\mathbf{x} + \mathbf{d}(t)\mathbf{y}.$$

Show that, if $\mathbf{v}$ is invertible on J, then $\mathbf{w} = \mathbf{u}\mathbf{v}^{-1}$ is an integral of the equation $d\mathbf{z}/dt = \mathbf{b}(t) + \mathbf{a}(t)\mathbf{z} - \mathbf{z}\mathbf{d}(t) - \mathbf{z}\mathbf{c}(t)\mathbf{z}$ (the "Ricatti equation"); and conversely. Deduce that every integral of this last equation on J, taking the value $\mathbf{x}_0$ at the point t_0, is of the form $(A(t)\mathbf{x}_0 + B(t)\mathbf{e})(C(t)\mathbf{x}_0 + D(t)\mathbf{e})^{-1}$, where $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are continuous maps from J into $\mathscr{L}(\mathrm{E})$ satisfying the identity $U.(\mathbf{xy}) = (U.\mathbf{x})\mathbf{y}$.

d) Let $\mathbf{w}_1$ be an integral of $d\mathbf{z}/dt = \mathbf{b}(t)+\mathbf{a}(t)\mathbf{z}-\mathbf{z}\mathbf{d}(t)-\mathbf{z}\mathbf{c}(t)\mathbf{z}$ on J; show that if $\mathbf{w}$ is another integral of this equation such that $\mathbf{w} - \mathbf{w}_1$ is invertible on J then $\mathbf{w} - \mathbf{w}_1$ can be expressed in terms of integrals of the equations $d\mathbf{x}/dt = -(\mathbf{d} + \mathbf{c}\mathbf{w}_1)\mathbf{x}$ and $d\mathbf{x}/dt = \mathbf{x}(\mathbf{a} - \mathbf{w}_1\mathbf{c})$ (consider the equation satisfied by $(\mathbf{w} - \mathbf{w}_1)^{-1}$, and use *b*)).

9) Let y_k $(1 \leqslant k \leqslant n)$ be n real functions defined on an interval $\mathrm{I} \subset \mathbf{R}$ and having a continuous $(n-1)^{th}$ derivative.

a) Show that if the n functions y_k are linearly dependent then the matrix $(y_k^{(h)}(t))_{0 \leqslant h \leqslant n-1,\ 1 \leqslant k \leqslant n}$ has rank $< n$ at every point $t \in \mathrm{I}$.

b) Conversely, if for every $t \in \mathrm{I}$ the matrix $(y_k^{(h)}(t))_{0 \leqslant h \leqslant n-1,\ 1 \leqslant k \leqslant n}$ has rank $< n$, show that in every nonempty open interval $\mathrm{J} \subset \mathrm{I}$ there exists a nonempty open interval $\mathrm{U} \subset \mathrm{J}$ such that the restrictions of the y_k to U are linearly dependent (if p is the smallest of the numbers $q < n$ such that the Wronskians of any q of the functions y_k are identically zero on J, consider a point $a \in \mathrm{J}$ where the Wronskian of $p-1$ of the functions y_k is not zero, and show that p of the functions y_k are integrals of a linear equation of order $p-1$ on a neighbourhood of a).

c) Put $y_1(t) = t^2$ for $t \in \mathbf{R}$, $y_2(t) = t^2$ for $t \geqslant 0$, $y_2(t) = -t^2$ for $t < 0$; show that y_1 and y_2 have continuous derivatives on $\mathbf{R}$ and that $y_1 y_2' - y_2 y_1' = 0$, but that y_1 and y_2 are not linearly dependent on $\mathbf{R}$.

* 10) Let $t \mapsto X(t)$ be a map from an interval $\mathrm{J} \subset \mathbf{R}$ into the space of complex matrices of order n. Suppose that the derivative $t \mapsto X'(t)$ exists and is continuous on J, and is such that $X(t)X'(t) = X'(t)X(t)$ for every $t \in \mathrm{J}$.

a) Suppose further that for each $t \in \mathrm{J}$ the eigenvalues of $X(t)$ are distinct. Show that then there exists a *constant* invertible matrix P_0 such that $P_0 X(t) P_0^{-1} = D(t)$, where $D(t)$ is a diagonal matrix, so that $X(t_1)$ and $X(t_2)$ commute for every pair of values t_1, t_2 of t in J. (Write $X(t) = P(t)D(t)P(t)^{-1}$ on a neighbourhood of each point of J and form the differential equation satisfied by $P(t)$.)

b) The matrix

$$X(t) = \begin{pmatrix} t^2 & t^3 & t^4 \\ -2t & -2t^2 & -2t^3 \\ 1 & t & t^2 \end{pmatrix}$$

commutes with its derivative, but the conclusion of *a*) does not hold.$_*$

HISTORICAL NOTE

(N.B. The roman numerals refer to the Bibliography at the end of this note.)

As we have seen (Historical Note to Chapters I-II-III), the problems that led to the integration of differential equations were among the first to be treated by the founders of the infinitesimal calculus of the XVIIth century (notably Descartes and Barrow). Since then the theory of differential equations has not ceased to tax the sagacity of mathematicians, and to be a favoured terrain for the application of the most varied methods of Analysis; the questions that it provokes are very far from being all solved, and the interest that attaches is accordingly so sustained that it constitutes one of the most permanent and fruitful points of contact between mathematics and the experimental sciences: these latter often find a valuable help there, and in exchange constantly provide new problems.

Of the numerous chapters which a modern complete study of differential equations should encompass, we have not wished to expound here but the two most elementary, treating existence theorems and linear equations, the variable being assumed to be *real*. This is why we also limit our brief historical account to these two aspects. From the beginning of the XVIIIth century mathematicians were convinced that the "general" integral of a differential equation of order n depends on n arbitrary constants, and that "in general" there exists one and only one integral with given values for the function and its first $n-1$ derivatives at a given value x_0 of the variable: a conviction which they justified by the process (going back to Newton) of calculating one by one the coefficients of the Taylor expansion of the solution at the point x_0, with the help of the differential equation itself and the first n coefficients. But up until Cauchy no one had studied the convergence of the series so obtained, nor proved that its sum was a solution of the differential equation; and of course, there was no question of nonanalytic differential equations. Among the various methods invented by Cauchy to prove the existence of integrals of differential equations, that which we have followed ((IV) and (IV *bis*)), generalized a little later by Lipschitz, is particularly interesting for the case of nonanalytic equations and approximation of integrals.

The linear differential equations were among the first to draw attention. Leibniz and Jakob Bernoulli integrated the first order linear equation by two quadratures ((I), v. II, p. 731); the general integral of the arbitrary linear equation with constant coefficients and arbitrary right hand side was obtained by Euler ((II), p. 296-354); d'Alembert solved the linear systems with constant coefficients in the same way. A little later, Lagrange (III) treated the general theory of linear equations of order n, recognised that the integral of the homogeneous equation is a linear function of the n constants of integration, introduced the adjoint equation, discovered the reduction of the order of the homogeneous equation when one knows particular solutions, and the method of variation of parameters to solve the inhomogeneous equation. The obscure points in Lagrange's theory (notably in what concerns the linear independence of the integrals) were clarified by Cauchy, whose exposition (IV *bis*) has remained quasi definitive, apart from the improvements in detail provided by matrix notation and the theory of elementary divisors.

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CHAPTER V
Local study of functions

§ 1. COMPARISON OF FUNCTIONS ON A FILTERED SET

Let E be a set filtered by a filter with base $\mathfrak{F}$ (*Gen. Top.*, I, p. 58); in this chapter we shall consider functions whose domain of definition is a subset of E belonging to the filter base $\mathfrak{F}$ (the subset depending on the function) and which take their values either in the field of real numbers $\mathbf{R}$, or, more generally, in a normed vector space over a valued field (*Gen. Top.*, IX, p. 170).

In applications, E will most often be a subspace of the real space $\mathbf{R}^n$, or the extended line $\overline{\mathbf{R}}$, and $\mathfrak{F}$ will be the trace on E of the filter of neighbourhoods of a closure point of E, or else the filter of complements of relatively compact subsets of E ("neighbourhoods of the point at infinity").

It will not in general suffice to know that such a function tends to a given limit along $\mathfrak{F}$ to be able to treat all the problems of "passage to the limit along $\mathfrak{F}$" for expressions formed from this function.

For example, when the real variable x tends to $+\infty$ the three functions x, x^2 and $\sqrt{x}$ all tend to $+\infty$, but, of the expressions

$$(x+1)^2 - x^2, \qquad (x+1) - x, \qquad \sqrt{x+1} - \sqrt{x},$$

the first tends to $+\infty$, the second to 1, the third to 0.

It is thus important to know not just the limit value of a function along $\mathfrak{F}$ (when this limit exists) but also the "manner" in which the function approaches its limit; in other words, one is led to classify the set of functions which tend to the same limit.

1. COMPARISON RELATIONS: I. WEAK RELATIONS

In what follows we denote by V a normed vector space over a valued field K, and by $\mathcal{H}(\mathfrak{F}, V)$ the set of functions with values in V, each of which is defined on a subset of E belonging to the filter base $\mathfrak{F}$. The relations which we shall define between such functions have a *local* character relative to the filter with base $\mathfrak{F}$: let us clarify what we mean by this. If $\mathbf{f}$ and $\mathbf{g}$ are two functions from $\mathcal{H}(\mathfrak{F}, V)$, recall that the relation "there is a set $Z \in \mathfrak{F}$ such that $\mathbf{f}$ and $\mathbf{g}$ are defined and equal on Z" is an *equivalence re-*

lation R_∞ on $\mathcal{H}(\mathfrak{F}, V)$ (*Gen. Top.*, I, p. 66). This being so, we shall say that a relation S involving a function $\mathbf{f}$ of $\mathcal{H}(\mathfrak{F}, V)$ is of *local* character (along $\mathfrak{F}$) relative to $\mathbf{f}$, if it is *compatible* (in $\mathbf{f}$) with the equivalence relation R_∞ (*Set Theory*, II, p. 117); we know that if $\tilde{\mathbf{f}}$ is the *germ* of $\mathbf{f}$ along $\mathfrak{F}$, the equivalence class of $\mathbf{f}$ modulo R_∞ (an element of the quotient set $\mathcal{H}_\infty(\mathfrak{F}, V) = \mathcal{H}(\mathfrak{F}, V)/\mathrm{R}_\infty$), then one can derive from S, by passage to the quotient, a relation between $\tilde{\mathbf{f}}$ and the other arguments of S, and that, conversely, every relation of this nature defines a relation of local character relative to $\mathbf{f}$.

Example. If f and g are two functions in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$, the relation "there is an $\mathrm{X} \in \mathfrak{F}$ such that f and g are defined on X and $f(t) \leqslant g(t)$ for every $t \in \mathrm{X}$" is of local character relative to f and g. We denote by $\tilde{f} \leqslant \tilde{g}$ the relation obtained by passing to the quotient (for f and g); we remark that if $\tilde{f} \leqslant \tilde{g}$ then there exist a function $f_1 \in \tilde{f}$ and a function $g_1 \in \tilde{g}$, defined on *all of* E, such that $f_1(t) \leqslant g_1(t)$ for all $t \in \mathrm{E}$.

Remarks. 1) Let V_i $(1 \leqslant i \leqslant n)$ be n normed vector spaces over K, and φ a function defined on $\mathrm{V}_1 \times \mathrm{V}_2 \times \cdots \times \mathrm{V}_n$, with values in V; by passing to the quotient along R_∞ the function φ defines a map from

$$\mathcal{H}_\infty(\mathfrak{F}, V_1) \times \cdots \times \mathcal{H}_\infty(\mathfrak{F}, V_n)$$

into $\mathcal{H}_\infty(\mathfrak{F}, V)$, which one most often denotes by $\varphi(\tilde{\mathbf{f}}_1, \ldots, \tilde{\mathbf{f}}_n)$ (*Gen. Top.*, I, p. 67). For example, taking φ to be the maps $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}+\mathbf{y}$ and $\mathbf{x} \mapsto \mathbf{x}\lambda$ $(\lambda \in \mathrm{K})$, one thus defines, for any two germs $\tilde{\mathbf{f}}$, $\tilde{\mathbf{g}}$ in $\mathcal{H}_\infty(\mathfrak{F}, V)$, the elements $\tilde{\mathbf{f}}+\tilde{\mathbf{g}}$ and $\tilde{\mathbf{f}}\lambda$, and one verifies immediately that the rules of composition $(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) \mapsto \tilde{\mathbf{f}}+\tilde{\mathbf{g}}$ and $(\lambda, \tilde{\mathbf{f}}) \mapsto \tilde{\mathbf{f}}\lambda$ define on $\mathcal{H}_\infty(\mathfrak{F}, V)$ a *vector space* structure over the field K; in this space $\tilde{0}$ is the class formed by functions equal to 0 on a set in $\mathfrak{F}$, and $-\tilde{\mathbf{f}}$ is the class formed by functions equal to $-\mathbf{f}$ on a set in $\mathfrak{F}$. In the same way, if V is an *algebra* over K, one defines on $\mathcal{H}_\infty(\mathfrak{F}, V)$ a second internal rule of composition $(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) \mapsto \tilde{\mathbf{f}}\tilde{\mathbf{g}}$ by taking $\varphi(\mathbf{x}, \mathbf{y}) = \mathbf{x}\mathbf{y}$; together with the two preceding rules this defines an *algebra* structure over K on $\mathcal{H}_\infty(\mathfrak{F}, V)$; if V has a unit element $\mathbf{e}$ then $\mathcal{H}_\infty(\mathfrak{F}, V)$ has as a unit element the class $\tilde{\mathbf{e}}$ formed by the functions equal to $\mathbf{e}$ on some set in $\mathfrak{F}$; for $\tilde{\mathbf{f}}$ to be *invertible* in $\mathcal{H}_\infty(\mathfrak{F}, V)$ it is necessary and sufficient that for some $\mathbf{f} \in \tilde{\mathbf{f}}$ there exists a $\mathrm{Z} \in \mathfrak{F}$ such that $\mathbf{f}(t)$ is invertible in V for every $t \in \mathrm{Z}$ (in which case this condition is satisfied by every function in the class $\tilde{\mathbf{f}}$).

2) With the same notation, let ψ be a map of a subset of $\prod_{i=1}^{n} \mathrm{V}_i$ into V; we denote by $\psi(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n)$ the function equal to $\psi(\mathbf{f}_1(t), \mathbf{f}_2(t), \ldots, \mathbf{f}_n(t))$ at every point $t \in \mathrm{E}$ where the $\mathbf{f}_i(t)$ are defined and where the point $(\mathbf{f}_i(t))$ belongs to the set where ψ is defined [1]. For example, $\mathbf{f}+\mathbf{g}$ is the function equal to $\mathbf{f}(t)+\mathbf{g}(t)$ at every point $t \in \mathrm{E}$ where $\mathbf{f}$ and $\mathbf{g}$ are both defined. Observe that the map $(\mathbf{f}, \mathbf{g}) \mapsto \mathbf{f}+\mathbf{g}$ *is not a group law* on $\mathcal{H}(\mathfrak{F}, V)$, since if $\mathbf{f}$ is not defined on all of E there is no function $\mathbf{g} \in \mathcal{H}(\mathfrak{F}, V)$ such that $\mathbf{f}+\mathbf{g} = 0$.

DEFINITION 1. *Given two real functions f, g belonging to $\mathcal{H}(\mathfrak{F}, \mathbf{R})$, which are $\geqslant 0$ on a set in $\mathfrak{F}$, we say that f is dominated by g, or that g dominates f (along $\mathfrak{F}$), and write $f \preccurlyeq g$ or $g \succcurlyeq f$, if there are a set $\mathrm{X} \in \mathfrak{F}$ and a number $k > 0$ such that $f(t) \leqslant k\,g(t)$ for every $t \in \mathrm{X}$* (in other words, if there exists a $k > 0$ such that $\tilde{f} \leqslant k\,\tilde{g}$)

[1] In particular, in all that follows, given a function $\mathbf{f}$ in $\mathcal{H}(\mathfrak{F}, V)$ we shall denote by $\|\mathbf{f}\|$ the function $t \mapsto \|\mathbf{f}(t)\|$ which belongs to $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ and is defined on the same set as $\mathbf{f}$: we expressly draw attention to the fact that in this chapter $\|\mathbf{f}\|$ is a *function* and not a *number*.

Given two normed vector spaces $\mathrm{V}_1, \mathrm{V}_2$*, and two functions* $\mathbf{f}_1, \mathbf{f}_2$ *belonging to* $\mathcal{H}(\mathfrak{F}, \mathrm{V}_1)$ *and* $\mathcal{H}(\mathfrak{F}, \mathrm{V}_2)$ *respectively, one says that* $\mathbf{f}_1$ *is dominated by* $\mathbf{f}_2$ *(along* $\mathfrak{F}$*), and writes* $\mathbf{f}_1 \preccurlyeq \mathbf{f}_2$ *or* $\mathbf{f}_2 \succcurlyeq \mathbf{f}_1$*, if* $\|\mathbf{f}_1\| \preccurlyeq \|\mathbf{f}_2\|$.

The relation $\mathbf{f}_1 \preccurlyeq \mathbf{f}_2$ is clearly of local character in $\mathbf{f}_1$ and $\mathbf{f}_2$; it is thus equivalent to the relation $\tilde{\mathbf{f}}_1 \preccurlyeq \tilde{\mathbf{f}}_2$ derived by passing to the quotient. When f and g are real functions one must be careful not to confuse the relations $\tilde{f} \preccurlyeq \tilde{g}$ and $\tilde{f} \leqslant \tilde{g}$.

Note that for every scalar $\lambda \neq 0$ the relation $\mathbf{f}_1 \preccurlyeq \mathbf{f}_2\lambda$ is *equivalent to* $\mathbf{f}_1 \preccurlyeq \mathbf{f}_2$. If $\mathbf{f}_1 \preccurlyeq \mathbf{f}_2$ there exists a set $\mathrm{X} \in \mathfrak{F}$ such that $\mathbf{f}_1(x) = 0$ for every point $x \in \mathrm{X}$ where $\mathbf{f}_2(x) = 0$.

Examples. 1) The relation $\mathbf{f} \preccurlyeq 1$ means that $\mathbf{f}$ is *bounded* on a set of $\mathfrak{F}$.

2) For every function $\mathbf{f}$ of $\mathcal{H}(\mathfrak{F}, V)$, and every scalar $\lambda \neq 0$, one has $\mathbf{f} \preccurlyeq \mathbf{f}\lambda$.
3) When x tends to $+\infty$ one has $\sin^2 x \preccurlyeq \sin x$.
4) When (x, y) tends to $(0, 0)$ in $\mathbf{R}^2$ one has

$$xy \preccurlyeq x^2 + y^2.$$

The following propositions are immediate consequences of def. 1:

PROPOSITION 1. *If* f, g, h *are three functions in* $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ *then the relations* $f \preccurlyeq g$ *and* $g \preccurlyeq h$ *imply* $f \preccurlyeq h$.

PROPOSITION 2. *Let* $\mathbf{f}_1, \mathbf{f}_2$ *be two functions in* $\mathcal{H}(\mathfrak{F}, V)$ *and* g *a function in* $\mathcal{H}(\mathfrak{F}, \mathbf{R})$*. The relations* $\mathbf{f}_1 \preccurlyeq g$ *and* $\mathbf{f}_2 \preccurlyeq g$ *imply* $\mathbf{f}_1 + \mathbf{f}_2 \preccurlyeq g$.

Further:

PROPOSITION 3. *Let* $\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}$ *be three normed spaces over the same valued field, and* $(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}.\mathbf{y}]$ *a bilinear map from* $\mathrm{V}_1 \times \mathrm{V}_2$ *into* V*. If* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *are functions in* $\mathcal{H}(\mathfrak{F}, \mathrm{V}_1)$ *and* $\mathcal{H}(\mathfrak{F}, \mathrm{V}_2)$ *respectively, and* g_1, g_2 *are two functions in* $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ *such that* $\mathbf{f}_1 \preccurlyeq g_1$ *and* $\mathbf{f}_2 \preccurlyeq g_2$ *then* $[\mathbf{f}_1.\mathbf{f}_2] \preccurlyeq g_1 g_2$.

Indeed (*Gen. Top.*, IX, p. 173, th. 1) there exists a number $a > 0$ such that $\|[\mathbf{f}_1.\mathbf{f}_2]\| \leqslant a\,\|\mathbf{f}_1\|\;\|\mathbf{f}_2\|$.

COROLLARY. *If* V *is a normed algebra, if* $\mathbf{f}_1, \mathbf{f}_2$ *are two functions in* $\mathcal{H}(\mathfrak{F}, V)$*, and* g_1, g_2 *are two functions in* $\mathcal{H}(\mathfrak{F}, \mathbf{R})$*, then the relations* $\mathbf{f}_1 \preccurlyeq g_1$*,* $\mathbf{f}_2 \preccurlyeq g_2$ *imply* $\mathbf{f}_1\mathbf{f}_2 \preccurlyeq g_1 g_2$.

The relation $\mathbf{f} \preccurlyeq \mathbf{g}$ between functions in $\mathcal{H}(\mathfrak{F}, V)$ is *transitive* by prop. 1; since it is *reflexive* the relation "$\mathbf{f} \preccurlyeq \mathbf{g}$ and $\mathbf{g} \preccurlyeq \mathbf{f}$" is an *equivalence relation* on $\mathcal{H}(\mathfrak{F}, V)$ (*Set Theory*, II, p. 113).

DEFINITION 2. *Given two functions* $\mathbf{f}$, $\mathbf{g}$ *of* $\mathcal{H}(\mathfrak{F}, V)$ *we say that* $\mathbf{f}$ *and* $\mathbf{g}$ *are similar* (*along* $\mathfrak{F}$), *and write* $\mathbf{f} \asymp \mathbf{g}$, *if* $\mathbf{f} \preccurlyeq \mathbf{g}$ *and* $\mathbf{g} \preccurlyeq \mathbf{f}$.

For every scalar $\lambda \neq 0$ the relation $\mathbf{f} \asymp \mathbf{g}$ is equivalent to $\mathbf{f} \asymp \mathbf{g}\lambda$. It implies the existence of a set $\mathrm{X} \in \mathfrak{F}$ such that that the subset of X of points where $\mathbf{f}(x) = 0$ is identical with the subset of X of points where $\mathbf{g}(x) = 0$.

Examples. 1) For a real function $f \in \mathcal{H}(\mathfrak{F}, \mathbf{R})$ the relation $f \asymp 1$ means that there are two numbers $a > 0$, $b > 0$ such that $a \leqslant |f(x)| \leqslant b$ on a set in $\mathfrak{F}$, or that the function $\log|f|$ is bounded on a set in $\mathfrak{F}$: one then says that f is *logarithmically bounded* on a set in $\mathfrak{F}$.

2) Let V be a normed space over a non-discrete valued field K, and let $\mathbf{f}(x) = \mathbf{a}_0 x^n + \mathbf{a}_1 x^{n-1} + \cdots + \mathbf{a}_n$ be a polynomial in the variable $x \in \mathrm{K}$, with coefficients in V, such that $\mathbf{a}_0 \neq 0$. For every vector $\mathbf{b} \neq 0$ one has $\mathbf{f}(x) \asymp \mathbf{b}x^n$ as $|x|$ tends to $+\infty$.

3) We have seen that $\sin^2 x \preccurlyeq \sin x$ as x tends to $+\infty$, but *we do not have* $\sin^2 x \asymp \sin x$, even though these functions vanish at the same points.

4) One has $x^2+xy+y^2 \asymp x^2+y^2$ when (x, y) tends to $(0, 0)$ in $\mathbf{R}^2$, but *not* $xy \asymp x^2+y^2$.

It follows immediately from prop. 3 of V, p. 213, that if f_1, f_2, g_1, g_2 are functions in $\mathcal{H}(\mathfrak{F}, \mathrm{K})$ (K being any valued field) the relations $f_1 \asymp g_1$ and $f_2 \asymp g_2$ imply that $f_1 f_2 \asymp g_1 g_2$.

We remark that in contrast the relations $f_1 \asymp g_1$ and $f_2 \asymp g_2$ *do not imply that* $f_1 + f_2 \asymp g_1 + g_2$, as is shown by the example $f_1(x) = g_1(x) = x^2$, $f_2(x) = -(x^2 + x)$, $g_2(x) = -(x^2 - 1)$, as the real variable x tends to $+\infty$.

The comparison relations $\mathbf{f} \preccurlyeq \mathbf{g}$, $\mathbf{f} \asymp \mathbf{g}$ are said to be *weak*. We say that two functions $\mathbf{f}, \mathbf{g}$ from $\mathcal{H}(\mathfrak{F}, V)$ are *weakly comparable* if they satisfy one (at least) of the relations $\mathbf{f} \preccurlyeq \mathbf{g}$, $\mathbf{g} \preccurlyeq \mathbf{f}$.

Remarks. 1) Two functions in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ need not be weakly comparable, as is shown by the example of the functions 1 and $x \sin x$ as x tends to $+\infty$.

2) Denote by R_0 the relation $\mathbf{f} \asymp \mathbf{g}$ on $\mathcal{H}(\mathfrak{F}, V)$, and by $\mathcal{H}_0(\mathfrak{F}, V)$ the quotient set $\mathcal{H}(\mathfrak{F}, \mathrm{V})/\mathrm{R}_0$; note that the relation R_∞ *implies* R_0. Passing to the quotient the relation $\mathbf{f} \preccurlyeq \mathbf{g}$ gives, by prop. 1 of V, p. 213, an *order relation* on $\mathcal{H}_0(\mathfrak{F}, V)$ (*Set Theory*, III, p. 134); the preceding example shows that $\mathcal{H}_0(\mathfrak{F}, V)$ *is not totally ordered* by this relation.

2. COMPARISON RELATIONS: II. STRONG RELATIONS

DEFINITION 3. *Given two real functions* f, g *belonging to* $\mathcal{H}(\mathfrak{F}, \mathbf{R})$, *which are* $\geqslant 0$ *on a set in* $\mathfrak{F}$, *one says that* f *is negligible relative to* g, *or that* g *is preponderant over* f (*along* $\mathfrak{F}$), *and one writes* $f \prec\!\!\prec g$ *or* $g \succ\!\!\succ f$ *if, for every* $\varepsilon > 0$, *there exists a set* $\mathrm{X} \in \mathfrak{F}$ *such that* $f(t) \leqslant \varepsilon g(t)$ *for every* $t \in \mathrm{X}$.

Given two normed spaces $\mathrm{V}_1, \mathrm{V}_2$ *and two functions* $\mathbf{f}_1, \mathbf{f}_2$ *belonging to* $\mathcal{H}(\mathfrak{F}, \mathrm{V}_1)$ *and* $\mathcal{H}(\mathfrak{F}, \mathrm{V}_2)$ *respectively, one says that* $\mathbf{f}_1$ *is negligible relative to* $\mathbf{f}_2$ (*along* $\mathfrak{F}$), *and one writes* $\mathbf{f}_1 \prec\!\!\prec \mathbf{f}_2$ *or* $\mathbf{f}_2 \succ\!\!\succ \mathbf{f}_1$, *if* $\|\mathbf{f}_1\| \prec\!\!\prec \|\mathbf{f}_2\|$.

For every scalar $\lambda \neq 0$ the relation $\mathbf{f}_1 \prec\!\!\prec \mathbf{f}_2\lambda$ is *equivalent* to $\mathbf{f}_1 \prec\!\!\prec \mathbf{f}_2$. The relation $\mathbf{f}_1 \prec\!\!\prec \mathbf{f}_2$ implies $\mathbf{f}_1 \preccurlyeq \mathbf{f}_2$ but is not equivalent to it.

Note that the relation $\mathbf{f}_1 \preccurlyeq \mathbf{f}_2$ by no means implies the relation "$\mathbf{f}_1 \prec\!\!\prec \mathbf{f}_2$ or $\mathbf{f}_1 \asymp \mathbf{f}_2$": one has $\sin x \preccurlyeq 1$ as x tends to $+\infty$, but neither of the relations $\sin x \prec\!\!\prec 1$ or $\sin x \asymp 1$ is true.

Examples. 1) The relation $\mathbf{f} \prec\!\!\prec 1$ means that $\mathbf{f}$ *tends to* 0 *along* $\mathfrak{F}$.

2) When α and β are two real numbers such that $\alpha < \beta$ one has $x^\alpha \prec\!\!\prec x^\beta$ as x tends to $+\infty$. Similarly, when m and n are two rational integers such that $m < n$ one has $z^m \prec\!\!\prec z^n$ as the complex number z tends to ∞.

3) As x tends to $+\infty$ one has $x^n \prec\!\!\prec e^x$ for every integer n (III, p. 105).

4) In $\mathbf{R}^2$ one has, as (x, y) tends to $(0, 0)$,

$$x^2 + y^2 \prec\!\!\prec |x| + |y|\,.$$

The following propositions can be deduced immediately from def. 3:

PROPOSITION 4. *If f, g, h are three functions in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ then the relations $f \preccurlyeq g$ and $g \prec\!\!\prec h$* (resp. *$f \prec\!\!\prec g$ and $g \preccurlyeq h$*) *imply $f \prec\!\!\prec h$.*

PROPOSITION 5. *Let $\mathbf{f}_1, \mathbf{f}_2$ be two functions in $\mathcal{H}(\mathfrak{F}, V)$ and g a function in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$. Then the relations $\mathbf{f}_1 \prec\!\!\prec g$ and $\mathbf{f}_2 \prec\!\!\prec g$ imply $\mathbf{f}_1 + \mathbf{f}_2 \prec\!\!\prec g$.*

Moreover, the same argument as in prop. 3 of V, p. 213 shows that:

PROPOSITION 6. *With the notation of prop.* 3, *the relations $\mathbf{f}_1 \prec\!\!\prec g_1$ and $\mathbf{f}_2 \preccurlyeq g_2$* (resp. *$\mathbf{f}_1 \preccurlyeq g_1$ and $\mathbf{f}_2 \prec\!\!\prec g_2$*) *imply $[\mathbf{f}_1 . \mathbf{f}_2] \prec\!\!\prec g_1 g_2$.*

Prop. 4 shows that the relation $\mathbf{f} \prec\!\!\prec \mathbf{g}$ between functions in $\mathcal{H}(\mathfrak{F}, V)$ is *transitive*; but *it is not reflexive*: to be precise, the relation $\mathbf{f} \prec\!\!\prec \mathbf{f}$ implies that $\mathbf{f}$ *vanishes on a set in* $\mathfrak{F}$ (in other words, $\mathbf{f}$ is equivalent to 0 modulo R_∞); indeed, for an ε such that $0 < \varepsilon < 1$ there exists an $\mathrm{X} \in \mathfrak{F}$ such that $\|\mathbf{f}(x)\| \leqslant \varepsilon \|\mathbf{f}(x)\|$ for every $x \in \mathrm{X}$, which is possible only if $\mathbf{f}(x) = 0$ for every $x \in \mathrm{X}$. It follows that the relation "$\mathbf{f} \prec\!\!\prec \mathbf{g}$ and $\mathbf{g} \prec\!\!\prec \mathbf{f}$" is transitive and symmetric, but not reflexive: so it is not an equivalence relation (it implies that there is a set $\mathrm{X} \in \mathfrak{F}$ such that $\mathbf{f}(x) = \mathbf{g}(x) = 0$ for every $x \in \mathrm{X}$).

PROPOSITION 7. *If $\mathbf{f}$ and $\mathbf{g}$ are two functions in $\mathcal{H}(\mathfrak{F}, V)$ then the relation $\mathbf{f} - \mathbf{g} \prec\!\!\prec \mathbf{f}$ is equivalent to $\mathbf{f} - \mathbf{g} \prec\!\!\prec \mathbf{g}$.*

Indeed, $\mathbf{f} - \mathbf{g} \prec\!\!\prec \mathbf{f}$ implies that for every $\varepsilon > 0$ there exists an $\mathrm{X} \in \mathfrak{F}$ such that $\|\mathbf{f}(x) - \mathbf{g}(x)\| \leqslant \varepsilon \|\mathbf{f}(x)\|$ for every $x \in \mathrm{X}$. But then $(1 - \varepsilon) \|\mathbf{f}(x)\| \leqslant \|\mathbf{g}(x)\|$, and consequently $\mathbf{f} \preccurlyeq \mathbf{g}$, whence (V, p. 215, prop. 4) $\mathbf{f} - \mathbf{g} \prec\!\!\prec \mathbf{g}$.

COROLLARY. *The relation $\mathbf{f} - \mathbf{g} \prec\!\!\prec \mathbf{f}$ is an equivalence relation on $\mathcal{H}(\mathfrak{F}, V)$.*

Indeed, if $\mathbf{f}-\mathbf{g} \prec\!\!\prec \mathbf{f}$ and $\mathbf{g}-\mathbf{h} \prec\!\!\prec \mathbf{g}$ then $\mathbf{f}-\mathbf{g} \prec\!\!\prec \mathbf{g}$, whence (V, p. 215, prop. 5) $\mathbf{f}-\mathbf{h} \prec\!\!\prec \mathbf{g}$, and consequently, since $\mathbf{g} \preccurlyeq \mathbf{f}$, we have $\mathbf{f}-\mathbf{h} \prec\!\!\prec \mathbf{f}$, which shows that this relation is transitive; it is symmetric by prop. 7, and is clearly reflexive, whence the corollary.

DEFINITION 4. *Given two functions* $\mathbf{f}$, $\mathbf{g}$ *of* $\mathcal{H}(\mathfrak{F}, V)$ *one says that* $\mathbf{f}$ *and* $\mathbf{g}$ *are equivalent* (*along* $\mathfrak{F}$)*, and writes* $\mathbf{f} \sim \mathbf{g}$*, if* $\mathbf{f}-\mathbf{g} \prec\!\!\prec \mathbf{f}$.

The relation $\mathbf{f} \sim \mathbf{g}$ implies $\mathbf{f} \asymp \mathbf{g}$ but is not equivalent to it.

Examples. 1) If $\mathbf{a}$ is a constant function, $\neq 0$ on E, the relation $\mathbf{f} \sim \mathbf{a}$ means that $\mathbf{f}$ *tends to* $\mathbf{a}$ *along* $\mathfrak{F}$.

2) Let V be a normed space over a non-discrete valued field K, and let $\mathbf{f}(x) = \mathbf{a}_0 x^n + \mathbf{a}_1 x^{n-1} + \cdots + \mathbf{a}_n$ be a polynomial in the variable $x \in$ K, with coefficients in V, such that $\mathbf{a}_0 \neq 0$. Then $\mathbf{f}(x) \sim \mathbf{a}_0 x^n$ as $|x|$ tends to $+\infty$.

3) When the real number x tends to $+\infty$ one has $\left(1+\dfrac{1}{x}\right)\sin x \sim \sin x$.

4) As the complex variable z tends to 0 one has $e^z - 1 \sim z$. More generally, if V is a normed space over a valued field K, and $\mathbf{f}$ is a function defined on a neighbourhood of $x_0 \in$ K, with values in V, and admitting a derivative $\mathbf{f}'(x_0) \neq 0$ at the point x_0, then, as x approaches x_0, we have $\mathbf{f}(x) - \mathbf{f}(x_0) \sim \mathbf{f}'(x_0)(x - x_0)$ (I, p. 3, def. 1).

5) As (x, y) approaches $(0, 0)$ in $\mathbf{R}^2$ one has

$$\sqrt{\sin^2 x + \sin^2 y} \sim \sqrt{x^2 + y^2}.$$

6) Let $f(x, y)$ be a polynomial with real coefficients in two real variables x, y, not having a constant term. If, as x approaches 0 while remaining > 0, there is a function $\varphi(x)$, continuous on an interval $[0, a]$, and such that $\varphi(0) = 0$ and $f(x, \varphi(x)) = 0$ for $0 \leqslant x \leqslant a$, one can show that there are a rational number r and a real number $\lambda \neq 0$ such that $\varphi(x) \sim \lambda x^r$ (V, p. 259, exerc. 3).

7) For every $x > 0$ let $\pi(x)$ denote the number of prime numbers which are $\leqslant x$; it has been proved that $\pi(x) \sim x/\log x$ [2] as x tends to $+\infty$.

Remark. Note that the relation $\mathbf{f} \sim \mathbf{g}$ by no means implies that the difference $\mathbf{f}-\mathbf{g}$ tends to 0 along $\mathfrak{F}$; this difference can even be *unbounded,* as is shown by the example $x^2 + x \sim x^2$ as x tends to $+\infty$.

PROPOSITION 8. *Let* K *be a valued field and* f_1, f_2, g_1, g_2 *four functions in* $\mathcal{H}(\mathfrak{F}, \mathrm{K})$*; then the relations* $f_1 \sim g_1$ *and* $f_2 \sim g_2$ *imply* $f_1 f_2 \sim g_1 g_2$.

Indeed, we have $f_1 f_2 - g_1 g_2 = f_1(f_2 - g_2) + (f_1 - g_1) g_2$; since $f_1 \preccurlyeq g_1$, $f_1 - g_1 \prec\!\!\prec g_1$ and $f_2 - g_2 \prec\!\!\prec g_2$, we have $f_1 f_2 - g_1 g_2 \prec\!\!\prec g_1 g_2$ (V, p. 215, prop. 5 and 6).

In contrast, we have given an example in V, p. 214 where one has $f_1 = g_1$, $f_2 \sim g_2$ and yet the relation $f_1 + f_2 \asymp g_1 + g_2$ does not hold (so neither *a fortiori* does

$$f_1 + f_2 \sim g_1 + g_2).$$

[2] See, for example, A. E. INGHAM, *The distribution of prime numbers* (Cambridge Tracts, n° 30), Cambridge University Press, 1932.

The comparison relations $\mathbf{f} \prec\!\!\prec \mathbf{g}$, $\mathbf{f} \sim \mathbf{g}$ are called *strong* relations. Two functions $\mathbf{f}, \mathbf{g}$ from $\mathcal{H}(\mathfrak{F}, V)$ are called *comparable* (or *strongly comparable* when one wants to avoid any possible confusion) if they satisfy one of the three relations: $\mathbf{f} \prec\!\!\prec \mathbf{g}$, $\mathbf{f} \succ\!\!\succ \mathbf{g}$, or "there exists a $\lambda \neq 0$ such that $\mathbf{f} \sim \mathbf{g}\lambda$".

Remarks. 1) Two functions can be weakly comparable though not strongly comparable, for example the functions 1 and $\sin x$ as x tends to $+\infty$.

2) In the definitions of the comparison relations $\mathbf{f}_1 \preccurlyeq \mathbf{f}_2$ and $\mathbf{f}_1 \prec\!\!\prec \mathbf{f}_2$ the *norms* on the spaces $\mathrm{V}_1, \mathrm{V}_2$ where $\mathbf{f}_1$ and $\mathbf{f}_2$ respectively take their values, are involved only apparently; in reality only the *topologies* on V_1 and V_2 are involved, for the relations $\mathbf{f}_1 \preccurlyeq \mathbf{f}_2$ and $\mathbf{f}_1 \prec\!\!\prec \mathbf{f}_2$ are replaced by equivalent relations when one replaces the norm on V_1 or V_2 by an *equivalent* norm (*Gen. Top.*, IX, p. 170, def. 7).

3. CHANGE OF VARIABLE

Let φ be a map from the set E' into E such that $\overset{-1}{\varphi}(\mathfrak{F})$ is a filter base on E'. It is clear that if $\mathbf{f}_1, \mathbf{f}_2$ are functions in $\mathcal{H}(\mathfrak{F}, V_1)$ and $\mathcal{H}(\mathfrak{F}, V_2)$ respectively, then $\mathbf{f}_1 \circ \varphi$, $\mathbf{f}_2 \circ \varphi$ belong to $\mathcal{H}(\overset{-1}{\varphi}(\mathfrak{F}), V_1)$ and $\mathcal{H}(\overset{-1}{\varphi}(\mathfrak{F}), V_2)$ respectively, and that the relation $\mathbf{f}_1 \preccurlyeq \mathbf{f}_2$ (resp. $\mathbf{f}_1 \prec\!\!\prec \mathbf{f}_2$) is equivalent to $\mathbf{f}_1 \circ \varphi \preccurlyeq \mathbf{f}_2 \circ \varphi$ (resp. $\mathbf{f}_1 \circ \varphi \prec\!\!\prec \mathbf{f}_2 \circ \varphi$).

4. COMPARISON RELATIONS BETWEEN STRICTLY POSITIVE FUNCTIONS

Let g be a function in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ which is *strictly positive on a set in* $\mathfrak{F}$. Comparison relations featuring g can be formulated in another way: the relation $\mathbf{f} \preccurlyeq g$ is equivalent to saying that $\|\mathbf{f}\|/g$ (which is defined on a set in $\mathfrak{F}$) is *bounded* on a set in $\mathfrak{F}$; the relation $\mathbf{f} \prec\!\!\prec g$ is equivalent to saying that $\|\mathbf{f}\|/g$ *tends to* 0 along $\mathfrak{F}$. If f is a function in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ the relation $f \asymp g$ means that f/g is *logarithmically bounded* on a set in $\mathfrak{F}$, and the relation $f \sim g$ that f/g *tends to* 1 along $\mathfrak{F}$. If f is a function in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ and *positive* on a set in $\mathfrak{F}$, to say that f and g are *comparable* implies that f/g *tends to a limit* (*finite or equal to* $+\infty$) along $\mathfrak{F}$.

PROPOSITION 9. *Let f and g be two functions in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$, strictly positive on a set in $\mathfrak{F}$. For f and g to be comparable it is necessary and sufficient that for every number $t \geqslant 0$, with at most one exception, the function $f - tg$ has constant sign*[3] *on a set in $\mathfrak{F}$.*

The condition is necessary. Indeed, if $f \prec\!\!\prec g$ one has $f - tg \sim -tg$ except for $t = 0$, so $f - tg$ is strictly negative on a set in $\mathfrak{F}$, except for $t = 0$; if $f \succ\!\!\succ g$ then $f - tg$ is strictly positive on a set in $\mathfrak{F}$ for all t; finally, if $f \sim kg$ (k constant > 0),

[3] Recall that we have defined the *sign* $\operatorname{sgn} x$ of a real number x as equal to $+1$ if $x > 0$, to -1 if $x < 0$, and to 0 if $x = 0$ (*Gen. Top.*, IV, p. 341). To say that a real function *has constant sign* on a set thus means that either it is > 0 at every point of this set, or is < 0 at every point of this set, or is identically zero on this set.

then $f - tg \sim (k-t)g$ except for $t = k$, so, except perhaps for $t = k$, $f - tg$ has the sign of $k - t$ on a set in $\mathfrak{F}$.

The condition is sufficient. Indeed, suppose that the ratio f/g has two distinct cluster points $\alpha < \beta$ along $\mathfrak{F}$. For *every* number t such that $\alpha < t < \beta$ there then exist in *every* set $X \in \mathfrak{F}$ two points x_1, x_2 such that $f(x_1)/g(x_1) < t$ and $f(x_2)/g(x_2) > t$; thus $f(x) - tg(x)$ does not have constant sign on X; we have arrived at a conclusion incompatible with the hypothesis. It follows that f/g can have *only one* cluster value (finite or infinite) along the filter with base $\mathfrak{F}$, and consequently, (*Gen. Top.*, I, p. 85, corollary) has this value as its *limit* along $\mathfrak{F}$.

PROPOSITION 10. *Let f and g be two functions in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ which are strictly positive on a set in $\mathfrak{F}$; for every α real and $\neq 0$ the relation $f \asymp g$* (resp. $f \sim g$) *is equivalent to $f^\alpha \asymp g^\alpha$* (resp. $f^\alpha \sim g^\alpha$)*; if $\alpha > 0$ the relation $f \preccurlyeq g$* (resp. $f \prec\!\!\prec g$) *is equivalent to $f^\alpha \preccurlyeq g^\alpha$* (resp. $f^\alpha \prec\!\!\prec g^\alpha$)*; if $\alpha < 0$ it is equivalent to $f^\alpha \succcurlyeq g^\alpha$* (resp. $f^\alpha \succ\!\!\succ g^\alpha$).

The proofs are immediate.

One notes that in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ the set Γ of functions strictly positive on a set in $\mathfrak{F}$ is such that Γ/R_∞ is a *multiplicative group* Γ_∞ in $\mathcal{H}_\infty(\mathfrak{F}, \mathbf{R})$; Γ/R_0 is identical to the quotient group of Γ_∞ by the subgroup of classes (mod. R_∞) of logarithmically bounded functions in Γ; on Γ/R_0 the order relation deduced from the relation $f \preccurlyeq g$ by passage to the quotient is *compatible* with the group structure of Γ/R_0 and thus makes it an ordered group.

PROPOSITION 11. *Let g be a function in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ such that* $\lim_{\mathfrak{F}} g = +\infty$*; the relation $f \prec\!\!\prec g$ implies $e^f \prec\!\!\prec e^g$; the relation $f \sim g$ implies* $\log f \sim \log g$.

Indeed, if $f \prec\!\!\prec g$ then $f - g = g\left(\dfrac{f}{g} - 1\right)$ tends to $-\infty$ along $\mathfrak{F}$. Similarly, if $f \sim g$ one has $\log f = \log g + \log \dfrac{f}{g}$, so $\log f - \log g$ tends to 0, and the same for $\dfrac{\log f}{\log g} - 1 = \dfrac{\log f - \log g}{\log g}$.

On the other hand, note that the relation $f \sim g$ *does not imply* $e^f \sim e^g$, nor even that $e^f \asymp e^g$, as is shown by the example where $f(x) = x^2$, $g(x) = x^2 + x$ as x tends to $+\infty$; similarly, the relation $f \prec\!\!\prec g$ *does not imply* $\log f \prec\!\!\prec \log g$, as is shown by the example $f(x) = x$, $g(x) = x^2$ as x tends to $+\infty$.

DEFINITION 5. *Let g be a function in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$, strictly positive on a set in $\mathfrak{F}$, and such that* $\lim_{\mathfrak{F}} g = 0$ *or* $\lim_{\mathfrak{F}} g = +\infty$. *One says that a function $f \in \mathcal{H}(\mathfrak{F}, \mathbf{R})$ is of order ρ (finite or infinite) relative to g if* $\lim_{\mathfrak{F}} \log(|f|)/\log g = \rho$.

Note that if f is of order ρ relative to g then f is of order $-\rho$ relative to $1/g$; one therefore need treat only the case where $g(x)$ tends to $+\infty$ along $\mathfrak{F}$.

PROPOSITION 12. *Let g be a function in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ such that* $\lim_{\mathfrak{F}} g = +\infty$*; let f be a function in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$.*

a) *For f to be of order $+\infty$ relative to g it is necessary and sufficient that $f \succ\!\succ g^{\alpha}$ for every $\alpha \geqslant 0$.*

b) *For f to be of order $-\infty$ relative to g it is necessary and sufficient that $f \prec\!\prec g^{-\alpha}$ for every $\alpha > 0$.*

c) *For f to be of finite order equal to ρ relative to g it is necessary and sufficient that, for every $\varepsilon > 0$, one has $g^{\rho-\varepsilon} \prec\!\prec f \prec\!\prec g^{\rho+\varepsilon}$.*

Let us prove c) for example. If the order of f relative to g is ρ then for every $\varepsilon > 0$ there exists a set $\mathrm{M} \in \mathfrak{F}$ such that, for every $x \in \mathrm{M}$, we have

$$\left(\rho - \frac{\varepsilon}{2}\right) \log g(x) \leqslant \log |f(x)| \leqslant \left(\rho + \frac{\varepsilon}{2}\right) \log g(x),$$

or $(g(x))^{\rho-\frac{\varepsilon}{2}} \leqslant |f(x)| \leqslant (g(x))^{\rho+\frac{\varepsilon}{2}}$; since $\lim_{\mathfrak{F}} g = +\infty$ one thus has $g^{\rho-\varepsilon} \prec\!\prec f \prec\!\prec g^{\rho+\varepsilon}$ for every $\varepsilon > 0$; the converse is immediate. The proofs of a) and b) are similar.

Note that if f is of *finite* order ρ relative to g then $fg^{-\rho}$ is of order 0 relative to g, and conversely; if f_1 (resp. f_2) is of order ρ_1 (resp. ρ_2) relative to g, and if $\rho_1 + \rho_2$ is defined, then $f_1 f_2$ is of order $\rho_1 + \rho_2$ relative to g.

Remarks. 1) Observe that though f may be of finite order ρ relative to g nevertheless the ratio f/g^{ρ} need not tend to a limit; for example, every *logarithmically bounded* function is of order 0 relative to g, but need not have a limit along $\mathfrak{F}$.

2) A function defined on a set in $\mathfrak{F}$ need not have a determinate order (finite or not) relative to g, for the functions having a determinate order relative to g are comparable to all the powers of g with at most one exception. Now f need not have this property, as one sees from the example $g(x) = x$, $f(x) = 1 + x^2 \sin^2 x$ (as x tends to $+\infty$). In this example f is comparable to g^{α} for $\alpha < 0$ and $\alpha > 2$; if one takes $f(x) = e^x \sin^2 x + e^{-x} \cos^2 x$ then f is not comparable to any power (positive or negative) of g.

5. NOTATION

Given a real function $f \in \mathcal{H}(\mathfrak{F}, \mathbf{R})$ it is often convenient in a formula to write $O(f)$ for a function *dominated* by f, and $o(f)$ for a *negligible* function relative to f. When, in a proof, there feature *several* functions dominated by the same function f (resp. negligible relative to f) we denote them by $O_1(f)$, $O_2(f)$, etc. (resp. $o_1(f)$, $o_2(f)$, etc.).

Many authors write $O(f)$ (resp. $o(f)$) indiscriminately for *all* functions in a proof that are dominated by f (resp. negligible relative to f), an abuse of language which may risk confusion.

With this notation we can express props. 1, 2, 3 (V, p. 213) as follows: if $g = O_1(f)$ and $h = O(g)$ then $h = O_2(f)$; one can write

$$\sum_{i=1}^{n} \lambda_i O_i(f) = O_{n+1}(f) \qquad (\lambda_i \text{ scalars}) \tag{1}$$

$$O(f)\,O(g) = O(fg). \tag{2}$$

Similarly, prop. 4 (V, p. 215) shows that if $g = O_1(f)$ and $h = o(g)$ (resp. $g = o_1(f)$ and $h = O(g)$ then $h = o_2(f)$, and props. 5 and 6 (V, p. 5) can be expressed in the form

$$\sum_{i=1}^{n} \lambda_i o_i(f) = o_{n+1}(f) \qquad (\lambda_i \text{ scalars}) \tag{3}$$

$$o(f)\,O(g) = o(fg). \tag{4}$$

The relation $f \sim g$ is equivalent to $f = g + o(g)$. The notation $O(1)$ (resp. $o(1)$) denotes a function bounded on a set in $\mathfrak{F}$ (resp. a function tending to 0 along $\mathfrak{F}$).

§ 2. ASYMPTOTIC EXPANSIONS

1. SCALES OF COMPARISON

Let E be a set filtered by a filter with base $\mathfrak{F}$, and K a non-discrete valued field (most often $\mathrm{K} = \mathbf{R}$ or $\mathrm{K} = \mathbf{C}$). In the set of functions in $\mathcal{H}(\mathfrak{F}, \mathrm{K})$ not equivalent to 0 modulo R_∞ (that is, those for which in every set in $\mathfrak{F}$ there is at least one point where the function does not vanish), the relation "$f \prec\!\!\prec g$ or $f = g$" is an *order relation*.

DEFINITION 1. *One says that a subset $\mathcal{E}$ of $\mathcal{H}(\mathfrak{F}, \mathrm{K})$ formed of functions not equivalent to* 0 *modulo* R_∞ *is a comparison scale when $\mathcal{E}$ is totally ordered by the relation "$f \prec\!\!\prec g$ or $f = g$".*

In other words, if f and g are functions in $\mathcal{E}$ then one (and only one) of the relations $f \prec\!\!\prec g$, $g \prec\!\!\prec f$, $f = g$ always holds. It follows that on $\mathcal{E}$ *the relation* $f \asymp g$ (and *a fortiori* $|f| \sim a\,|g|$, where a is a number > 0) *implies* $f = g$.

Every subset of a comparison scale is clearly also a comparison scale.

Examples. 1) For x real and tending to $+\infty$ the set of functions x^α (α an arbitrary real number) is a comparison scale. The same is true for the functions $(x-a)^\alpha$ when $\mathfrak{F}$ is the set of open intervals with left-hand endpoint a.

2) For z complex tending to ∞ the set of functions z^n (n a rational integer) is a comparison scale; so are the functions $(z-a)^n$ when $\mathfrak{F}$ is the trace on the complement of the point $a \in \mathbf{C}$ of the filter of neighbourhoods of this point.

3) Let F be normed space; the family of functions $\|\mathbf{x} - \mathbf{a}\|^\alpha$ (α an arbitrary real number) is a comparison scale when $\mathfrak{F}$ is the trace on the complement of $\mathbf{a}$ of the filter of neighbourhoods of this point. Note that if p and q are two distinct norms on F, the union of the two comparison scales $\big(p(\mathbf{x}-\mathbf{a})\big)^\alpha$ and $\big(q(\mathbf{x}-\mathbf{a})\big)^\alpha$ is not in general a comparison scale.

4) For x real tending to $+\infty$ the family $\mathcal{E}$ of functions of the form $\exp(p(x))$, where p runs through the set of *polynomials without constant term* (with real coefficients), is a comparison scale: it suffices to remark that the quotient of two functions in $\mathcal{E}$ is again in $\mathcal{E}$, and that a function $\exp(p(x))$ must tend either to 0 or $+\infty$ if $p \neq 0$; indeed, $p(x) \sim \alpha x^n$ where $n > 0$ and $\alpha \neq 0$; if $\alpha > 0$ then $p(x) > \frac{1}{2}\alpha x^n$ for x sufficiently large; if $\alpha < 0$ then $p(x) < \frac{1}{2}\alpha x^n$ for x sufficiently large; in the first case $\exp(p(x))$ tends to $+\infty$, in the second to 0.

5) For x real tending to $+\infty$ the set $\mathcal{E}$ of functions of the form $x^\alpha(\log x)^\beta$ (defined for $x > 1$), where α and β are arbitrary real numbers, is a comparison scale. Indeed, here again the quotient of two functions in $\mathcal{E}$ is a function in $\mathcal{E}$; it is enough to show that if α and β are not both zero then $x^\alpha(\log x)^\beta$ tends to 0 or $+\infty$; this is clear if $\alpha = 0, \beta \neq 0$; if $\alpha > 0$ one has $(\log x)^{-\beta} \prec\!\!\prec x^\alpha$, and if $\alpha < 0$ one has $(\log x)^\beta \prec\!\!\prec x^{-\alpha}$ for any β, whence the proposition.

Note that this last comparison scale is a totally ordered set (for the relation "$f \prec\!\!\prec g$ or $f = g$") whose order structure is isomorphic to the *lexicographic* order on $\mathbf{R}^2$ (*Set Theory*, III, p. 157); recall that in this structure the relation $(\alpha, \beta) < (\gamma, \delta)$ means "$\alpha < \gamma$, or $\alpha = \gamma$ and $\beta < \delta$").

Similarly, the scale formed by the functions $\exp(p(x))$, where p runs through the set P_0 of polynomials with no constant term, has order structure isomorphic to the order structure of P_0, in which the relation $p < q$ implies that the dominant term of the polynomial $q - p$ has a coefficient > 0 (*cf. Alg.*, VI. 19, *Example* 2).

Let φ be a map from a set F into E, such that $\overset{-1}{\varphi}(\mathfrak{F})$ is a filter base on F. If $\mathcal{E}$ is a comparison scale on E (for the filter base $\mathfrak{F}$) then the functions $f \circ \varphi$, as f runs through $\mathcal{E}$, form a comparison scale on F (for the filter base $\overset{-1}{\varphi}(\mathfrak{F})$).

2. PRINCIPAL PARTS AND ASYMPTOTIC EXPANSIONS

Let $\mathcal{E}$ be a comparison scale formed by functions with values in a non-discrete valued field K. Let V be a normed space over K, and let $\mathbf{f}$ be a function in $\mathcal{H}(\mathfrak{F}, V)$; if there are a function $g \in \mathcal{E}$ and an element $\mathbf{a} \neq 0$ in V such that $\mathbf{f} \sim \mathbf{a}g$, one says that $\mathbf{a}g$ is a *principal part* of $\mathbf{f}$ relative to the scale $\mathcal{E}$. From def. 1 of V, p. 10, $\mathbf{f}$ can have *only one* principal part relative to $\mathcal{E}$, for if g_1, g_2 are two functions in $\mathcal{E}$ and $\mathbf{a}_1$, $\mathbf{a}_2$ are two $\neq 0$ elements of V, then the relation $\mathbf{a}_1 g_1 \sim \mathbf{a}_2 g_2$ implies $|g_1| \asymp |g_2|$ and consequently $g_1 = g_2$, whence $(\mathbf{a}_2 - \mathbf{a}_1)g_1 \prec\!\!\prec g_1$, and since g_1 is not identically zero on any set in $\mathcal{E}$ this implies $\mathbf{a}_2 = \mathbf{a}_1$.

If $\mathbf{f}$ has a principal part relative to a comparison scale $\mathcal{E}$ it has *the same* principal part relative to every comparison scale $\mathcal{E}' \supset \mathcal{E}$.

Examples. 1) For x real (resp. complex) tending to $+\infty$ (resp. ∞), every polynomial $\mathbf{a}_0 x^n + \mathbf{a}_1 x^{n-1} + \ldots + \mathbf{a}_n$ with coefficients in V, such that $\mathbf{a}_0 \neq 0$, has principal part $\mathbf{a}_0 x^n$ with respect to the scale x^n (or any scale containing the x^n). It follows that every rational fraction $\dfrac{a_0 x^n + \cdots + a_m}{b_0 x^n + \cdots + b_n}$ with real or complex coefficients such that $a_0 b_0 \neq 0$ has principal part $\dfrac{a_0}{b_0} x^{m-n}$ with respect to the same scale.

2) A function may be comparable to all the functions of a scale and yet not admit a principal part with respect to this scale. For example, for x real tending to $+\infty$, $\sqrt{x}$ has no principal part with respect to the scale x^n where n is a rational integer; $\log x$ has no principal part with respect to the scale of x^α (α arbitrary real); $\exp(\sqrt{\log x})$ and $x^x = e^{x \log x}$ have no principal part with respect to the scale of the $x^\alpha (\log x)^\beta$, nor with respect to the scale of the $\exp(p(x))$ (p a polynomial with no constant term).

The concept of principal part admits extensive generalization. Suppose that a function $\mathbf{f} \in \mathcal{H}(\mathfrak{F}, V)$ has a principal part $\mathbf{a}_1 g_1$ with respect to a scale $\mathcal{E}$; the relation $\mathbf{f} \sim \mathbf{a}_1 g_1$ is equivalent to $\mathbf{f} - \mathbf{a}_1 g_1 \prec\!\!\prec g_1$ (V, p. 216, def. 4); to study the function $\mathbf{f}$ more closely one is thus led to consider the function $\mathbf{f} - \mathbf{a}_1 g_1$. If this function has a principal part $\mathbf{a}_2 g_2$ with respect to $\mathcal{E}$ one must have $g_2 \prec\!\!\prec g_1$ and $\mathbf{f} - \mathbf{a}_1 g_1 - \mathbf{a}_2 g_2 \prec\!\!\prec g_2$.

More generally, suppose that the scale $\mathcal{E}$ is written parametrically in the form (g_α) where α runs through a set of indices A endowed with a totally ordered structure isomorphic to the *opposite* of the order structure of $\mathcal{E}$: the relation $\alpha < \beta$ is thus equivalent to $g_\beta \prec\!\!\prec g_\alpha$. In these circumstances:

DEFINITION 2. *One says that a function* $\mathbf{f} \in \mathcal{H}(\mathfrak{F}, V)$ *admits an asymptotic expansion to precision* g_α (relative to the scale $\mathcal{E}$) *if there exists a family* $(\mathbf{a}_\lambda)_{\lambda \leqslant \alpha}$ *of elements of* V, *all but a finite number of them being* 0, *such that* $\mathbf{f} - \sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda \prec\!\!\prec g_\alpha$. *One says that* $\sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda$ *is an asymptotic expansion of* $\mathbf{f}$ *to precision* g_α, *that the* $\mathbf{a}_\lambda g_\lambda$ ($\lambda \leqslant \alpha$) *are its terms, the* $\mathbf{a}_\lambda$ *its coefficients, and the function* $\mathbf{r}_\alpha = \mathbf{f} - \sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda$ *the remainder of this expansion*.

To express the fact that $\sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda$ is an asymptotic expansion of $\mathbf{f}$ to precision g_α one most often restricts oneself to writing

$$\mathbf{f} = \sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda + o(g_\alpha) \quad \Big(\text{or } \mathbf{f} = \sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda + o_k(g_\alpha)$$

if there are several functions in the proof) following the notation of V, p. 219 and 220.

Of two asymptotic expansions (of two functions, distinct or not) relative to the same scale $\mathcal{E}$, one says that the one with precision of greater index is the *more precise*.

If $\sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda$ is an asymptotic expansion of $\mathbf{f}$ to precision g_α, then, for every $\beta < \alpha$, $\sum_{\lambda \leqslant \beta} \mathbf{a}_\lambda g_\lambda$ is an asymptotic expansion of $\mathbf{f}$ to precision g_β (V, p. 215, prop. 5): one says that it is obtained by *reducing the precision* of the given expansion $\sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda$ of $\mathbf{f}$ to g_β.

If $\sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda$ and $\sum_{\lambda \leqslant \alpha} \mathbf{b}_\lambda g_\lambda$ are asymptotic expansions to the *same* precision g_α of two functions $\mathbf{f}_1$, $\mathbf{f}_2$, then $\sum_{\lambda \leqslant \alpha} (\mathbf{a}_\lambda + \mathbf{b}_\lambda) g_\lambda$ is an asymptotic expansion of $\mathbf{f}_1 + \mathbf{f}_2$ to precision g_α (V, p. 215, prop. 5); and for every scalar c, $\sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda c g_\lambda$ is an asymptotic expansion of $\mathbf{f}_1 c$ to precision g_α. It follows that if a function admits an asymptotic expansion to precision g_α then this expansion is *unique:* it is sufficient to see that the function 0 does not admit an asymptotic expansion with precision g_α having coefficients $\neq 0$. For, if $0 = \sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda + \mathbf{r}_\alpha$, and if γ were the least of the indices $\lambda \leqslant \alpha$ such that $\mathbf{a}_\lambda \neq 0$, one would have $\mathbf{a}_\gamma g_\gamma = -\sum_{\gamma < \lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda - \mathbf{r}_\alpha \prec\!\!\prec g_\gamma$, which is absurd.

To say that a function $\mathbf{f}$ admits an asymptotic expansion to precision g_α, all of whose coefficients are *zero*, is equivalent to saying that $\mathbf{f} \prec\!\!\prec g_\alpha$. If $\mathbf{f}$ admits an asymptotic expansion $\sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda$ to precision g_α whose coefficients are not all zero, and if γ is the smallest of the indices λ such that $\mathbf{a}_\lambda \neq 0$, then $\mathbf{a}_\gamma g_\gamma$ is the *principal part* of $\mathbf{f}$ relative to the scale $\mathcal{E}$, for one has $\mathbf{f} - \mathbf{a}_\gamma g_\gamma = \sum_{\gamma < \lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda + \mathbf{r}_\alpha \prec\!\!\prec g_\gamma$; similarly, if $\mu \leqslant \alpha$ is an index such that $\mathbf{a}_\mu \neq 0$, then $\mathbf{a}_\mu g_\mu$ is the principal part of $\mathbf{f} - \sum_{\lambda < \mu} \mathbf{a}_\lambda g_\lambda$.

The most important asymptotic expansions in applications are those relative to the scale of the x^{-n} (resp. of the z^{-n}), where n is a positive or negative integer, as x tends to $+\infty$ or to $-\infty$ (resp. when the complex number z tends to ∞), or relative to the scale of the $(x - c)^n$ (resp. $(z - c)^n$) when the real number x tends to c from the right or left (resp. when the complex number z tends to c). We saw in I, p. 21 that every vector function of a real variable x which is k times differentiable at a point $c \in \mathbf{R}$ admits a Taylor expansion of order k at this point, that is, an asymptotic expansion to precision $(x - c)^k$ with respect to the scale of the $(x - c)^n$.

3. SUMS AND PRODUCTS OF ASYMPTOTIC EXPANSIONS

If $\mathbf{f}_1$, $\mathbf{f}_2$ admit asymptotic expansions to precision g_α and g_β respectively, relative to a comparison scale $\mathcal{E}$, one deduces expansions to precision $g_{\min(\alpha,\beta)}$ by *limiting* the two expansions to this precision; we have seen in V, p. 222 how one obtains an asymptotic expansion for $\mathbf{f}_1 + \mathbf{f}_2$ to precision $g_{\min(\alpha,\beta)}$.

Let V_1, V_2 and V be three normed spaces over the field K, and let $(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}.\mathbf{y}]$ be a *continuous bilinear map* from $V_1 \times V_2$ into V; we suppose further, for the rest of this section, that the scale $\mathcal{E}$ is such that the *product* of any two functions in $\mathcal{E}$ is again in $\mathcal{E}$ (which is true for all the comparison scales given as examples (in V, p. 220)).

Now let $\mathbf{f}_1$, $\mathbf{f}_2$ be two functions in $\mathcal{H}(\mathfrak{F}, V_1)$ and $\mathcal{H}(\mathfrak{F}, V_2)$ respectively, having asymptotic expansions $\mathbf{f}_1 = \sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda + \mathbf{r}_\alpha$ and $\mathbf{f}_2 = \sum_{\mu \leqslant \beta} \mathbf{b}_\mu g_\mu + \mathbf{r}_\beta$ to precision g_α and g_β respectively, with respect to the scale $\mathcal{E}$. Suppose further that neither the $\mathbf{a}_\lambda$ nor the $\mathbf{b}_\mu$ are all zero, and let $\mathbf{a}_\gamma g_\gamma$ and $\mathbf{b}_\delta g_\delta$ be the principal parts of $\mathbf{f}_1$ and $\mathbf{f}_2$. By hypothesis, one can write $g_\gamma g_\beta = g_\rho$ and $g_\delta g_\alpha = g_\sigma$; let us show that the sum $\sum [\mathbf{a}_\lambda.\mathbf{b}_\mu]g_\lambda g_\mu$ taken over all pairs (λ, μ) such that $g_{\min(\rho,\sigma)} \prec\!\prec g_\lambda g_\mu$, is an *asymptotic expansion of* $[\mathbf{f}_1.\mathbf{f}_2]$ *to precision* $g_{\min(\rho,\sigma)}$. Now the difference between $[\mathbf{f}_1.\mathbf{f}_2]$ and this sum is the sum of a finite number of terms, each of which is either of the form $[\mathbf{a}_\lambda.\mathbf{b}_\mu]g_\lambda g_\mu$ with $g_\lambda g_\mu \prec\!\prec g_{\min(\rho,\sigma)}$, or of the form $[\mathbf{a}_\lambda.\mathbf{r}_\beta]g_\lambda$ where $\lambda \geqslant \gamma$, or of the form $[\mathbf{r}_\alpha.\mathbf{b}_\mu]g_\mu$ where $\mu \geqslant \delta$; but since $[\mathbf{x}.\mathbf{y}]$ is continuous, one has from (V, p. 213, prop. 3 and V, p. 215, prop. 6) that $[\mathbf{a}_\lambda.\mathbf{r}_\beta]g_\lambda \preccurlyeq \mathbf{r}_\beta g_\lambda \prec\!\prec g_\beta g_\gamma = g_\rho$ for $\lambda \geqslant \gamma$, and similarly $[\mathbf{r}_\alpha.\mathbf{b}_\mu]g_\mu \preccurlyeq \mathbf{r}_\alpha g_\mu \prec\!\prec g_\alpha g_\delta = g_\sigma$ for $\mu \geqslant \delta$, whence the proposition (V, p. 215, prop. 5).

If all the $\mathbf{a}_\lambda$ are zero one has $[\mathbf{f}_1.\mathbf{f}_2] \prec\!\prec g_\alpha g_\delta$; in other words, one has an asymptotic expansion of $[\mathbf{f}_1.\mathbf{f}_2]$ with zero terms, to precision $g_\alpha g_\delta$; similarly if all the $\mathbf{a}_\lambda$ and $\mathbf{b}_\mu$ are zero one has an asymptotic expansion of $[\mathbf{f}_1.\mathbf{f}_2]$ with zero terms to precision $g_\alpha g_\beta$.

We shall apply the preceding result principally to the case where V is a *normed algebra* over K and the bilinear function $[\mathbf{x}.\mathbf{y}]$ is the product $\mathbf{xy}$ in this algebra; the most important cases are those where V is equal to $\mathbf{R}$ or to $\mathbf{C}$.

In particular, if f_i $(1 \leqslant i \leqslant n)$ are n functions in $\mathcal{H}(\mathfrak{F}, \mathrm{K})$ each of which admits an asymptotic expansion with respect to $\mathcal{E}$ one can obtain an asymptotic expansion with respect to $\mathcal{E}$ for every *polynomial* $\sum\limits_{(\nu_i)} \mathbf{a}_{\nu_1\nu_2\ldots\nu_n} f_1^{\nu_1} \ldots f_n^{\nu_n}$ in the f_i with coefficients in a normed space V; furthermore, the preceding rules allow one to determine the precision of the expansion obtained, if one knows those of the expansions of the functions f_i.

4. COMPOSITION OF ASYMPTOTIC EXPANSIONS

Let f be a function in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ (resp. $\mathcal{H}(\mathfrak{F}, \mathbf{C})$), admitting an asymptotic expansion to precision g_α with respect to a scale $\mathcal{E}$, and *having limit* 0 along the filter with base $\mathfrak{F}$. On the other hand let $\mathbf{h}$ be a function with values in a normed space V over $\mathbf{R}$ (resp. $\mathbf{C}$), defined on a neighbourhood of the point 0 in $\mathbf{R}$ (resp. $\mathbf{C}$), and *n times differentiable* on this neighbourhood; then

$$\mathbf{h}(t) = \mathbf{c}_0 + \mathbf{c}_1 t + \cdots + \mathbf{c}_n t^n + o(t^n)$$

on this neighbourhood (I, p. 21), whence, on a suitable set in $\mathfrak{F}$

$$\mathbf{h} \circ f = \mathbf{c}_0 + \mathbf{c}_1 f + \cdots + \mathbf{c}_n f^n + o(f^n).$$

We have seen, in n° 3, how to form an asymptotic expansion for $\mathbf{c}_0 + \mathbf{c}_1 f + \cdots + \mathbf{c}_n f^n$ to a precision g_ρ determined by the precision of the expansion of f; moreover, suppose that the coefficients of the asymptotic expansion of f are not all zero, and that $a_\gamma g_\gamma$ is the principal part of f, and let $g_\sigma = g_\gamma^n$; if $\sigma < \rho$ one will have an expansion of $\mathbf{h} \circ f$ to precision g_σ on limiting the expansion of $\sum\limits_{k=0}^{n} \mathbf{c}_k f^k$ to this precision; if, on the contrary, $\rho \leqslant \sigma$, then the expansion of $\sum\limits_{k=0}^{n} \mathbf{c}_k f^k$ is also an expansion of $\mathbf{h} \circ f$ to precision g_ρ.

If all the terms of the asymptotic expansion of f are zero, and if $g_\alpha \lll 1$, then $f \lll g_\alpha$ and so $f^k \lll g_\alpha^k \lll g_\alpha$ for every integer $k > 0$; if $\mathbf{c}_m$ is the first coefficient of index > 0 which is not zero (assuming that the $\mathbf{c}_k$ for indices $k > 0$ are not all 0), then $\mathbf{c}_0$ is an asymptotic expansion of $\mathbf{h} \circ f$ to precision g_α^m.

In the remainder of this section we shall restrict ourselves to the case where the functions in $\mathcal{E}$ have real values and are *strictly positive* on a set in $\mathfrak{F}$, and we shall consider asymptotic expansions only of functions in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$. Suppose first that for every function $g \in \mathcal{E}$ and every real number ν, g^ν again belongs to $\mathcal{E}$: this condition is fulfilled, for example, by the scale of the x^α or by that of the $x^\alpha \, |\log x|^\beta$ (α and β being arbitrary real numbers) on a neighbourhood of $+\infty$ or a neighbourhood of 0 in $\mathbf{R}$. This property implies that the *quotient* of two functions in $\mathcal{E}$ again belongs to $\mathcal{E}$. This being so, from an asymptotic expansion relative to $\mathcal{E}$ of a function $f \in \mathcal{H}(\mathfrak{F}, \mathbf{R})$, to precision g_α, one can derive an expansion for $|f|^\nu$ for every real number ν. Let us restrict ourselves to the case where the coefficients of the expansion of f are not all zero, and let $a_\gamma g_\gamma$ be the principal part of f; one can write $|f|^\nu = \left|a_\gamma\right|^\nu g_\gamma^\nu (1+h)^\nu$, with

$$h = \sum_{\gamma < \lambda \leqslant \alpha} \frac{a_\lambda}{a_\gamma} \frac{g_\lambda}{g_\gamma} + o\left(\frac{g_\alpha}{g_\gamma}\right).$$

Under our hypotheses $\sum\limits_{\gamma < \lambda \leqslant \alpha} \dfrac{a_\lambda}{a_\gamma} \dfrac{g_\lambda}{g_\gamma}$ is an asymptotic expansion of h, to precision g_α / g_γ; since h tends to 0 along $\mathfrak{F}$ the method described above gives an asymptotic expansion of $(1+h)^\nu$, then an expansion of $|f|^\nu$ on multiplying by $\left|a_\gamma\right|^\nu g_\gamma^\nu$.

With the same hypotheses on f one can write

$$\log |f| = \log \left|a_\gamma g_\gamma\right| + \log(1+h)$$

and $\log(1+h)$ can be expanded, as has been said above, the function $\log(1+t)$ being indefinitely differentiable on a neighbourhood of 0; if, further, $\log g_\gamma$ admits an asymptotic expansion with respect to $\mathcal{E}$, or with respect to a scale $\mathcal{E}_1 \supset \mathcal{E}$, one obtains an asymptotic expansion for $\log |f|$ on adding the two asymptotic expansions.

Example. We have $(1+x)^{1/x} = \exp\left(\dfrac{1}{x}\log(1+x)\right)$; when x tends to $+\infty$ we have $\log(1+x) = \log x + \log\left(1 + \dfrac{1}{x}\right)$, whence the asymptotic expansion of $\dfrac{1}{x}\log(1+x)$ with respect to the scale of the $x^\alpha (\log x)^\beta$:

$$\frac{1}{x}\log(1+x) = \frac{\log x}{x} + \frac{1}{x^2} - \frac{1}{2x^3} + o_1\left(\frac{1}{x^3}\right).$$

From this expansion, and from the Taylor expansion

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + o(u^3)$$

on a neighbourhood of $u = 0$, one deduces the asymptotic expansion

$$(1+x)^{1/x} = 1 + \frac{\log x}{x} + \frac{1}{2}\frac{(\log x)^2}{x^2} + \frac{1}{x^2} + \frac{1}{6}\frac{(\log x)^3}{x^3} + \frac{\log x}{x^3} - \frac{1}{2x^3} + o_2\left(\frac{1}{x^3}\right)$$

with respect to the scale $x^\alpha(\log x)^\beta$ by the methods explained above.

Keeping the same hypotheses and notation, the asymptotic expansion of e^f does not pose new problems except when $f \succ\!\succ 1$; one must then distinguish two cases, as $g_\alpha \succ\!\succ 1$ or $g_\alpha \preccurlyeq 1$. In the first case, giving an expansion for f does not allow one to obtain a principal part for e^f relative to $\mathcal{E}$, because one does not know in general whether the remainder r_α tends to 0, that is, whether e^{r_α} tends to 1. On the contrary, if $g_\alpha \preccurlyeq 1$ then $r_\alpha \prec\!\prec 1$ and so $e^f \sim \exp\left(\sum\limits_{\lambda \leqslant \alpha} a_\lambda g_\lambda\right)$. One can make this result more precise: let $a_\gamma g_\gamma$ be the principal part of f and let δ be the index (such that $\gamma < \delta \leqslant \alpha$) for which $g_\delta = 1$; we put $f_1 = \sum\limits_{\lambda \leqslant \delta} a_\lambda g_\lambda$, $f_2 = \sum\limits_{\delta < \lambda \leqslant \alpha} a_\lambda g_\lambda + r_\alpha$; we have $f = f_1 + f_2$, so $e^f = e^{f_1} e^{f_2}$, and the general method explained at the start of this subsection allows one to form an asymptotic expansion of e^{f_2} (starting from the Taylor expansion of e^t at the point $t = 0$). One will then have an asymptotic expansion for e^f if $e^{f_1} = \prod\limits_{\lambda \leqslant \delta} \exp(a_\lambda g_\lambda)$ belongs to $\mathcal{E}$, or to a scale $\mathcal{E}_1$ containing $\mathcal{E}$.

Example. We have $x^{x^{1/x}} = \exp\left(\log x . \exp\left(\frac{1}{x}\log x\right)\right)$; when x tends to $+\infty$ one has $\log x \prec\!\prec x$, whence the asymptotic expansion of $\log x . \exp\left(\frac{1}{x}\log x\right)$ with respect to the scale $x^\alpha(\log x)^\beta$:

$$\log x . \exp\left(\frac{1}{x}\log x\right) = \log x + \frac{(\log x)^2}{x} + \frac{1}{2}\frac{(\log x)^3}{x^2} + o\left(\frac{(\log x)^3}{x^2}\right).$$

All the terms of this expansion, starting from the second, tend to 0; from this expansion and from the Taylor expansion $e^u = 1 + u + u^2/2 + o(u^2)$ on a neighbourhood of $u = 0$ one deduces

$$x^{x^{1/x}} = x + (\log x)^2 + \frac{1}{2}\frac{(\log x)^4}{x} + \frac{1}{2}\frac{(\log x)^3}{x} + o\left(\frac{(\log x)^3}{x}\right).$$

5. ASYMPTOTIC EXPANSIONS WITH VARIABLE COEFFICIENTS

One can generalize the concept of principal part, and that of an asymptotic expansion, in the following way. Let $\mathcal{E}$ be a comparison scale formed of real (resp. complex) functions such that, for each of them, there is a set in $\mathfrak{F}$ on which the function *does not vanish at any point*. Further, let $\mathcal{C}$ be a set of functions in $\mathcal{H}(\mathfrak{F}, V)$ satisfying the following three conditions:

($\mathrm{CO_I}$) *For every function* $\mathbf{a} \in \mathcal{C}$ *one has* $\mathbf{a} \preccurlyeq 1$.
($\mathrm{CO_{II}}$) *The relation* $\mathbf{a} \prec\!\!\prec 1$ *for a function* $\mathbf{a} \in \mathcal{C}$ *implies* $\mathbf{a} = 0$.
($\mathrm{CO_{III}}$) $\mathcal{C}$ *is a vector space over* $\mathbf{R}$ (resp. $\mathbf{C}$).

Now let $\mathbf{f}$ be any function in $\mathcal{H}(\mathfrak{F}, V)$; if there exist a function $g \in \mathcal{E}$ and a nonzero function $\mathbf{a} \in \mathcal{C}$ such that $\mathbf{f} - \mathbf{a}g \prec\!\!\prec g$ one will say that $\mathbf{a}g$ is a *principal part* of $\mathbf{f}$, relative to the comparison scale $\mathcal{E}$ and to the *domain of coefficients* $\mathcal{C}$. If such a principal part exists, it is *unique*: for suppose that there are two principal parts $\mathbf{a}_1 g_1$ and $\mathbf{a}_2 g_2$; one cannot have $g_1 \prec\!\!\prec g_2$ since from ($\mathrm{CO_I}$) one could deduce that $\mathbf{a}_1 g_1 \prec\!\!\prec g_2$ and $\mathbf{f} - \mathbf{a}_1 g_1 \prec\!\!\prec g_1 \prec\!\!\prec g_2$, so $\mathbf{f} \prec\!\!\prec g_2$; but one would also have $\mathbf{a}_2 g_2 \prec\!\!\prec g_2$ and consequently $\mathbf{a}_2 \prec\!\!\prec 1$ contradicting the hypothesis $\mathbf{a}_2 \neq 0$ and ($\mathrm{CO_{II}}$). So one must have $g_1 = g_2$; from the relations $\mathbf{f} - \mathbf{a}_1 g_1 \prec\!\!\prec g_1, \mathbf{f} - \mathbf{a}_2 g_1 \prec\!\!\prec g_1$ one deduces that $(\mathbf{a}_2 - \mathbf{a}_1) g_1 \prec\!\!\prec g_1$ whence $\mathbf{a}_2 - \mathbf{a}_1 \prec\!\!\prec 1$, and consequently $\mathbf{a}_2 = \mathbf{a}_1$, by ($\mathrm{CO_{II}}$) and ($\mathrm{CO_{III}}$).

Example. For x real tending to $+\infty$ the *periodic* bounded functions on $\mathbf{R}$, having the same period τ, satisfy conditions ($\mathrm{CO_I}$), ($\mathrm{CO_{II}}$) and ($\mathrm{CO_{III}}$): if $\lim_{x \to +\infty} a(x) = 0$ then for every $\varepsilon > 0$ there is an x_0 such that $|a(x)| \leqslant \varepsilon$ for every $x \geqslant x_0$; one deduces that $|a(x)| \leqslant \varepsilon$ for $0 \leqslant x \leqslant \tau$ too, since there exists an integer n such that $x + n\tau \geqslant x_0$, and such that $a(x) = a(x + n\tau)$; since ε is arbitrary one has $a(x) = 0$ on $[0, \tau]$, hence everywhere.

With the notation of V, p. 222, we will say that $\sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda$, where the $\mathbf{a}_\lambda$ belong to $\mathcal{C}$ and all but a finite number of them are zero, is an *asymptotic expansion of* $\mathbf{f}$ *with coefficients in* $\mathcal{C}$, to precision g_α, if $\mathbf{f} - \sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda \prec\!\!\prec g_\alpha$; for every index μ such that $\mathbf{a}_\mu \neq 0$ then $\mathbf{a}_\mu g_\mu$ is the principal part of $\mathbf{f} - \sum_{\lambda < \mu} \mathbf{a}_\lambda g_\lambda$, relative to $\mathcal{E}$ and to $\mathcal{C}$, which proves the uniqueness of the asymptotic expansion of $\mathbf{f}$ (to precision g_α) when it exists.

The methods given in n° 3 (V, p. 223) for forming an asymptotic expansion for $\mathbf{f}_1 + \mathbf{f}_2$ or for $[\mathbf{f}_1 . \mathbf{f}_2]$, starting from given asymptotic expansions for $\mathbf{f}_1$ and $\mathbf{f}_2$, can again be applied to expansions with variable coefficients, so long as the $[\mathbf{a}_\lambda . \mathbf{b}_\mu]$ belong to the domain of coefficients $\mathcal{C}$ corresponding to the normed space V or admit an asymptotic expansion with coefficients in $\mathcal{C}$.

§ 3. ASYMPTOTIC EXPANSIONS OF FUNCTIONS OF A REAL VARIABLE

In this section we shall consider only the case where the set E is an *open interval* in the extended real line $\overline{\mathbf{R}}$, and $\mathfrak{F}$ is a base for the trace on E of the filter of neighbourhoods of the left or right-hand endpoint α of E; further, we shall study above all the (finite) *real* functions defined on a set in $\mathfrak{F}$ (depending on the function under consideration).

Using one of the changes of variable $x' = -x$, $x' = \dfrac{1}{x-\alpha}$, $x' = -\dfrac{1}{x-\alpha}$ as needed, one can always reduce to the case where E is an interval of the form $]a, +\infty[$ so that $\mathfrak{F}$ will be formed of intervals $[t, +\infty[$ with $t > a$. We shall restrict ourselves principally to this latter case, and leave the task of translating most of the propositions we obtain (by the above changes of variable) to the reader, except for a few particularly important results.

It will be convenient to employ an abuse of language and refer to the sets in $\mathfrak{F}$ as "neighbourhoods of $+\infty$".

1. INTEGRATION OF COMPARISON RELATIONS: I. WEAK RELATIONS

PROPOSITION 1. *Let* $\mathbf{f}$ *be a regulated vector function,* g *a regulated function* $\geqslant 0$ *on an interval* $[a, +\infty[$*, such that* $\int_a^{+\infty} g(t)\,dt > 0$*. The relation* $\mathbf{f} \preccurlyeq g$ *as* x *tends to* $+\infty$ *implies that* $\int_a^x \mathbf{f}(t)\,dt \preccurlyeq \int_a^x g(t)\,dt$*. If the integral* $\int_a^{+\infty} g(t)\,dt$ *converges then the integral* $\int_a^{+\infty} \mathbf{f}(t)\,dt$ *converges absolutely.*

Indeed, by hypothesis there exist a $b \geqslant a$ and a number $c' > 0$ such that

$$\|\mathbf{f}(x)\| \leqslant c' g(x) \qquad \text{for } x \geqslant b,$$

whence

$$\left\| \int_b^x \mathbf{f}(t)\,dt \right\| \leqslant \int_b^x \|\mathbf{f}(t)\|\,dt \leqslant c' \int_b^x g(t)\,dt;$$

since, moreover, one may suppose b so large that $\int_a^b g(t)\,dt > 0$, there exists a $c'' > 0$ such that $\left\| \int_a^b \mathbf{f}(t)\,dt \right\| \leqslant c'' \int_a^b g(t)\,dt$; on putting $c = \max(c', c'')$ one thus has

$$\left\| \int_a^x \mathbf{f}(t)\,dt \right\| \leqslant c \int_a^x g(t)\,dt$$

for every $x \geqslant b$, whence the proposition.

COROLLARY 1. *If* f *and* g *are regulated functions and* $\geqslant 0$ *on the interval* $[a, +\infty[$*, and such that* $f \succcurlyeq g$*, and if* $\int_a^{+\infty} g(t)\,dt = +\infty$*, then* $\int_a^{+\infty} f(t)\,dt = +\infty$*.*

COROLLARY 2. *If* f *and* g *are* $\geqslant 0$ *and not identically zero on* $[a, +\infty[$*, and such that* $f \asymp g$*, then* $\int_a^x f(t)\,dt \asymp \int_a^x g(t)\,dt$*.*

2. APPLICATION: LOGARITHMIC CRITERIA FOR CONVERGENCE OF INTEGRALS

By choosing the function g suitably one can deduce criteria for deciding whether the integral $\int_a^{+\infty} f(t)\,dt$ of a function $f \geqslant 0$ converges or is infinite from prop. 1 of V, p. 228, and cor. 1 thereto: it suffices to choose for g a function whose primitive is known. In particular, since x^μ has primitive $\dfrac{x^{\mu+1}}{\mu+1}$ when $\mu \neq -1$, and $\log x$ when $\mu = -1$, we have the following criterion:

PROPOSITION 2 ("logarithmic criterion of order 0"). *Let f be a regulated function $\geqslant 0$ on an interval $[a, +\infty[$; if $f(x) \preccurlyeq x^\mu$ for some $\mu < -1$ then the integral $\int_a^{+\infty} f(t)\,dt$ converges; if $f(x) \succcurlyeq x^\mu$ for some $\mu \geqslant -1$ then the integral $\int_a^{+\infty} f(t)\,dt$ is infinite.*

This criterion is not decisive when $1/x^{1+\alpha} \prec\!\prec f(x) \prec\!\prec 1/x$ for *every* exponent $\alpha > 0$, as, for example, when $f(x) = 1/x(\log x)^\mu$ $(\mu > 0)$. But in this last case f has primitive $\dfrac{1}{1-\mu}(\log x)^{1-\mu}$ if $\mu \neq 1$ and $\log\log x$ when $\mu = 1$. Thus:

PROPOSITION 3 ("logarithmic criterion of order 1"). *Let f be a regulated function $\geqslant 0$ on an interval $[a, +\infty[$; if $f(x) \preccurlyeq 1/x(\log x)^\mu$ for some $\mu > 1$ then the integral $\int_a^{+\infty} f(t)\,dt$ converges; if $f(x) \succcurlyeq 1/x(\log x)^\mu$ for some $\mu \leqslant 1$ then the integral $\int_a^{+\infty} f(t)\,dt$ is infinite.*

Generally, for every integer $n \geqslant 0$, let us denote by $l_n(x)$ the function defined inductively (for x large enough) by the relations $l_0(x) = x$, $l_n(x) = \log(l_{n-1}(x))$ for $n \geqslant 1$; one says that $l_n(x)$ is the n^{th} *iterated logarithm of x* (*cf. Appendix*). One verifies immediately that $\dfrac{1}{1-\mu}\big(l_n(x)\big)^{1-\mu}$ is a primitive of

$$\frac{1}{xl_1(x)l_2(x)\dots l_{n-1}(x)\big(l_n(x)\big)^\mu}$$

for $\mu \neq 1$, and $l_{n+1}(x)$ is a primitive of $\dfrac{1}{xl_1(x)l_2(x)\dots l_{n-1}(x)l_n(x)}$. Hence:

PROPOSITION 4 ("logarithmic criterion of order n"). *Let f be a regulated function $\geqslant 0$ on an interval $[a, +\infty[$; if, for some $\mu > 1$, $f(x) \preccurlyeq \dfrac{1}{xl_1(x)l_2(x)\dots l_{n-1}(x)\big(l_n(x)\big)^\mu}$, then the integral $\int_a^{+\infty} f(t)\,dt$ converges; if $f(x) \succcurlyeq \dfrac{1}{xl_1(x)l_2(x)\dots l_{n-1}(x)\big(l_n(x)\big)^\mu}$ for some $\mu \leqslant 1$, then the integral $\int_a^{+\infty} f(t)\,dt$ is infinite.*

Each logarithmic criterion is thus applicable to functions for which the criteria of lower order are not decisive (*cf.* V, p. 264, exerc. 5 *b*) and V, p. 265, exerc. 8).

Because of its usefulness we translate the criterion of order 0 for integrals $\int_\alpha^a f(t)\,dt$ where f is regulated and $\geqslant 0$ on a non-compact interval $]\alpha, a]$:

PROPOSITION 5 ("logarithmic criterion of order 0"). *If on a neighbourhood of α one has $f(x) \preccurlyeq 1/(x-\alpha)^\mu$ for some $\mu < 1$ then the integral $\int_\alpha^a f(t)\,dt$ converges; if $f(x) \succcurlyeq 1/(x-\alpha)^\mu$ for some $\mu \geqslant 1$ then the integral $\int_\alpha^a f(t)\,dt$ is infinite.*

We leave it to the reader to translate the logarithmic criterion of order n similarly.

The application of the logarithmic criteria is immediate if one knows how to obtain the *principal part* of f with respect to a comparison scale containing the functions that feature in these criteria: if f_1 is the principal part, the integral $\int_\alpha^{+\infty} f(t)\,dt$ converges or is infinite together with $\int_\alpha^{+\infty} f_1(t)\,dt$, and the logarithmic criteria apply immediately to this last integral.

Examples. 1) The function $t^p(1-t)^q$ is not bounded on $]0, 1[$ when $p < 0$ or $q < 0$; by the logarithmic criterion of order 0 applied on a neighbourhood of the points 0 and 1 the integral $\int_0^1 t^p(1-t)^q\,dt$ converges if and only if $p > -1$ and $q > -1$. If so, this integral is called the *Eulerian integral of the first kind* and is denoted by $\mathbf{B}(p+1, q+1)$ (*cf.* VII, p. 312).

2) Consider the integral $\int_0^\infty t^{x-1}e^{-t}\,dt$. Since $e^{-t} \sim 1$ on a neighbourhood of 0, one must have $x > 0$ if this integral is to converge; this condition is also sufficient since on a neighbourhood of $+\infty$ one has $e^{-t} \lll t^{-\mu}$ for any $\mu > 0$. When $x > 0$ the integral is called the *Eulerian integral of the second kind* and is denoted by $\Gamma(x)$ (*cf.* VII, p. 311).

3. INTEGRATION OF COMPARISON RELATIONS: II. STRONG RELATIONS

PROPOSITION 6. *Let $\mathbf{f}$ be a regulated vector function, and g a regulated function $\geqslant 0$ on $[a, +\infty[$.*

1° *If the integral $\int_a^{+\infty} g(t)\,dt$ converges then the relation $\mathbf{f} \lll g$* (resp. $\mathbf{f} \sim \mathbf{c}g$*, where $\mathbf{c}$ is constant*) *implies that $\int_x^{+\infty} \mathbf{f}(t)\,dt \lll \int_x^{+\infty} g(t)\,dt$* (resp. $\int_x^{+\infty} \mathbf{f}(t)\,dt \sim \mathbf{c}\int_x^{+\infty} g(t)\,dt$).

2° *If the integral $\int_a^{+\infty} g(t)\,dt$ is infinite then the relation $\mathbf{f} \lll g$* (resp. $\mathbf{f} \sim \mathbf{c}g$) *implies that*

$$\int_\alpha^x \mathbf{f}(t)\,dt \lll \int_\beta^x g(t)\,dt \qquad \text{(resp. } \int_\alpha^x \mathbf{f}(t)\,dt \sim \mathbf{c}\int_\beta^x g(t)\,dt),$$

for any α and β in $[a, +\infty[$.

It is enough to prove the proposition for the relation $\mathbf{f} \lll g$ since, if $\mathbf{c} \neq 0$, the relation $\mathbf{f} \sim \mathbf{c}g$ is equivalent to $\mathbf{f} - \mathbf{c}g \lll g$.

The first part is an immediate consequence of the mean value theorem, since if $\|\mathbf{f}(x)\| \leqslant \varepsilon g(x)$ for $x \geqslant x_0$ one deduces that

$$\left\| \int_x^{+\infty} \mathbf{f}(t)\,dt \right\| \leqslant \int_x^{+\infty} \|\mathbf{f}(t)\|\;dt \leqslant \varepsilon \int_x^{+\infty} g(t)\,dt \quad \text{for } x \geqslant x_0.$$

In the second place, suppose that $\int_a^{+\infty} g(t)\,dt = +\infty$. If $\|\mathbf{f}(x)\| \leqslant \varepsilon g(x)$ for $x \geqslant x_0 \geqslant \max(\alpha, \beta)$, one has

$$\begin{aligned} \int_\alpha^x \|\mathbf{f}(t)\|\;dt &= \int_\alpha^{x_0} \|\mathbf{f}(t)\|\;dt + \int_{x_0}^x \|\mathbf{f}(t)\|\;dt \\ &\leqslant \int_\alpha^{x_0} \|\mathbf{f}(t)\|\;dt + \varepsilon \int_{x_0}^x g(t)\,dt \\ &= \varepsilon \int_\beta^x g(t)\,dt + \left(\int_\alpha^{x_0} \|\mathbf{f}(t)\|\;dt - \varepsilon \int_\beta^{x_0} g(t)\,dt \right). \end{aligned}$$

Now there exists an $x_1 \geqslant x_0$ such that for every $x \geqslant x_1$

$$\left| \int_\alpha^{x_0} \|\mathbf{f}(t)\|\;dt - \varepsilon \int_\beta^{x_0} g(t)\,dt \right| \leqslant \varepsilon \int_\beta^x g(t)\,dt$$

whence, for $x \geqslant x_1$

$$\left\| \int_\alpha^x \mathbf{f}(t)\,dt \right\| \leqslant \int_\alpha^x \|\mathbf{f}(t)\|\;dt \leqslant 2\varepsilon \int_\beta^x g(t)\,dt$$

which completes the proof, since $\varepsilon > 0$ is arbitrary.

In other words, one can *integrate* the two terms of a strong relation $\mathbf{f} \prec\!\!\prec g$, $\mathbf{f} \sim \mathbf{a}g$, when g is *positive* on an interval $[a, +\infty[$, and the relation persists between the primitives of the two terms, provided one takes care to integrate from x to $+\infty$ if $\int_a^{+\infty} g(t)\,dt$ converges, and from α to x (for any α in $[a, +\infty[$) in the opposite case.

Note that props. 1 (V, p. 228) and 6 (V, p. 230) remain valid when $\mathfrak{F}$ is a base for the trace filter of the intervals $[t, +\infty[$ (where $t > a$) on the *complement of a countable set* (*cf.* I, p. 15, th. 2).

Examples. 1) On applying prop. 6 of V, p. 230, to the relation $1/x \prec\!\!\prec x^{\alpha-1}$ where $\alpha > 0$, one again obtains the relation $\log x \prec\!\!\prec x^\alpha$ for every $\alpha > 0$, which is equivalent to the relation $y^{1/\alpha} \prec\!\!\prec e^y$ proved in III, p. 105.

2) We have $\left(\dfrac{e^x}{x}\right)' = \dfrac{e^x}{x}\left(1 - \dfrac{1}{x}\right) \sim e^x/x$; since e^x/x tends to $+\infty$ with x, one deduces from prop. 6 of V, p. 230, that $\displaystyle\int_1^x \frac{e^t}{t}\,dt \sim e^x/x$.

Remark. When g is not assumed to remain $\geqslant 0$ on an interval $[a, +\infty[$ (or to remain $\leqslant 0$ on such an interval), and $\int_a^{+\infty} g(t)\,dt$ is not convergent, the relation $f \sim g$ *does not necessarily imply* that $\int_a^x f(t)\,dt \sim \int_a^x g(t)\,dt$, as is shown by the example where $g(x) = \sin x$ and $f(x) = \left(1 + \dfrac{\sin x}{x}\right)\sin x$; here

$$\int_{n\pi}^{(n+1)\pi} \frac{\sin^2 t}{t}\,dt \geqslant \frac{1}{(n+1)\pi} \int_0^\pi \sin^2 t\,dt \geqslant \frac{1}{2} \int_{n+1}^{n+2} \frac{dt}{t},$$

whence

$$\int_{\pi}^{n\pi} \frac{\sin^2 t}{t}\,dt \geqslant \frac{1}{2}\int_{2}^{n+1}\frac{dt}{t}$$

and the integral $\int_1^{+\infty} dt/t$ is infinite, though $\displaystyle\int_{\frac{\pi}{2}}^{x} g(t)\,dt = -\cos x$ remains bounded (see V, p. 260, exerc. 4).

4. DIFFERENTIATION OF COMPARISON RELATIONS

Propositions 1 (V, p. 228) and 6 (V, p. 230) *do not have converses*: the existence of a comparison relation $\mathbf{f} \preccurlyeq g$, $\mathbf{f} \prec\!\!\prec g$, $\mathbf{f} \sim \mathbf{c}g$ between two functions that are differentiable on a neighbourhood of $+\infty$ *does not imply* the same comparison relation between their derivatives, even for real *monotone* functions f and g.

For example, the function $x^2 + x\sin x + \cos x$ is monotone and equivalent to x^2, but its derivative $x(2+\cos x)$ is not equivalent to $2x$.

On the other hand, one can derive comparison relations when one assumes *a priori* that the derivatives of the functions considered are *comparable* (V, p. 217). In general, we shall say that two real functions f and g defined on an interval $[a, +\infty[$ are *comparable to order* k on a neighbourhood of $+\infty$ if, on a neighbourhood of $+\infty$, they each have a regulated k^{th} derivative except at a countable set of points, and if, on this neighbourhood, $f^{(k)}$ and $g^{(k)}$ *have constant sign* (on the set on which they are defined) and are *comparable*.

We agree to say that two *comparable* real functions (V, p. 217) are *comparable of order* 0.

PROPOSITION 7. *If two real functions* f, g, *are comparable of order* 1, *then they are comparable*; *further, the relation* $f \prec\!\!\prec g$ (resp. $f \sim cg$, *c constant*) *implies* $f' \prec\!\!\prec g'$ (resp. $f' \sim cg'$) *unless* g *is equivalent to a nonzero constant.*

Now, since f' and g' are of constant sign on an interval $[x_0, +\infty[$, both f and g are monotone on this interval, so tend to a finite or infinite limit as x tends to $+\infty$. It is clear that f and g are comparable as x tends to $+\infty$ if one of these limits is finite and $\neq 0$, or if one is zero and the other infinite. If both f and g tend to 0 one can write $f(x) = -\int_x^{+\infty} f'(t)\,dt$, $g(x) = -\int_x^{+\infty} g'(t)\,dt$; since f' and g' are comparable the same is true for f and g and the comparison relation between f and g is the same as that between f' and g', by prop. 6 (V, p. 20). Similarly, if f and g both have an infinite limit one has $f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt$, $g(x) = g(x_0) + \int_{x_0}^{x} g'(t)\,dt$; again prop. 6 (V, p. 230) shows that f and g are comparable and that the comparison relation between f and g is the same as that between f' and g'. To complete the proof it remains to consider the case where g tends to $\pm\infty$ and f to a constant; one cannot then have $f' \succcurlyeq g'$, since then one could deduce from prop. 1 (V, p. 218) that the integral $\int_{x_0}^{+\infty} g'(t)\,dt$ was convergent; since f' and g' have been assumed comparable, one must have $f' \prec\!\!\prec g'$.

COROLLARY. *If two real functions f, g, are comparable of order $k \geqslant 1$, then they are comparable of order p for $0 \leqslant p \leqslant k$; further, the relation $f \prec\!\!\prec g$* (resp. $f \sim cg$) *implies $f^{(k)} \prec\!\!\prec g^{(k)}$* (resp. $f^{(k)} \sim cg^{(k)}$) *unless one of the derivatives $g^{(p)}$* $(0 \leqslant p \leqslant k-1)$ *is equivalent to a constant $\neq 0$.*

Indeed, since $f^{(k)}$ and $g^{(k)}$ have constant sign on an interval $[x_0, +\infty[$, it follows that $f^{(k-1)}$ and $g^{(k-1)}$ are monotone on this interval, so have constant sign on a neighbourhood of $+\infty$; further, prop. 7 of V, p. 232, shows that $f^{(k-1)}$ and $g^{(k-1)}$ are comparable, so the corollary follows from applying prop. 7 recursively.

Remarks. 1) The restriction on g in prop. 7 is essential. For example, one has $\dfrac{1}{x} \prec\!\!\prec 1+\dfrac{1}{x}$ even though the derivatives of the two sides are equivalent; similarly $1+\dfrac{1}{x} \sim 1+\dfrac{1}{x^2}$, but $1/x^2 \succ\!\!\succ 2/x^3$.

2) Though f and g are comparable of order k a function f_1 equivalent to f need not be comparable of order k to g; however it will be so if one assumes that f_1 is comparable of order k to f and that none of the derivatives $f^{(p)}$ $(0 \leqslant p \leqslant k-1)$ is equivalent to a nonzero constant.

3) If f and g are comparable of order k this need not be so for hf and hg even for a monotone function h as simple as $h(x) = x$ (V, p. 260, exerc. 3); likewise, $1/f$ and $1/g$ are not necessarily comparable of order k (V, p. 259, exerc. 1).

5. PRINCIPAL PART OF A PRIMITIVE

Let f be a regulated real function $\neq 0$ having constant sign on an interval $[a, +\infty[$; the following proposition allows one to obtain the principal part of the primitive $\int_x^{+\infty} f(t)\,dt$ if $\int_a^{+\infty} f(t)\,dt$ converges in certain cases, and of the primitive $\int_a^x f(t)\,dt$ if $\int_a^{+\infty} f(t)\,dt$ is infinite:

PROPOSITION 8. *Put* $\mathrm{F}(x) = \int_x^{+\infty} f(t)\,dt$ *if* $\int_a^{+\infty} f(t)\,dt$ *converges, and* $\mathrm{F}(x) = \int_a^x f(t)\,dt$ *if* $\int_a^{+\infty} f(t)\,dt$ *is infinite. Suppose that* $\log|f|$ *and* $\log x$ *are comparable of order* 1.

1° *If f is of finite order $\mu \neq -1$ with respect to x one has*

$$\mathrm{F}(x) \sim \frac{1}{|\mu+1|} x f(x). \tag{1}$$

2° *If f is of infinite order relative to x and if f/f' and x are comparable of order 1, then one has*

$$\mathrm{F}(x) \sim \frac{(f(x))^2}{|f'(x)|}. \tag{2}$$

One should note that the hypothesis implies that $f(x)$ has a determinate order relative to x (V, p. 219).

1° If f is of order $\mu \neq 0$ relative to x one has $\log|f| \sim \mu \log x$, so, since $\log|f|$ and $\log x$ are comparable of order 1, by prop. 7 of V, p. 232 one has $f'/f \sim$

μ/x, whence $xf' \sim \mu f$. If $\mu > -1$ one has $f(x) \succ\!\succ x^{\mu-\varepsilon}$ for every $\varepsilon > 0$, and hence (V, p. 229, prop. 2) the integral $\int_a^{+\infty} f(t)\,dt$ is infinite. One can write $\mathrm{F}(x) = \int_a^x f(t)\,dt = xf(x) - af(a) - \int_a^x tf'(t)\,dt$, whence again

$$\int_a^x \big(f(t) + tf'(t)\big)\,dt = xf(x) - af(a);$$

since $\mu \neq -1$ we have $f(x) + xf'(x) \sim (\mu+1)f(x)$, so (V, p. 230, prop. 6)

$$\int_a^x \big(f(t) + tf'(t)\big)\,dt \sim (\mu+1)\mathrm{F}(x),$$

which proves (1) in this case. If $\mu = 0$ one has likewise that $xf'(x) \prec\!\prec f(x)$, which again gives $f(x) + xf'(x) \sim f(x)$. One argues similarly when $\mu < -1$, in the case where $\int_a^{+\infty} f(t)\,dt$ converges.

2° If f is of order $+\infty$ relative to x one has $\log|f| \succ\!\succ \log x$, so (V, p. 232, prop. 7) $f'/f \succ\!\succ 1/x$, or again, putting $g(x) = f(x)/f'(x)$, $g(x) \prec\!\prec x$; further, since $f(x) \succ\!\succ x^\alpha$ for every $\alpha > 0$ the integral $\int_a^{+\infty} f(t)\,dt$ is infinite. One can write

$$\mathrm{F}(x) = \int_a^x f(t)\,dt = \int_a^x g(t)f'(t)\,dt = g(x)f(x) - g(a)f(a) - \int_a^x f(t)g'(t)\,dt;$$

since g and x are comparable of order 1, from the relation $g(x) \prec\!\prec x$ one can deduce (V, p. 232, prop. 7) that $g'(x) \prec\!\prec 1$, hence that $fg' \prec\!\prec f$, and consequently (V, p. 230, prop.6)

$$\int_a^x f(t)g'(t)\,dt \prec\!\prec \mathrm{F}(x),$$

which establishes the relation (2). The proof is similar when f is of order $-\infty$ relative to x, in the case where $\int_a^{+\infty} f(t)\,dt$ converges.

Let $\mathcal{E}$ be a comparison scale (for real x tending to $+\infty$) formed of nonzero real functions that are of constant sign on a neighbourhood of $+\infty$, such that $x \in \mathcal{E}$ and such that the product and quotient of two functions in $\mathcal{E}$ again belongs to $\mathcal{E}$ (V, p. 221 and p. 224). If a regulated function f with constant sign on a neighbourhood of $+\infty$ has a principal part cg relative to $\mathcal{E}$, then $\int_x^{+\infty} f(t)\,dt$ (resp. $\int_a^x f(t)\,dt$ according to the case) will be equivalent to $c\int_x^{+\infty} g(t)\,dt$ (resp. $c\int_a^x g(t)\,dt$); if the function g satisfies the hypotheses of prop. 8 of V, p. 233, and if (when the formula (2) of V, p. 233, applies) one knows a principal part of g' relative to $\mathcal{E}$, one will thus have a principal part of $\int_x^{+\infty} f(t)\,dt$ (resp. $\int_a^x f(t)\,dt$) relative to $\mathcal{E}$.

Examples. 1) The function $1/\log x$ is of order 0 relative to x and satisfies the conditions of prop. 8 of V, p. 233; thus

$$\int_a^x \frac{dt}{\log t} \sim \frac{x}{\log x}.$$

2) The function e^{x^2} is of order $+\infty$ relative to x and satisfies the conditions of prop. 8, so

$$\int_a^x e^{t^2}\,dt \sim \frac{1}{2x}e^{x^2}.$$

In the Appendix (V, p. 252) we shall define a set of functions for which prop. 7 and prop. 8 always apply.

Remark. Prop. 8 does not apply directly to a function f of order -1 relative to x. But then one can write $f(x) = f_1(x)/x$, with f_1 of order 0 relative to x. Suppose, for example, that $\int_a^{+\infty} f(t)\,dt$ is infinite; then

$$\mathrm{F}(x) = \int_a^x f(t)\,dt = \int_a^x \frac{1}{t} f_1(t)\,dt = \int_{\log a}^{\log x} f_1(e^u)\,du.$$

If the function $f_1(e^y)$ satisfies the hypotheses of prop. 8 and has an order $\neq -1$ relative to y (that is, if $f_1(x)$ has an order $\neq -1$ relative to $\log x$) the formulae (1) and (2) again allow us to obtain a principal part for $\mathrm{F}(x)$. For example, let $f(x) = \dfrac{\exp(\sqrt{\log x})}{x \log x}$; since $\exp(\sqrt{\log x})$ is of order 0 relative to x, f is of order -1; here one has $f_1(e^y) = e^{\sqrt{y}}/y$, and this function is of order $+\infty$ relative to y; prop. 8 applies and gives $\int_\alpha^y e^{\sqrt{u}}/u\,du \sim 2e^{\sqrt{y}}/\sqrt{y}$; on reverting to the variable x we have $\displaystyle\int_a^x \frac{\exp(\sqrt{\log t})}{t \log t}\,dt \sim \frac{2\exp(\sqrt{\log x})}{\sqrt{\log x}}$.

6. ASYMPTOTIC EXPANSION OF A PRIMITIVE

Let $\mathcal{E}$ be a comparison scale on a neighbourhood of $+\infty$ formed of real functions $\neq 0$ of constant sign on a neighbourhood of $+\infty$; let $\mathbf{f}$ be a regulated vector function defined on an interval $[a, +\infty[$, with values in a complete normed space E, admitting an asymptotic expansion

$$\mathbf{f} = \sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda g_\lambda + \mathbf{r}_\alpha$$

to precision g_α relative to $\mathcal{E}$. Suppose further that every primitive $\int_a^x g(t)\,dt$ of a function $g \in \mathcal{E}$ admits an asymptotic expansion with respect to $\mathcal{E}$. In these circumstances we shall see that one can obtain an asymptotic expansion of $\mathbf{F}(x) = \int_a^x \mathbf{f}(t)\,dt$ with respect to $\mathcal{E}$. We distinguish two cases:

1° $\int_a^{+\infty} g_\alpha(t)\,dt$ is infinite; then one has $\int_a^x \mathbf{r}_\alpha(t)\,dt \lll \int_a^x g_\alpha(t)\,dt$ (V, p. 230, prop. 6); by hypothesis one can obtain an asymptotic expansion of $\sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda \int_a^x g_\lambda(t)\,dt$ to a certain precision g_ρ (V, p. 222); if $c g_\sigma$ is the principal part of $\int_a^x g_\alpha(t)\,dt$ one will thus have an asymptotic expansion of $\int_a^x \mathbf{f}(t)\,dt$ to precision $g_{\min(\rho,\sigma)}$, with all the terms having indefinitely increasing norms.

2° $\int_a^{+\infty} g_\alpha(t)\,dt$ converges; let β then be the smallest of the indices $\lambda \leqslant \alpha$ such that $\mathbf{a}_\lambda \neq 0$ and such that $\int_a^{+\infty} g_\lambda(t)\,dt$ converges; the integral

$$\mathbf{C} = \int_a^{+\infty} \Big(\mathbf{f}(t) - \sum_{\lambda < \beta} \mathbf{a}_\lambda g_\lambda(t)\Big)\,dt$$

is then convergent, and one can write

$$\mathbf{F}(x) = \sum_{\lambda < \beta} \mathbf{a}_\lambda \int_a^x g_\lambda(t)\,dt + \mathbf{C} - \sum_{\beta \leqslant \lambda \leqslant \alpha} \mathbf{a}_\lambda \int_x^{+\infty} g_\lambda(t)\,dt - \int_x^{+\infty} \mathbf{r}_\alpha(t)\,dt.$$

Then $\int_x^{+\infty} \mathbf{r}_\alpha(t)\,dt \prec\!\prec \int_x^{+\infty} g_\alpha(t)\,dt$; if cg_σ is the principal part of $\int_x^{+\infty} g_\alpha(t)\,dt$, and if one has an asymptotic expansion of

$$\sum_{\lambda<\beta} \mathbf{a}_\lambda \int_a^x g_\lambda(t)\,dt + \mathbf{C} - \sum_{\beta\leqslant\lambda\leqslant\alpha} \mathbf{a}_\lambda \int_x^{+\infty} g_\lambda(t)\,dt$$

to precision g_ρ, one will as a result have an asymptotic expansion of $\mathbf{F}$ to precision $g_{\min(\rho,\sigma)}$.

So it all amounts to finding asymptotic expansions with respect to $\mathcal{E}$ of *primitives of functions in* $\mathcal{E}$. We have seen how, subject to certain hypotheses on $\mathcal{E}$, prop. 8 of V, p. 233 gives the principal part of such a primitive. Further, the proof of prop. 8 gives the expression for the difference of the two sides of formula (1) (resp. (2)) of V, p. 233, in the form of a primitive of the function $\dfrac{1}{|\mu+1|}\big(xf'(x)+f(x)\big) - f(x)$ (resp. $f(x)g'(x)$ with $g = f/f'$); on forming the principal part of this new primitive, as an asymptotic expansion of the right-hand side of (1) (resp. (2)), one obtains the right-hand side of the sought-for expansion (see V, p. 247-255).

Examples. 1) Let $f(x) = 1/\log x$ $(x > 1)$; we have seen that $\int_a^x dt/\log t \sim x/\log x$, and that the difference $\int_a^x \frac{dt}{\log t} - \frac{x}{\log x}$ is a primitive of $1/(\log x)^2$; one can apply prop. 8 again to this function, so obtaining $\int_a^x dt/(\log t)^2 \sim x/(\log x)^2$. By recursion one thus obtains the expansion

$$\int_a^x \frac{dt}{\log t} = \frac{x}{\log x} + \frac{x}{(\log x)^2} + \frac{2x}{(\log x)^3} + \cdots + (n-1)!\frac{x}{(\log x)^n} + o\left(\frac{x}{(\log x)^n}\right).$$

Note that, irrespective of n, all the terms of this expansion tend to $+\infty$ with x.

2) Let $f(x) = \dfrac{e^x}{x^2+1}$; one can write $f(x) = \dfrac{e^x}{x^2} - \dfrac{e^x}{x^4} + o_1\left(\dfrac{e^x}{x^4}\right)$. Prop. 8 gives the expansions

$$\int_a^x \frac{e^t}{t^2}\,dt = \frac{e^x}{x^2} + \frac{2e^x}{x^3} + \frac{6e^x}{x^4} + o_2\left(\frac{e^x}{x^4}\right), \quad \int_a^x \frac{e^t}{t^4}\,dt = \frac{e^x}{x^4} + o_3\left(\frac{e^x}{x^4}\right)$$

whence, on adding,

$$\int_a^x \frac{e^t}{t^2+1}\,dt = \frac{e^x}{x^2} + 2\frac{e^x}{x^3} + 5\frac{e^x}{x^4} + o_4\left(\frac{e^x}{x^4}\right).$$

§ 4. APPLICATION TO SERIES WITH POSITIVE TERMS

1. CONVERGENCE CRITERIA FOR SERIES WITH POSITIVE TERMS

In this section by a *series with positive terms* we shall understand (by an abuse of language) a series (u_n) of real terms such that $u_n \geqslant 0$ *starting from some particular* *value of* n. Everything we shall say about such series will extend immediately by a change of sign to series all of whose terms are $\leqslant 0$ *starting from some particular value of* n. We have seen (II, p. 64, *Example* 3) that to every sequence $(\mathbf{u}_n)_{n\geqslant 1}$ of points in a normed space E one can associate a step function $\mathbf{u}$ defined on $[1, +\infty[$ by the conditions $\mathbf{u}(x) = \mathbf{u}_n$ for $n \leqslant x < n+1$: then the series $(\mathbf{u}_n)$ converges if and only if the integral $\int_1^{+\infty} \mathbf{u}(t)\,dt$ converges.

Let (u_n) and (v_n) be two series with positive terms, and u and v the associated step functions: the relation $u_n \leqslant v_n$ for $n \geqslant n_0$ is equivalent to $u(x) \leqslant v(x)$ for $x \geqslant n_0$. Thus each of the relations $u_n \preccurlyeq v_n$, $u_n \lll v_n$, $u_n \sim v_n$ is equivalent to $u(x) \preccurlyeq v(x)$, $u(x) \lll v(x)$ and $u(x) \sim v(x)$ respectively; this remark allows us to translate propositions 1 (V, p. 228) and 6 (V, p. 230) as follows:

PROPOSITION 1. *Let* (u_n) *and* (v_n) *be two series with positive terms. If* $u_n \preccurlyeq v_n$ *and if the series* (v_n) *converges, then* (u_n) *converges; if* $u_n \succcurlyeq v_n$ *and if* $\sum\limits_{n=1}^{\infty} v_n = +\infty$, *then* $\sum\limits_{n=1}^{\infty} u_n = +\infty$.

PROPOSITION 2. *Let* (u_n) *and* (v_n) *be two series with positive terms*:

1° *If the series* (v_n) *converges then the relation* $u_n \lll v_n$ (resp. $u_n \sim v_n$) *implies* $\sum\limits_{p=n}^{\infty} u_p \lll \sum\limits_{p=n}^{\infty} v_p$ (resp. $\sum\limits_{p=n}^{\infty} u_p \sim \sum\limits_{p=n}^{\infty} v_p$).

2° *If* $\sum\limits_{n=1}^{\infty} v_n = +\infty$ *then the relation* $u_n \lll v_n$ (resp. $u_n \sim v_n$) *implies* $\sum\limits_{p=1}^{n} u_p \lll \sum\limits_{p=1}^{n} v_p$ (resp. $\sum\limits_{p=1}^{n} u_p \sim \sum\limits_{p=1}^{n} v_p$).

One obtains convenient convergence criteria by taking for the comparison series (v_n) in prop. 1 a series whose terms are of the form $v_n = f(n)$, where f is a function $\geqslant 0$ *defined for every real number* $x > x_0$ and decreasing on the interval $[x_0, +\infty[$; indeed:

PROPOSITION 3 (Cauchy-Maclaurin criterion). *If* f *is a function* $\geqslant 0$ *and decreasing on* $[x_0, +\infty[$, *then the series with general term* $v_n = f(n)$ *converges if and only if the integral* $\int_{x_0}^{+\infty} f(t)\,dt$ *converges*.

To prove this it is sufficient to note that if v is the step function associated with the series (v_n) then $v(x+1) \leqslant f(x) \leqslant v(x)$ for every $x \geqslant x_0$ since f is decreasing; the proposition is thus a consequence of the comparison principle (II, p. 66, prop. 3).

Since the functions which feature in the logarithmic convergence criteria for integrals (V, p. 229, prop. 2, 3 and 4) are decreasing on an interval $[x_0, +\infty[$, applying prop. 1 and 3 of V, p. 237, gives the following criteria:

PROPOSITION 4 ("logarithmic criterion of order 0"). *Let* (u_n) *be a series with positive terms; if* $u_n \preccurlyeq n^{\mu}$ *for some* $\mu < -1$ *then the series* (u_n) *converges; if* $u_n \succcurlyeq n^{\mu}$ *for some* $\mu \geqslant -1$ *then the series* (u_n) *has an infinite sum.*

PROPOSITION 5 ("logarithmic criterion of order p"). *Let* (u_n) *be a series with positive terms. If* $u_n \preccurlyeq \dfrac{1}{nl_1(n)l_2(n)\dots l_{p-1}(n)\big(l_p(n)\big)^{\mu}}$ *for some* $\mu > 1$ *then the series* (u_n) *converges; if* $u_n \succcurlyeq \dfrac{1}{nl_1(n)l_2(n)\dots l_{p-1}(n)\big(l_p(n)\big)^{\mu}}$ *for some* $\mu \leqslant 1$ *then the series* (u_n) *has an infinite sum.*

If $0 \leqslant q < 1$ one has $q^n \preccurlyeq n^{-\mu}$ for any $\mu > 0$; applying the logarithmic criterion of order 0 again proves the convergence of the *geometric series* $\sum_{n=0}^{\infty} q^n$ for $|q| < 1$ (*Gen. Top.*, IV, p. 364). If one applies prop. 1 with $v_n = q^n$ one obtains a criterion which may be put in the following form ("*Cauchy's criterion*"): *Let* (u_n) *be a series with positive terms; if* $\limsup_{n\to\infty}(u_n)^{1/n} < 1$ *then the series* (u_n) *converges; if* $\limsup_{n\to\infty}(u_n)^{1/n} > 1$ *then the series* (u_n) *has infinite sum*. Indeed, if $\limsup_{n\to\infty}(u_n)^{1/n} = a < 1$ then $u_n \preccurlyeq q^n$ for every q such that $a < q < 1$. If, on the other hand, $\limsup_{n\to\infty}(u_n)^{1/n} = a > 1$ then $u_n \geqslant q^n > 1$ for infinitely many values of n, for any q such that $1 < q < a$; since u_n does not tend to 0 one has $\sum_{n=1}^{\infty} u_n = +\infty$.

This criterion is very useful in the theory of *entire series,* which we shall study later; but it even so it does not permit one to decide the convergence of the series $(1/n^{\alpha})$, in other words, its field of application is much more restricted than that of the logarithmic criteria.

2. ASYMPTOTIC EXPANSION OF THE PARTIAL SUMS OF A SERIES

For x real tending to $+\infty$ let $\mathcal{E}$ be a comparison scale formed by functions each of which is defined on a *whole interval* $[x_0, +\infty[$ (depending on the function) and is $\geqslant 0$ on this interval. Let $(\mathbf{u}_n)$ be a series whose terms belong to a complete normed space E, such that $\mathbf{u}_n$ admits an asymptotic expansion to precision g_{α} with respect to the scale $\mathcal{E}'$ of restrictions to $\mathbf{N}$ of the functions in $\mathcal{E}$:

$$\mathbf{u}_n = \sum_{\lambda\leqslant\alpha} \mathbf{a}_{\lambda} g_{\lambda}(n) + \mathbf{r}_{\alpha}(n).$$

Suppose that every partial sum $\sum_{m=1}^{n} g(m)$, where $g \in \mathcal{E}$, admits an asymptotic expansion relative to $\mathcal{E}'$. One can then obtain an asymptotic expansion of the $\mathbf{s}_n = \sum_{m=1}^{n} \mathbf{u}_m$ with respect to $\mathcal{E}'$; again we distinguish two cases:

1° $\sum_{n=1}^{\infty} g_\alpha(n) = +\infty$. Then (V, p. 237, prop. 2) one has $\sum_{m=1}^{n} \mathbf{r}_\alpha(m) \prec\!\prec \sum_{m=1}^{n} g_\alpha(m)$; by hypothesis one can obtain an asymptotic expansion of

$$\sum_{\lambda \leqslant \alpha} \mathbf{a}_\lambda \left(\sum_{m=1}^{n} g(m) \right)$$

(V, p. 222) to a certain precision g_σ; if $cg_\sigma(n)$ is the principal part of $\sum_{m=1}^{n} g_\alpha(m)$, one will have an asymptotic expansion of $\mathbf{s}_n$ to precision $g_{\min(\rho,\sigma)}$.

2° $\sum_{n=1}^{\infty} g_\alpha(n)$ converges; then let β be the smallest of the indices $\lambda \leqslant \alpha$ such that $\mathbf{a}_\lambda \neq 0$ and such that $\sum_{n=1}^{\infty} g_\lambda(n)$ converges; the series

$$\mathbf{C} = \sum_{n=1}^{\infty} \left(\mathbf{u}_n - \sum_{\lambda < \beta} \mathbf{a}_\lambda g_\lambda(n) \right)$$

then converges absolutely, and one can write

$$\mathbf{s}_n = \sum_{\lambda < \beta} \mathbf{a}_\lambda \left(\sum_{m=1}^{n} g_\lambda(m) \right) + \mathbf{C} - \sum_{\beta \leqslant \lambda \leqslant \alpha} \mathbf{a}_\lambda \left(\sum_{m=n+1}^{\infty} g_\lambda(m) \right) - \sum_{m=n+1}^{\infty} \mathbf{r}_\alpha(m).$$

Further, $\sum_{m=n+1}^{\infty} \mathbf{r}_\alpha(m) \prec\!\prec \sum_{m=n+1}^{\infty} g_\alpha(m)$; if $cg_\sigma(n)$ is the principal part of $\sum_{m=n+1}^{\infty} g_\alpha(m)$, and if one has an asymptotic expansion of

$$\sum_{\lambda < \beta} \mathbf{a}_\lambda \left(\sum_{m=1}^{n} g_\lambda(m) \right) + \mathbf{C} - \sum_{\beta \leqslant \lambda \leqslant \alpha} \mathbf{a}_\lambda \left(\sum_{m=n+1}^{\infty} g_\lambda(m) \right)$$

to precision g_ρ one thus obtains an asymptotic expansion of $\mathbf{s}_n$ to precision $g_{\min(\rho,\sigma)}$.

One is thus led to the particular case of series $\big(g(n)\big)$ where $g \in \mathcal{E}$. We shall see how, subject to certain conditions, one can straight away obtain a principal part of $s_n = \sum_{m=1}^{n} g(m)$ (when $\sum_{n=1}^{\infty} g(n) = +\infty$) or of $r_n = \sum_{m=n+1}^{\infty} g(m)$ (when $\sum_{n=1}^{n} g(n) < +\infty$).

PROPOSITION 6. *Let g be a real function, > 0 and monotone, defined on an interval $[x_0, +\infty[$ (where $x_0 \leqslant 1$), and such that $\log g$ and x are comparable to order 1.*

1° *If g is of infinite order relative to e^x one has*

$$s_n = \sum_{m=1}^{n} g(m) \sim g(n) \qquad \textit{if} \quad \sum_{n=1}^{\infty} g(n) = +\infty; \tag{1}$$

$$r_n = \sum_{m=n+1}^{\infty} g(m) \sim g(n+1) \qquad \textit{if} \quad \sum_{n=1}^{\infty} g(n) < +\infty. \tag{2}$$

2° *If g is of finite order μ with respect to e^x one has*

$$s_n = \sum_{m=1}^{n} g(m) \sim \frac{\mu}{1-e^{-\mu}} \int_{x_0}^{n} g(t)\,dt \qquad \textit{if} \quad \sum_{n=1}^{\infty} g(n) = +\infty; \tag{3}$$

$$r_n = \sum_{m=n+1}^{\infty} g(m) \sim \frac{\mu}{1-e^{-\mu}} \int_{n}^{\infty} g(t)\,dt \qquad \textit{if} \quad \sum_{n=1}^{\infty} g(n) < +\infty \tag{4}$$

(the number $\dfrac{\mu}{1-e^{-\mu}}$ is to be replaced by 1 in (3) and (4) when $\mu = 0$).

1° If g is of order $+\infty$ relative to e^x one has $\log g \succ\!\succ x$ whence $g'/g \succ\!\succ 1$, or $g' \succ\!\succ g$, by the hypothesis; g is therefore increasing and tends to $+\infty$ with x, whence $\sum_{n=1}^{\infty} g(n) = +\infty$. If u is the step function associated with the series $\big(g(n)\big)$ (V, p. 237), one has $u(x) \leqslant g(x)$ starting from a certain value of x, so $u \preccurlyeq g$ and consequently

$$s_{n-1} = \int_1^n u(t)\,dt \preccurlyeq \int_1^n g(t)\,dt \prec\!\prec \int_1^n g'(t)\,dt \sim g(n);$$

since $s_n = s_{n-1} + g(n)$ one has $s_n \sim g(n)$. The proof is similar when g is of order $-\infty$ relative to e^x; we thus obtain formula (2).

2° If g is of finite order μ relative to e^x one can write $g(x) = e^{\mu x} h(x)$ where h is of order 0 relative to e^x; further, by hypothesis, $\log g \sim \mu x$ for $\mu \neq 0$ ($\log g \prec\!\prec x$ for $\mu = 0$) implies $h' \prec\!\prec h$. Suppose first that $\sum_{n=1}^{\infty} g(n) = +\infty$ (which implies that $\mu \geqslant 0$; the converse always holds if $\mu > 0$, since then $g(x)$ tends to $+\infty$ with x); let us evaluate the principal part of $\int_{n-1}^{n} g(t)\,dt$. One can write

$$\begin{aligned}
\int_{n-1}^{n} g(t)\,dt &= \int_{n-1}^{n} e^{\mu t} h(t)\,dt \\
&= h(n) \int_{n-1}^{n} e^{\mu t}\,dt + \int_{n-1}^{n} e^{\mu t}\big(h(t) - h(n)\big)\,dt \\
&= \frac{1-e^{-\mu}}{\mu} g(n) + \int_{n-1}^{n} e^{\mu t}\big(h(t) - h(n)\big)\,dt.
\end{aligned}$$

Now, the relation $h' \prec\!\prec h$ implies that for every $\varepsilon > 0$ there is an n_0 such that the relation $x \geqslant n_0$ implies that $\left|h'(x)/h(x)\right| \leqslant \varepsilon$; one deduces from the mean value theorem that $-\varepsilon \leqslant \log|h(t)/h(n)| \leqslant \varepsilon$, for $n-1 \leqslant t \leqslant n$, if $n \geqslant n_0$, whence

$$|h(t) - h(n)| \leqslant \big(e^{\varepsilon} - 1\big)h(n)$$

and consequently

$$\left|\int_{n-1}^{n} e^{\mu t}\big(h(t) - h(n)\big)\,dt\right| \leqslant \big(e^{\varepsilon} - 1\big)e^{\mu n}h(n) = \big(e^{\varepsilon} - 1\big)g(n)$$

since $e^{\mu t}$ is increasing. Since $e^{\varepsilon} - 1$ becomes arbitrarily small with ε one sees that one can write

$$\int_{n-1}^{n} g(t)\,dt = \frac{1 - e^{-\mu}}{\mu}\,g(n) + o\big(g(n)\big)$$

$\left(\dfrac{1 - e^{-\mu}}{\mu}\text{ being replaced by 1 when } \mu = 0\right)$. The proposition is then a consequence of prop. 2 of V, p. 237. One argues similarly when $\sum_{n=1}^{\infty} g(n)$ is finite.

By applying prop. 6 of V, p. 239, repeatedly one can then sometimes obtain an *asymptotic expansion* for $s_n = \sum_{m=1}^{n} g(m)$. First suppose that g is of order $+\infty$ relative to e^x; for every *fixed* value of p one can write, by prop. 6,

$$s_n = g(n) + g(n-1) + \cdots + g(n-p) + o\big(g(n-p)\big)$$

and it suffices to expand (relative to $\mathcal{E}'$) each of the functions $g(n-k)$ $(0 \leqslant k \leqslant p)$, limiting the precision of the expansions to the principal part of $g(n-p)$, to obtain an expansion for the s_n.

Example. Let $g(x) = x^x = \exp(x\log x)$, of order $+\infty$ relative to e^x. Taking $p = 2$ one has

$$(n-1)\log(n-1) = (n-1)\log n - 1 + \frac{1}{2n} + o\left(\frac{1}{n}\right)$$

whence (V, p. 225)

$$(n-1)^{n-1} = \frac{1}{e}n^{n-1} + \frac{1}{2e}n^{n-2} + o_1\left(n^{n-2}\right)$$

and similarly

$$(n-2)^{n-2} = \frac{1}{e^2}n^{n-2} + o_2\left(n^{n-2}\right);$$

consequently

$$s_n = n^n + \frac{1}{e}n^{n-1} + \left(\frac{1}{2e} + \frac{1}{e^2}\right)n^{n-2} + o_3\left(n^{n-2}\right).$$

One proceeds similarly (for r_n) when g is of order $-\infty$ relative to e^x.

Now if g is of finite order μ relative to e^x, and if, for example, $\sum_{n=1}^{\infty} g(n) = +\infty$, one can write

$$s_n = \frac{\mu}{1-e^{-\mu}} \int_1^n g(t)\,dt + \sum_{m=1}^{n} f_1(m)$$

where $f_1(n) = g(n) - \dfrac{\mu}{1-e^{-\mu}} \displaystyle\int_1^n g(t)\,dt \prec\prec g(n)$ by prop. 6 of V, p. 239. If one has a principal part $cg_1(n)$ of $f_1(n)$ relative to $\mathcal{E}'$, and if one can again apply prop. 6 to the function g_1 one will obtain a primitive equivalent to $\sum_{m=1}^{n} f_1(m)$ if $\sum_{n=1}^{\infty} g_1(n) = +\infty$, and equivalent to $\sum_{m=n+1}^{\infty} f_1(m)$ in the opposite case (in the latter case one writes $\sum_{m=1}^{n} f_1(m) = \mathrm{C} - \sum_{m=n+1}^{\infty} f_1(m)$, with $\mathrm{C} = \sum_{n=1}^{\infty} f_1(n)$).

Step by step one can thus eventually obtain an expression for s_n as the sum of a certain number of primitives each of which is negligible with respect to the previous, of a term remaining negligible relative to the last primitive written, and finally a constant (the case where the remainder term tends to 0). It then remains to expand each of the primitives obtained with respect to $\mathcal{E}'$ (*cf.* V, p. 235).

Example. Let $g(n) = \dfrac{1}{n}$; then

$$s_n = \sum_{m=1}^{n} \frac{1}{m} \sim \int_1^n \frac{dt}{t} = \log n$$

then

$$\frac{1}{n} - (\log n - \log(n-1)) \sim -\frac{1}{2n^2}$$

whence

$$s_n = \log n + \gamma + \frac{1}{2n} + o\left(\frac{1}{n}\right).$$

The constant γ which appears in this formula plays an important rôle in Analysis (*cf.* chap. VI and VII); it is known as *Euler's constant;* one has

$$\gamma = 0.577\,215\,664\,\ldots$$

to within $1/10^9$.

We shall see in VI, p. 288, how the *Euler-Maclaurin summation formula* gives an asymptotic expansion of *arbitrary* order for s_n (or for r_n) in the most important cases.

3. ASYMPTOTIC EXPANSION OF THE PARTIAL PRODUCTS OF AN INFINITE PRODUCT

One knows (*Gen. Top.*, V, p. 22 and 23) that for the infinite product with general factor $1 + u_n$ ($u_n > -1$) to be convergent (resp. commutatively convergent) it is necessary and sufficient that the series with general term $\log(1 + u_n)$ should be convergent (resp. commutatively convergent), and that one then has the relation

$$\log \prod_{n=1}^{\infty} (1 + u_n) = \sum_{n=1}^{\infty} \log(1 + u_n).$$

When the infinite product converges one knows that u_n tends to 0; thus $\log(1 + u_n) \sim u_n$; now one knows that for a series of real numbers to be commutatively convergent it is necessary and sufficient that it should be absolutely convergent (*Gen. Top.*, IV, p. 372, prop. 5); by prop. 1 one thus recovers the fact that the product with general factor $1 + u_n$ is commutatively convergent if and only if the series with general term u_n is absolutely convergent (*Gen. Top.*, IV, p. 368, th. 4).

A similar argument applies to an infinite product whose general factor is a *complex* number $1 + u_n$ ($u_n \neq -1$). Indeed, for such a product to be commutatively convergent it is necessary and sufficient (*Gen. Top.*, VIII, p. 115, prop. 2) that the infinite product with general factor $|1 + u_n|$ should be so, and further, if θ_n is the amplitude of $1 + u_n$ (taken between $-\pi$ and $+\pi$), that the series of the θ_n should be commutatively convergent. Since u_n tends to 0, $\log(1 + u_n)$ is defined starting from some particular value of n (III, p. 100) and one has

$$\log(1 + u_n) = \log|1 + u_n| + i\theta_n;$$

thus, for the product with general factor $1+u_n$ to be commutatively convergent it is necessary and sufficient that the series with general term $|\log(1 + u_n)|$ be absolutely convergent (*Gen. Top.*, VII, p. 84, th. 1); now $\log(1 + u_n) \sim u_n$ (I, p. 18, prop. 5), so one again obtains the condition that the series with general term u_n should be absolutely convergent (*Gen. Top.*, VIII, p. 116, th. 1).

The relation between infinite products and series of real numbers sometimes allows one to obtain an asymptotic expansion for the partial product $p_n = \prod_{k=1}^{n}(1+u_k)$; it suffices to have an asymptotic expansion for the partial sum $s_n = \sum_{k=1}^{n} \log(1 + u_k)$, then to expand $p_n = \exp(s_n)$; one is thus brought back to the two problems examined earlier (V, p. 238 and p. 226).

Example: Stirling's formula. Let us seek an asymptotic expansion for $n!$; this leads us to expand $s_n = \sum_{p=1}^{n} \log p$, and then $\exp(s_n)$. The method of n° 2 gives successively

$$s_n = \sum_{p=1}^{n} \log p \sim \int_1^n \log t \, dt = n \log n - n + 1$$

then

$$\log n - \int_{n-1}^{n} \log t \, dt = \log n - (n \log n - (n-1)\log(n-1) - 1) \sim \frac{1}{2n}$$

whence

$$s_n = n \log n - n + \tfrac{1}{2} \log n + o(\log n).$$

Then

$$\log n - \int_{n-1}^{n} \log t \, dt - \tfrac{1}{2}(\log n - \log(n-1)) \sim -\frac{1}{12n^2}$$

whence

$$s_n = n \log n - n + \tfrac{1}{2} \log n + k + \frac{1}{12n} + o_1\left(\frac{1}{n}\right) \qquad (k \text{ constant})$$

and one finally deduces (V, p. 226)

$$n! = e^k \, n^{n+1/2} \, e^{-n} \left(1 + \frac{1}{12n} + o_2(\left(\frac{1}{n}\right)\right). \tag{5}$$

We shall show in VII, p. 322, that $e^k = \sqrt{2\pi}$. The formula (5) (with this value of k) is called *Stirling's formula*. In the same way, for every real number a not an integer > 0, one shows that

$$(a+1)(a+2)\ldots(a+n) \sim \mathrm{K}(a) \, n^{n+a+\frac{1}{2}} \, e^{-n}. \tag{6}$$

We shall determine the function $\mathrm{K}(a)$ too (VII, p. 18). From formulae (5) and (6) one derives in particular that

$$\binom{a}{n} \sim (-1)^n \, \varphi(a) \, n^{-a-1} \tag{7}$$

for every real number a not an integer > 0, where $\varphi(a)$ is a function of a which will be specified in VII, p. 322.

4. APPLICATION: CONVERGENCE CRITERIA OF THE SECOND KIND FOR SERIES WITH POSITIVE TERMS

Quite often one meets series (u_n) for which $u_n > 0$ from a certain point on, and u_{n+1}/u_n has an asymptotic expansion which is easy to determine. It is convenient, for such series, to have criteria (called *criteria of the second kind*) allowing one to determine whether the series is convergent from the form of u_{n+1}/u_n alone. The following is such a criterion:

PROPOSITION 7 ("Raabe's test"). *Let (u_n) be a series with terms > 0 from some point on. If, from a certain stage, $u_{n+1}/u_n \leqslant 1 - \dfrac{\alpha}{n}$ for some $\alpha > 1$, then the series (u_n) converges; if, from a certain point on, $u_{n+1}/u_n \geqslant 1 - \dfrac{1}{n}$, then the series (u_n) has infinite sum.*

Indeed, if $u_{n+1}/u_n \leqslant 1 - \dfrac{\alpha}{n}$ with $\alpha > 1$, for all $n \geqslant n_0$, then one has $u_n \preccurlyeq p_n = \prod\limits_{k=n_0}^{n} \left(1 - \dfrac{\alpha}{k}\right)$. Now, $\log\left(1 - \dfrac{\alpha}{n}\right) = -\dfrac{\alpha}{n} - \dfrac{\alpha^2}{2n^2} + o\left(\dfrac{1}{n^2}\right)$, whence $\log p_n =$

$-\alpha \log n + k + o(1/n)$ (k constant), and $p_n \sim e^k \dfrac{1}{n^\alpha}$; since $\alpha > 1$ the logarithmic criterion of order 0 allows one to finish.

If, on the other hand, $u_{n+1}/u_n \geqslant 1 - \dfrac{1}{n}$ from a certain point, then the same calculation proves that $u_n \succcurlyeq \dfrac{1}{n}$, whence the proposition.

One can prove in the same way, using the logarithmic criteria of order > 0, the following criterion of the second kind:

PROPOSITION 8. *Let (u_n) be a series with terms > 0 from a certain point on. If, from a certain point, one has*

$$\frac{u_{n+1}}{u_n} \leqslant 1 - \frac{1}{n} - \frac{1}{nl_1(n)} - \dots - \frac{1}{nl_1(n)l_2(n)\dots l_{p-1}(n)} - \frac{\alpha}{nl_1(n)l_2(n)\dots l_p(n)}$$

for some $\alpha > 1$, then the series (u_n) converges; if, from some point on, one has

$$\frac{u_{n+1}}{u_n} \geqslant 1 - \frac{1}{n} - \frac{1}{nl_1(n)} - \dots - \frac{1}{nl_1(n)l_2(n)\dots l_p(n)}$$

then the series (u_n) has an infinite sum.

Example. Consider the *hypergeometric series,* with general term

$$u_n = \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\,\beta(\beta+1)\dots(\beta+n-1)}{1.2\dots n.\,\gamma(\gamma+1)\dots(\gamma+n-1)}$$

where α, β, γ are arbitrary real numbers, not integers $\leqslant 0$; it is clear that u_n is > 0 from a certain stage, or < 0 from a certain stage on. One has

$$\begin{aligned}\frac{u_{n+1}}{u_n} &= \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)} \\ &= \left(1 + \frac{\alpha+\beta}{n} + \frac{\alpha\beta}{n^2}\right)\left(1 + \frac{\gamma+1}{n} + \frac{\gamma}{n^2}\right)^{-1} \\ &= 1 + \frac{\alpha+\beta-\gamma-1}{n} \\ &\quad + \frac{\alpha\beta - (\alpha+\beta)(\gamma+1) + \gamma^2 + \gamma + 1}{n^2} + o\left(\frac{1}{n^2}\right).\end{aligned}$$

Raabe's test shows that the series converges for $\alpha + \beta < \gamma$, and has infinite sum for $\alpha + \beta > \gamma$; when $\alpha + \beta = \gamma$ the series also has an infinite sum, as is shown by prop. 8.

Remarks. 1) As a particular instance of Raabe's test one sees that if $\limsup_{n\to\infty} u_{n+1}/u_n < 1$ the series (u_n) converges; if, on the other hand, $\liminf_{n\to\infty} u_{n+1}/u_n > 1$, the series (u_n) has an infinite sum (*d'Alembert's test*).

2) The criteria of the second kind can be applied only to series whose general term behaves in a very regular way as n tends to $+\infty$; in other words, their scope is much more restricted than those of the logarithmic criteria, and it would be a blunder to try to use them beyond the special cases to which they are particularly suited. For example,

for the series (u_n) defined by $u_{2m} = 2^{-m}$, $u_{2m+1} = 3^{-m}$ one has $u_{2m+1}/u_{2m} = \left(\frac{2}{3}\right)^m$, $u_{2m+2}/u_{2m+1} = \frac{1}{2}\left(\frac{3}{2}\right)^m$; the first of these ratios tends to 0 and the second to $+\infty$ as m increases indefinitely, so no criterion of the second kind is applicable; nevertheless, since $u_n \preccurlyeq 2^{-n/2}$, it is immediate that the series converges.

Even when u_{n+1}/u_n has a simple expression, a direct evaluation of a principal part for u_n often leads to a result as quickly as the criteria of the second kind. For example, for the hypergeometric series, Stirling's formula shows immediately that $u_n \sim a\, n^{\alpha+\beta-\gamma-1}$, where a is a constant $\neq 0$, and the logarithmic criterion of order 0 is then applicable.

APPENDIX
Hardy Fields. (H) Functions

1. HARDY FIELDS

Let $\mathfrak{F}$ be the filter base on $\mathbf{R}$ formed by the intervals of the form $[x_0, +\infty[$. Recall that we have defined an equivalence relation R_∞: on the set $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ of real functions defined on sets belonging to $\mathfrak{F}$ "there exists a set $\mathrm{M} \in \mathfrak{F}$ such that $f(x) = g(x)$ on M" (V, p. 211), and that the quotient set $\mathcal{H}(\mathfrak{F}, \mathbf{R})/\mathrm{R}_\infty$ is endowed with the structure of a *ring* with unit element.

DEFINITION 1. *Given a subset* $\mathfrak{K}$ *of* $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ *one says that* $\mathfrak{K}/\mathrm{R}_\infty$ (the canonical image of $\mathfrak{K}$ in $\mathcal{H}(\mathfrak{F}, \mathbf{R})/\mathrm{R}_\infty$) *is a Hardy field, if* $\mathfrak{K}$ *satisfies the following conditions:*

1° $\mathfrak{K}/\mathrm{R}_\infty$ *is a subfield of the ring* $\mathcal{H}(\mathfrak{F}, \mathbf{R})/\mathrm{R}_\infty$.

2° *Every function in* $\mathfrak{K}$ *is continuous and differentiable on an interval* $[a, +\infty[$ (depending on the function), *and the class with respect to* R_∞ *of its derivative belongs to* $\mathfrak{K}/\mathrm{R}_\infty$.

The hypothesis that $\mathfrak{K}/\mathrm{R}_\infty$ is a *field* is equivalent to the following conditions: if $f \in \mathfrak{K}$ and $g \in \mathfrak{K}$ then $f + g$ and fg are equal to functions in $\mathfrak{K}$ on some set in $\mathfrak{F}$; further, if f is not identically zero on a set in $\mathfrak{F}$ then there exists a set M in $\mathfrak{F}$ on which f *does not vanish*, and $1/f$ is equal to a function from $\mathfrak{K}$ on M; by condition 2° one can always assume that M is taken so that f is *continuous* on M, and consequently *has constant sign* on this interval.

By an abuse of language, if $\mathfrak{K}$ is such that $\mathfrak{K}/\mathrm{R}_\infty$ is a Hardy field we shall say in what follows that $\mathfrak{K}$ is itself a *Hardy field*.

Examples. 1) Every Hardy field contains the field of *rational constants* (the smallest field of characteristic 0, *cf. Alg.*, V, §1), which one can identify with the field $\mathbf{Q}$; moreover, since two constants cannot be congruent modulo R_∞ unless they are equal, $\mathbf{Q}/\mathrm{R}_\infty$ is identical to $\mathbf{Q}$. The *real constants* also form a Hardy field, which one can identify with $\mathbf{R}$.

2) A very important example of a Hardy field is the *set of rational functions with real coefficients,* which we shall denote by $\mathbf{R}(x)$, by an abuse of language; if $f(x) = p(x)/q(x)$ is a rational function with real coefficients, not identically zero, it is continuous, differentiable, and $\neq 0$ on an interval $[a, +\infty[$, where a is strictly greater than the largest of the real roots of the polynomials $p(x)$ and $q(x)$; thus every element of $\mathbf{R}(x)/\mathrm{R}_\infty$ other than 0 is invertible. We note again that two rational

functions cannot be congruent modulo R_∞ unless they are equal, so $\mathbf{R}(x)/\mathrm{R}_\infty$ can again be identified with $\mathbf{R}(x)$.

2. EXTENSION OF A HARDY FIELD

Given a Hardy field $\mathfrak{K}$ we shall see how one can form new Hardy fields $\mathfrak{K}' \supset \mathfrak{K}$ such that $\mathfrak{K}'/\mathrm{R}_\infty$ is obtained by *adjoining* (in the algebraic sense of the term, *cf. Alg.*, V, §2) new elements, of a form which we shall make precise, to $\mathfrak{K}/\mathrm{R}_\infty$.

Lemma 1. Let $a(x)$, $b(x)$ be two continuous real functions that do not change sign on an interval $[x_0, +\infty[$. If, on this interval, the function $y(x)$ is continuous and differentiable, and satisfies the identity

$$y' = ay + b \tag{1}$$

then there exists an interval $[x_1, +\infty[$ on which y does not change sign.

Indeed, let us put $z(x) = y(x)\exp\left(-\int_{x_0}^{x} a(t)\,dt\right)$ (*cf.* IV, p. 183); then, by (1), $z'(x) = b(x)\exp\left(-\int_{x_0}^{x} a(t)\,dt\right)$. If $b(x) \geqslant 0$ for $x \geqslant x_0$ then z is increasing on this interval, so either is < 0 on all this interval, or is zero on an interval $[x_1, +\infty[$, or else is > 0 on an interval $[x_1, +\infty[$; since y has the same sign as z the proposition is proved in this case. The argument is similar if $b(x) \leqslant 0$ for $x \geqslant x_0$.

Remark. This very elementary property does not extend to linear differential equations of order > 1; for example, the function $y = \sin x$ satisfies $y'' + y = 0$, but changes sign on every neighbourhood of $+\infty$.

Lemma 2. Let $a(x)$ and $b(x)$ be two functions belonging to a given Hardy field $\mathfrak{K}$ and $y(x)$ a function satisfying the identity (1) *on an interval $[x_0, +\infty[$ where a and b are defined and continuous. If $p(u)$ is a polynomial in u whose coefficients are functions of x belonging to $\mathfrak{K}$, defined and differentiable on $[x_0, +\infty[$, then there exists an interval $[x_1, +\infty[$ on which the function $p(y)$ does not change sign.*

The proposition is trivial if $p(u)$ has coefficients which are identically zero on $[x_0, +\infty[$, or if $p(u)$ is of degree 0 in u, since any function in $\mathfrak{K}$ is of constant sign on an interval $[x_1, +\infty[$. Suppose that $p(u)$ is of degree $n > 0$; the leading coefficient c of $p(u)$ is then $\neq 0$ on an interval $[\alpha, +\infty[$; one can thus write $p(u) = c(u^n + c_1u^{n-1} + \cdots + c_n)$ where $c, c_1, c_2, \ldots, c_n$ are *functions* belonging to $\mathfrak{K}$ and differentiable on $[\alpha, +\infty[$; so it is enough to prove the lemma for $c = 1$. We argue by induction on n; one has

$$\begin{aligned}\frac{d}{dx}\big(p(y)\big) = (ay+b)\big(ny^{n-1} + (n-1)c_1y^{n-2} + \cdots + c_{n-1}\big)\\ + c_1'y^{n-1} + \cdots + c_n' = na\,p(y) + q(y)\end{aligned}$$

where $q(u)$ is a polynomial of degree $\leqslant n-1$, with coefficients in $\mathfrak{K}$. By hypothesis the functions $na(x)$ and $q(y(x))$ do not change sign on an interval $[\beta, +\infty[$; the lemma is thus a consequence of lemma 1.

THEOREM 1. *Let $a(x)$ and $b(x)$ be two functions belonging to a given Hardy field $\mathfrak{K}$ and $y(x)$ a function satisfying* (1) *on an interval $[x_0, +\infty[$. When $r(u) = p(u)/q(u)$ runs through the set of rational fractions in u with coefficients in $\mathfrak{K}$ such that $q(y)$ is not identically zero on a neighbourhood of $+\infty$ the set $\mathfrak{K}(y)$ of functions $r(y)$ forms a Hardy field.*

Indeed, by lemma 2, there exists an interval $[x_1, +\infty[$ on which $r(y)$ is defined, continuous, and does not change sign, from which it follows immediately that $\mathfrak{K}(y)/\mathrm{R}_\infty$ is a field; moreover, since

$$\frac{d}{dx}\big(r(y)\big) = r'(y)y' = r'(y)(ay+b)$$

$\big($where $r'(y) = \big(p'(y)q(y) - p(y)q'(y)\big)/\big(q(y)\big)^2$ is defined by hypothesis on a neighbourhood of $+\infty\big)$, the derivative of every function in $\mathfrak{K}(y)$ belongs to $\mathfrak{K}(y)$, which proves that $\mathfrak{K}(y)$ satisfies the conditions of def. 1 of V, p. 247.

It is clear that $\mathfrak{K}(y)/\mathrm{R}_\infty$ is obtained by the algebraic *adjunction* to $\mathfrak{K}/\mathrm{R}_\infty$ of the class of y modulo R_∞. Again one says that $\mathfrak{K}(y)$ is obtained by *adjoining* y to $\mathfrak{K}$.

COROLLARY 1. *If y is a function in $\mathfrak{K}$, not identically zero on a neighbourhood of $+\infty$, then $\mathfrak{K}(\log|y|)$ is a Hardy field.*

Indeed, $(\log|y|)' = y'/y$ is equal to a function in $\mathfrak{K}$ on an interval $[x_0, +\infty[$.

COROLLARY 2. *If y is any function in $\mathfrak{K}$, then $\mathfrak{K}(e^y)$ is a Hardy field.*

Indeed, $\big(e^y\big)' = e^y y'$, and y' is equal to a function in $\mathfrak{K}$ on an interval $[x_0, +\infty[$.

COROLLARY 3. *If $\mathfrak{K}$ contains the real constants, and if y is a function in $\mathfrak{K}$, not identically zero on a neighbourhood of $+\infty$, then $\mathfrak{K}(|y|^\alpha)$ is a Hardy field for every real number α.*

Indeed, $\dfrac{d}{dx}\big(|y|^\alpha\big) = |y|^\alpha(\alpha y'/y)$, and $\alpha y'/y$ is equal to a function in $\mathfrak{K}$ on an interval $[x_0, +\infty[$.

Finally, we note that if y is the *primitive* of any function in $\mathfrak{K}$ then $\mathfrak{K}(y)$ is again a Hardy field.

3. COMPARISON OF FUNCTIONS IN A HARDY FIELD

PROPOSITION 1. *Two functions in the same Hardy field are comparable to any order* (V, p. 232).

Indeed, if f belongs to a Hardy field $\mathfrak{K}$, then for every integer $n > 0$ there exists an interval $[x_0, +\infty[$ on which f is n times differentiable, its n^{th} derivative being equal to a function in $\mathfrak{K}$ on this interval. It is therefore enough to show that any two functions f, g of $\mathfrak{K}$ are *comparable*. This is evident if one of the functions is identically zero on a neighbourhood of $+\infty$; one may therefore restrict oneself to the case where they are both strictly positive on a neighbourhood of $+\infty$. But then, for every real number t, $f - tg$ is equal to a function in $\mathfrak{K}$ on a neighbourhood of $+\infty$, so has constant sign on a neighbourhood of $+\infty$, which proves the proposition (V, p. 217, prop. 9).

One deduces immediately from this proposition that, if a Hardy field $\mathfrak{K}$ contains the real constants (as we shall always assume in what follows), and if f and g are any two functions in $\mathfrak{K}$ then any two of the functions e^f, e^g, $\log|f|$, $\log|g|$, $|f|^\alpha$, $|g|^\alpha$ (α an arbitrary real), $\int_a f$, $\int_a g$ (a being any real number in an interval $[x_0, +\infty[$ where f and g are regulated) are *comparable* (when they are defined); indeed, any two of these functions belong to a given Hardy field obtained by adjoining them successively to $\mathfrak{K}$.

Similarly, every function $f(x)$ in a Hardy field $\mathfrak{K}$ is comparable to x, since x and $f(x)$ belong to the Hardy field obtained by adjoining x to $\mathfrak{K}$. One thus concludes (in particular) that f is comparable to any order to every power x^α, as well as to $\log x$ and to e^x.

One also sees that if f and g belong to the same Hardy field $\mathfrak{K}$, if $g(x) > 0$ on an interval $[x_0, +\infty[$, and if $g(x)$ tends to 0 or to $+\infty$ as x tends to $+\infty$, then the *order* of f relative to g (V, p. 219) is always defined.

Prop. 8 of V, p. 233, is therefore applicable to every function f in a Hardy field, and proves that:

1° if f is of order $+\infty$ relative to x then $\displaystyle\int_a^x f(t)\,dt \sim \big(f(x)\big)^2/f'(x)$.

2° if f is of order $\mu > -1$ relative to x then $\displaystyle\int_a^x f(t)\,dt \sim \frac{1}{\mu+1}xf(x)$.

3° if f is of order $\mu < -1$ relative to x then $\displaystyle\int_x^{+\infty} f(t)\,dt \sim -\frac{1}{\mu+1}xf(x)$.

4° if f is of order $-\infty$ relative to x then $\displaystyle\int_x^{+\infty} f(t)\,dt \sim -\big(f(x)\big)^2/f'(x)$.

Further, we have the following proposition:

PROPOSITION 2. *Let f be a function belonging to a Hardy field $\mathfrak{K}$.*

1° *If f is of infinite order relative to x, then, for every integer $n > 0$,*

$$f^{(n)}(x) \sim \frac{(f'(x))^n}{(f(x))^{n-1}}. \tag{2}$$

2° *If f is of finite order μ relative to x then, for every $n > 0$,*

$$\begin{aligned} f^{(n)}(x) &\sim \mu(\mu-1)\dots(\mu-n+1)\frac{f(x)}{x^n} \\ &\sim \frac{(\mu-1)\dots(\mu-n+1)}{\mu^{n-1}}\frac{(f'(x))^n}{(f(x))^{n-1}} \end{aligned} \tag{3}$$

except if μ is an integer $\geqslant 0$ and $n > \mu$.

1° If f is of infinite order relative to x one has $\log|f| \succ\!\succ \log x$, so, since $\log|f|$ and $\log x$ are comparable to any order, $f'/f \succ\!\succ 1/x$. Put $g = f'/f$; since g is equal to a function in $\mathfrak{K}$ on a neighbourhood of $+\infty$ one deduces from $1/g \prec\!\prec x$, that $g'/g^2 \prec\!\prec 1$, and so $g'/g \prec\!\prec g = f'/f$, or again $fg' \prec\!\prec gf'$. From the relation $f' = fg$ one deduces by differentiating that

$$f'' = fg' + gf' \sim gf'$$

or again $f''/f' \sim f'/f$. The same argument, applied to $f^{(n)}$ instead of f, shows, by induction on n, that $f^{(n)}/f^{(n-1)} \sim f'/f$; whence relation (2).

2° If f is of finite order μ relative to x, and if $\mu \neq 0$, one has $\log|f| \sim \mu\log x$, whence, on differentiating, $f'(x) \sim \mu\dfrac{f(x)}{x}$; one deduces that f' is of order $\mu - 1$ relative to x, which allows one to apply the same argument by induction on n so long as $\mu \neq n$, whence formula (3) when μ is not an integer $\geqslant 0$ and $< n$.

When f is of integral order $p \geqslant 0$ relative to x one can write $f(x) = x^p f_1(x)$, where f_1 is of order 0 relative to x. By prop. 2 one has

$$f^{(p)} \sim p!f_1.$$

To evaluate the derivatives of order $n > p$ one may restrict oneself to the case $p = 0$. Then one has $\log|f| \prec\!\prec \log x$, whence $f'(x)/f(x) \prec\!\prec 1/x$, in other words $xf'(x) \prec\!\prec f(x)$; if f is not equivalent to a constant $k \neq 0$ one has, on differentiating this relation (V, p. 232, prop. 7), $xf''(x) + f'(x) \prec\!\prec f'(x)$, which implies that $xf''(x) \sim -f'(x)$. Taking account of this formula, one sees by induction on n that $f^{(n)}$ is of order $\leqslant -n$ relative to x, and that

$$f^{(n)} \sim (-1)^{n+1}(n-1)!\,\frac{f'(x)}{x^{n-1}}. \tag{4}$$

If f is equivalent to a constant $k \neq 0$ one has $f(x) = k + f_2(x)$ with $f_2 \prec\!\prec 1$, and it reduces to studying the derivatives of f_2.

4. (H) FUNCTIONS

PROPOSITION 3. *If $\mathfrak{K}_0$ is a Hardy field there exists a Hardy field $\mathfrak{K}$ containing $\mathfrak{K}_0$ and such that, for every function $z \in \mathfrak{K}$, not identically zero on a neighbourhood of $+\infty$, both e^z and* $\log|z|$ *belong to $\mathfrak{K}$.*

Denote by $\mathfrak{K}$ the set of functions $f \in \mathcal{H}(\mathfrak{F}, \mathbf{R})$ having the following properties: for each function $f \in \mathfrak{K}$ there is a finite number of Hardy fields $\mathfrak{K}_1, \mathfrak{K}_2, \dots, \mathfrak{K}_n$ (the number n and the fields $\mathfrak{K}_i$ depending on f) such that $f \in \mathfrak{K}_n$ and, for $0 \leqslant i \leqslant n-1$, one has $\mathfrak{K}_{i+1} = \mathfrak{K}_i(u_{i+1})$ where u_{i+1} is equal either to e^{z_i} or to $\log|z_i|$ with z_i belonging to $\mathfrak{K}_i$ and not vanishing identically on a neighbourhood of $+\infty$. One says that $u_1, u_2, \dots, u_n$ form a *definition sequence* for the field $\mathfrak{K}_n$ and of the function f; the same function can naturally admit several definition sequences.

By def. 1 of V, p. 247, every function $f \in \mathfrak{K}$, not identically zero on a neighbourhood of $+\infty$, has constant sign and is differentiable on an interval $[x_0, +\infty[$; if $f \in \mathfrak{K}_n$, then $1/f$ and f' are equal to functions in $\mathfrak{K}_n$, thus to functions in $\mathfrak{K}$, on a neighbourhood of $+\infty$. To see that $\mathfrak{K}$ is a Hardy field it is enough to prove that if f and g are two functions in $\mathfrak{K}$ then $f - g$ and fg are equal to functions in $\mathfrak{K}$ on a neighbourhood of $+\infty$. Now let $u_1, u_2, \dots, u_m$ be a definition sequence for f, and $v_1, v_2, \dots, v_n$ a definition sequence for g. The sequence $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ obtained by concatenating the sequences (u_i) and (v_i) is again a definition sequence of a Hardy field $\mathfrak{K}_{m+n}$ and this field contains f and g, so $f - g$ and fg are equal to functions in $\mathfrak{K}_{m+n}$ on a neighbourhood of $+\infty$.

One says that the Hardy field $\mathfrak{K}$ defined in the proof of prop. 3 is the (H) *extension* of the Hardy field $\mathfrak{K}_0$.

If $\mathfrak{K}'$ is another Hardy field possessing the properties stated in prop. 3, it follows from the construction of $\mathfrak{K}$ that $\mathfrak{K}/\mathrm{R}_\infty$ is *contained* in $\mathfrak{K}'/\mathrm{R}_\infty$. By abuse of language, one says that the (H) extension of the Hardy field $\mathfrak{K}_0$ is the *smallest* Hardy field $\mathfrak{K}$ having these properties.

DEFINITION 2. *The field of* (H) *functions is called the* (H) *extension of the Hardy field* $\mathbf{R}(x)$ *of rational functions with real coefficients. Any function belonging to this extension is called an* (H) *function.*

Following this definition, if f is an (H) function, not identically zero on a neighbourhood of $+\infty$, then e^f and $\log|f|$ are also (H) functions. More generally, if g is a second (H) function, and $u_1, u_2, \dots, u_n$ a definition sequence for g, and if $f(x)$ tends to $+\infty$ with x, one sees by induction on n that the composite functions $u_1 \circ f, u_2 \circ f, \dots, u_n \circ f$ and $g \circ f$ are (H) functions.

5. EXPONENTIALS AND ITERATED LOGARITHMS

We have already (V, p. 229) defined the *iterated logarithms* $l_n(x)$ by the conditions $l_0(x) = x, l_n(x) = \log(l_{n-1}(x))$ for $n \geqslant 1$. In the same way one defines the *iterated exponentials* $e_n(x)$ by the conditions $e_0(x) = x, e_n(x) = \exp(e_{n-1}(x))$ for $n \geqslant 1$. It is immediate, by induction on n, that $l_n(x)$ is the inverse function of $e_n(x)$, defined for $x > e_{n-1}(0)$, and that $e_m(e_n(x)) = e_{m+n}(x), l_m(l_n(x)) = l_{m+n}(x)$. By the relations $\log x \prec\!\!\prec x^\mu \prec\!\!\prec e^x$ for every $\mu > 0$, one has, for $n \geqslant 1$,

$$l_n(x) \prec\!\!\prec (l_{n-1}(x))^\mu \qquad \text{for every } \mu > 0 \tag{5}$$

$$\begin{aligned} e_{n-1}(x^{1+\beta}) \prec\!\!\prec e_n(x^{1+\delta}) \prec\!\!\prec e_n((1-\gamma)x) \prec\!\!\prec (e_n(x))^\mu \\ \prec\!\!\prec e_n((1+\alpha)x) \prec\!\!\prec e_n(x^{1+\beta}) \end{aligned} \tag{6}$$

for all $\mu > 0, \alpha > 0, \beta > 0, 0 < \gamma < 1, 0 < \delta < 1$, these numbers being otherwise arbitrary (*cf.* V, p. 218, prop. 11).

We have already seen (V, p. 229) that, for $n \geqslant 1$, one has

$$\frac{d}{dx}\big(l_n(x)\big) = \prod_{i=0}^{n-1} \frac{1}{l_i(x)}. \tag{7}$$

Similarly for $n \geqslant 1$

$$\frac{d}{dx}\big(e_n(x)\big) = \prod_{i=1}^{n} e_i(x). \tag{8}$$

whence, by prop. 8 of V, p. 233, for every $\mu > 0$

$$\int_a^x e_n(t^\mu)\,dt \sim \frac{x}{\mu} e_n(x^\mu) \prod_{i=0}^{n-1} \frac{1}{e_i(x^\mu)} \tag{9}$$

$$\int_x^{+\infty} \frac{dt}{e_n(t^\mu)} \sim \frac{x}{\mu} \prod_{i=0}^{n} \frac{1}{e_i(x^\mu)}. \tag{10}$$

One can show that if f is *any* (H) function such that $f \succ\!\!\succ 1$ then there exist two integers m and n such that

$$l_m(x) \prec\!\!\prec f(x) \prec\!\!\prec e_n(x)$$

(V, p. 263, exerc. 1 and p. 264, exerc. 5) On the other hand, one can define increasing functions $g(x)$ (which are no longer (H) functions) such that $g(x) \succ\!\!\succ e_n(x)$ for *every* $n > 0$, or $1 \prec\!\!\prec g(x) \prec\!\!\prec l_m(x)$ for *every* $m > 0$ (V, p. 265, exerc. 8, 9 and 10).

With the help of the iterated logarithms we shall show that one can define a *comparison scale* $\mathcal{E}$ (for x tending to $+\infty$) of (H) functions, which are > 0 on a neighbourhood of $+\infty$ and satisfy the following conditions:

a) the product of any two functions in $\mathcal{E}$ belongs to $\mathcal{E}$;

b) $f^\mu \in \mathcal{E}$ for every function $f \in \mathcal{E}$ and every real number μ;

c) for every function $f \in \mathcal{E}$, $\log f$ is a linear combination of a finite number of functions in $\mathcal{E}$;

d) for every function $f \in \mathcal{E}$, apart from the constant 1, e^f is equivalent to a function in $\mathcal{E}$.

First we consider the set $\mathcal{E}_0$ of functions of the form $\prod\limits_{m=0}^{\infty} \big(l_m(x)\big)^{\alpha_m}$, where the α_m are real numbers, zero apart from for a finite number of indices m; it is immediate, from (5) (V, p. 253) that these functions form a *comparison scale* which satisfies conditions *a*), *b*) and *c*). Now we define, by recursion on n, the set $\mathcal{E}_n$ (for $n \geqslant 1$) formed by the constant 1 and by the functions of the form $\exp\left(\sum\limits_{k=1}^{p} a_k f_k\right)$, where p is an arbitrary integer > 0, the functions f_k $(1 \leqslant k \leqslant p)$ are functions in $\mathcal{E}_{n-1}$ such that $f_1 \succ\!\succ f_2 \succ\!\succ \cdots \succ\!\succ f_p \succ\!\succ 1$, and the a_k are real numbers $\neq 0$; we show by induction that $\mathcal{E}_n$ is a *comparison scale* satisfying *a*), *b*) and *c*) and containing $\mathcal{E}_{n-1}$. In the first place, the relation $\mathcal{E}_{n-1} \subset \mathcal{E}_n$ holds for $n = 1$, since the logarithm of any nonconstant function in $\mathcal{E}_0$ is of the form $\sum\limits_{k=1}^{p} a_k f_k$, where the f_k are iterated logarithms, and so $\succ\!\succ 1$; on the other hand, if $\mathcal{E}_{n-2} \subset \mathcal{E}_{n-1}$ one deduces from the definition of $\mathcal{E}_n$ that $\mathcal{E}_{n-1} \subset \mathcal{E}_n$; this definition furthermore shows that $\mathcal{E}_n$ satisfies *a*), *b*) and *c*). It remains to see that $\mathcal{E}_n$ is a comparison scale: since the quotient of two functions in $\mathcal{E}_n$ again belongs to $\mathcal{E}_n$ it suffices to prove that every function f of $\mathcal{E}_n$, apart from the constant 1, cannot be equivalent to a constant $\neq 0$. Now one has $\log f = \sum\limits_{k=1}^{p} a_k f_k \sim a_1 f_1$ by construction, and since $f_1 \succ\!\succ 1$, $\log f$ tends to $\pm\infty$, so f tends to 0 or to $+\infty$ as x tends to $+\infty$.

This being so, if $\mathcal{E}$ is the *union* of the $\mathcal{E}_n$ for $n \geqslant 0$, then $\mathcal{E}$ is a comparison scale, for two functions in $\mathcal{E}$ belong to the same scale $\mathcal{E}_n$; for the same reason, $\mathcal{E}$ satisfies *a*), and it is clear that it also satisfies *b*) and *c*). Finally, if $f \in \mathcal{E}$ there exists an n such that $f \in \mathcal{E}_n$; if f is not the constant 1 then $f(x)$ tends to 0 or to $+\infty$ as x tends to $+\infty$; in the first case $e^f \sim 1$ and in the second, e^f belongs to $\mathcal{E}_{n+1}$ by definition, and so to $\mathcal{E}$.

Remark. Despite the practical usefulness of the scale $\mathcal{E}$ which we have just defined, it is easy to give examples of (H) functions which *have no principal part* with respect to $\mathcal{E}$. Indeed, if f is an (H) function such that $f \sim ag$, where a is a constant > 0 and $g \in \mathcal{E}$, then $\log f - \log g - \log a$ tends to 0 with $1/x$, so $\log f$ admits, relative to $\mathcal{E}$, an asymptotic expansion whose *remainder tends to* 0, by property *c*). Now, if one considers for example the (H) function $f(x) = e_2\left(x + \dfrac{1}{x}\right)$ one has $\log f(x) = \exp\left(x + \dfrac{1}{x}\right)$, so the asymptotic expansions of $\log f$ relative to $\mathcal{E}$ are of the form

$$\log f(x) = e^x + \frac{e^x}{x} + \frac{1}{2!}\,\frac{e^x}{x^2} + \cdots + \frac{1}{n!}\,\frac{e^x}{x^n} + o\left(\frac{e^x}{x^n}\right) \qquad (n \text{ an integer } > 0).$$

It is clear that the remainder in this expansion is equivalent to $\dfrac{1}{(n+1)!}\,\dfrac{e^x}{x^{n+1}}$, so does not tend to 0. Hence f does not have a principal part relative to $\mathcal{E}$.

6. INVERSE FUNCTION OF AN (H) FUNCTION

If f is an (H) function, then f is monotone and continuous on an interval $[x_0, +\infty[$, so the inverse function φ of the restriction of f to this interval is monotone and continuous on a neighbourhood of the point $a = \lim_{x\to+\infty} f(x)$; but, if a is equal to $+\infty$ (resp. $-\infty$, finite), one can show that $\varphi(y)$ (resp. $\varphi(-y)$, $\varphi\left(a+\dfrac{1}{y}\right)$ or $\varphi\left(a-\dfrac{1}{y}\right)$) *is not in general equal to an* (H) *function* on a neighbourhood of $+\infty$. Nevertheless we shall see that in certain important cases one can obtain an (H) function *equivalent* to $\varphi(y)$ (resp. $\varphi(-y)$, $\varphi\left(a+\dfrac{1}{y}\right)$, $\varphi\left(a-\dfrac{1}{y}\right)$) and sometimes even an asymptotic expansion of this function relative to the scale $\mathcal{E}$ defined in V, p. 254.

We shall use the following proposition:

PROPOSITION 4. *Let p and q be two* (H) *functions which are strictly positive on an interval* $[x_0, +\infty[$.

1° *If $q \prec\!\!\prec p/p'$ one has $p(x+q(x)) \sim p(x)$.*

2° *If both $q \prec\!\!\prec p/p'$ and $q(x) \prec\!\!\prec x$ then $p(x-q(x)) \sim p(x)$.*

The two parts of this proposition are clear if $p \sim k$ (constant $\neq 0$); one may therefore suppose that $p(x) \prec\!\!\prec 1$ (otherwise one applies the argument to $1/p$). One then deduces $p'(x) \prec\!\!\prec 1$.

1° One can write $p(x+q(x)) = p(x) + q(x)p'(x+\theta q(x))$ with $0 \leqslant \theta \leqslant 1$ (I, p. 14, corollary). Since $|p'(x)|$ tends to 0 as x tends to $+\infty$, and is equal to an (H) function on a neighbourhood of $+\infty$, it is decreasing on an interval $[x_1, +\infty[$, so, for $x \geqslant x_1$, one has $|p'(x+\theta q(x))| \leqslant |p'(x)|$; since $qp' \prec\!\!\prec p$ one has $p(x+q(x)) \sim p(x)$.

2° The condition $q(x) \prec\!\!\prec x$ ensures that $x - q(x)$ tends to $+\infty$ with x. Again one has $p(x-q(x)) = p(x) - q(x)p'(x-\theta p(x))$ with $0 \leqslant \theta \leqslant 1$. The same argument as in the first part of the proof shows that, for x sufficiently large, $|p'(x-\theta q(x))| \leqslant |p'(x-q(x))|$. It reduces to showing that $q(x)\dfrac{p'(x-q(x))}{p(x-q(x))}$ tends to 0 as x tends to $+\infty$. The proposition is true if $p'/p \succ\!\!\succ 1$ since then $|p'/p|$ is an (H) function, increasing for x large enough; thus $q(x)\dfrac{|p'(x-q(x))|}{|p(x-q(x))|} \leqslant q(x)\dfrac{|p'(x)|}{|p(x)|}$, and $qp' \prec\!\!\prec p$ by hypothesis. It is also true if $p'/p \sim k$ (k constant $\neq 0$), for then

$$\frac{p'(x-q(x))}{p(x-q(x))} \sim \frac{p'(x)}{p(x)}$$

since $x - q(x)$ tends to $+\infty$. It only remains to examine the case where $p'/p \prec\!\prec 1$. First suppose that $p(x)$ is of *finite* order relative to x, and so (V, p. 232, prop. 7) $p'(x)/p(x) \prec\!\prec 1/x$. Then one has

$$\frac{p'(x - q(x))}{p(x - q(x))} = \frac{1}{x - q(x)} O_1(1),$$

so

$$q(x)\frac{p'(x - q(x))}{p(x - q(x))} = \frac{q(x)}{x}\left(1 - \frac{q(x)}{x}\right)^{-1} O_1(1) = \frac{q(x)}{x} O_2(1),$$

and one sees in this case that the proposition is true subject *only* to the hypothesis that $q(x) \prec\!\prec x$. Finally we examine the case where $1/x \prec\!\prec p'(x)/p(x) \prec\!\prec 1$; the function $r = p'/p$ is then of finite order relative to x; since, by the preceding remark, prop. 4 of V, p. 255, is applicable to such a function, one has $p'(x-q(x))/p(x-q(x)) \sim p'(x)/p(x)$, and the hypothesis $qp' \prec\!\prec p$ allows us to complete the proof.

Remark. The conditions imposed on $q(x)$ may not be improved, as the following examples show:

a) $\quad p(x) = e^x, \qquad q(x) = 1 = \dfrac{p(x)}{p'(x)}, \qquad p(x + q(x)) = ep(x)$

b) $\quad p(x) = \log x, \qquad q(x) = x - \log x \prec\!\prec \dfrac{p(x)}{p'(x)} = x \log x,$

$$p(x - q(x)) = \log\log x \prec\!\prec p(x).$$

We shall first study the inverses of (H) functions of a particular sort:

PROPOSITION 5. *Let g be an* (H) *function not equivalent to a constant $\neq 0$, and such that $g(x) \prec\!\prec x$, and let $u(x)$ be the inverse function of $x - g(x)$, defined on a neighbourhood of $+\infty$. Let (u_n) be the sequence of functions defined, by recursion on n, by the conditions $u_0(x) = x$, $u_n(x) = x + g(u_{n-1}(x))$ for $n \geqslant 1$; then $u_n \succ\!\succ 1$ and*

$$u(x) - u_n(x) \sim g(x)\big(g'(x)\big)^n. \tag{11}$$

Let us put $y = u(x)$, $y_n = u_n(x)$; then $x = y - g(y)$, $y_0 = x$ and $y_n = x + g(y_{n+1})$. One first deduces that $x/y = 1 - \dfrac{g(y)}{y}$; since y tends to $+\infty$ with x, the hypothesis $g(x) \prec\!\prec x$ shows that $y = u(x) \sim x = y_0$; further,

$$y - x = g(y) = g(x) + (y - x)g'(z)$$

where z belongs to the interval with endpoints x, y; when x tends to $+\infty$ so does z, and since $g(x) \prec\!\prec x$, $g' \prec\!\prec 1$, thus $g'(z)$ tends to 0, and consequently

$$y - x = g(x) + o(y - x)$$

whence

$$u(x) - x \sim g(x). \tag{12}$$

In the second place we show, by recursion on n, that when x tends to $+\infty$ one has $u_n \succ\!\succ 1$, and

$$u(x) - u_n(x) \prec\!\prec u(x) - u_{n-1}(x). \tag{13}$$

Indeed, $y - y_n = g(y) - g(y_{n-1}) = (y - y_{n-1})g'(z_{n-1})$, where z_{n-1} belongs to the interval with endpoints y and y_{n-1}; by the induction hypothesis z_{n-1} tends to $+\infty$ with x, and so $g'(z_{n-1})$ tends to 0, which proves (13). One deduces from this relation and from (12) that $u(x) - u_n(x) \prec\!\prec u(x) - x \sim g(x) \prec\!\prec x \sim u(x)$, whence $u_n(x) \sim u(x)$ and consequently $u_n \succ\!\succ 1$. Finally, the relation $u(x) - u_n(x) \prec\!\prec u(x) - x$ can also be written $(u(x) - x) - (u_n(x) - x) \prec\!\prec u(x) - x$, whence

$$u_n(x) - x \sim u(x) - x \sim g(x). \tag{14}$$

To prove (11), first remark that if $t(x)$ is a function such that $t(x) - x \sim g(x)$, then one has $g'(t(x)) \sim g'(x)$. Indeed, for any $\varepsilon > 0$, for x sufficiently large g' is monotone, so $g'(t(x))$ lies between $g'(x + (1 + \varepsilon)g(x))$ and $g'(x + (1 - \varepsilon)g(x))$. Prop. 4 of V, p. 255, shows that $g'(t(x)) \sim g'(x)$, provided one has established the relation $g \prec\!\prec g'/g''$. Now, if g is of infinite order relative to x one has (V, p. 251, prop. 2) $g''/g' \sim g'/g$, and, since $g' \prec\!\prec 1$, $g \prec\!\prec g/g' \sim g'/g''$; if g is of finite order μ relative to x one necessarily has $\mu \leqslant 1$; if $\mu < 1$, since g is not equivalent to a constant $\neq 0$, formulae (3) and (4) (V, p. 251) show that $g''/g' \sim k/x$ (k constant $\neq 0$), whence again $g \prec\!\prec g'/g''$; finally, if $\mu = 1$ then g' is of order 0 relative to x, so $g''/g' \prec\!\prec 1/x$, and so again $g \prec\!\prec g'/g''$.

This being so, since z_{n-1} lies between y and y_n, it follows from (14) that $z_{n-1} - x \sim g(x)$, whence $g'(z_{n-1}) \sim g'(x)$ by the above; thus

$$y - y_n \sim (y - y_{n-1})\, g'(x),$$

whence we obtain (11) by induction on n.

Remarks. 1) If g is of *order* < 1 *relative to* x, the function $u(x) - u_n(x)$ *tends to* 0 *with* x once n is sufficiently large. Indeed, in the opposite case one would have $gg'^n \succ\!\succ 1$ for *every* n, so g would be of infinite order relative to $1/g'$; in other words, one would have $\log|g| \succ\!\succ \log|g'|$, whence, on differentiating, $g'/g \succ\!\succ g''/g'$. But if g is of order $\mu < 1$ relative to x one has $g'/g \sim g''/g'$ when $\mu = -\infty$, $\dfrac{g'}{g} \sim \dfrac{\mu}{\mu - 1}\dfrac{g''}{g'}$ when $\mu \neq 0$, and finally $g'/g \prec\!\prec g''/g'$ when $\mu = 0$ (V, p. 251, n° 3).

In contrast, if g is of order 1 relative to x one can have $gg'^n \succ\!\succ 1$ for every integer $n > 0$, as the example $g(x) = x/\log x$ shows.

2) When $g(x)$ is an (H) function equivalent to a constant $k \neq 0$ one has $g(x) = k + g_1(x)$, with $g_1 \prec\!\prec 1$; the function $u_1(x) = u(x) - k$ is the inverse of the function $x - g_1(x + k)$, and one is brought back to the case treated in prop. 5 of V, p. 256.

To have an asymptotic expansion for the function u it thus suffices to have such an expansion for the function u_n: if g admits an asymptotic expansion relative to the scale under consideration one is thus led (by the definition of (H) functions) to the problems examined in V, p. 223 to 227.

The most general case, as follows, reduces to the case treated in prop. 5 of V, p. 256: the function $y = u(x)$ is assumed to satisfy the relation

$$\varphi(x) = \psi(y) - g(y) \tag{15}$$

where φ is an (H) function, ψ is an (H) function such that $\psi \succ\!\succ 1$ and such that the inverse function θ of ψ is also an (H) function, and g is an (H) function such that $g \prec\!\prec \psi$. Let now $v(x)$ be the inverse function of $x - g(\theta(x))$; one has $u = \theta \circ v \circ \varphi$, and $g(\theta(x)) \prec\!\prec x$; if one knows an asymptotic expansion for v thanks to prop. 5 of V, p. 256, one can then deduce an asymptotic expansion for u as in V, p. 223 to 227.

Examples. 1) Let us seek an asymptotic expansion for the inverse function $v(x)$ of $x^5 + x$ (for x tending to $+\infty$); putting $x^5 = t$ one is led to seek an expansion of the inverse function $u(t)$ of $t + t^{1/5}$ (for t tending to $+\infty$), that is, to apply prop. 5 of V, p. 256, to the case where $g(t) = -t^{1/5}$. Let us, for example, calculate, $u_2(t)$; one has

$$u_2(t) = t - \left(t - t^{1/5}\right)^{1/5} = t - t^{1/5} + \frac{1}{5}t^{-3/5} + \frac{2}{25}t^{-7/5} + o_1\left(t^{-7/5}\right).$$

Moreover, by (11) (V, p. 256)

$$u(t) - u_2(t) \sim -\frac{1}{25}t^{-7/5}$$

whence

$$u(t) = t - t^{1/5} + \frac{1}{5}t^{-3/5} + \frac{1}{25}t^{-7/5} + o_2\left(t^{-7/5}\right)$$

and one deduces the expansion

$$v(x) = u(x)^{1/5} = x^{1/5} - \frac{1}{5}x^{-3/5} - \frac{1}{25}x^{-7/5} + o_3\left(x^{-7/5}\right)$$

sought.

2) Let us seek an asymptotic expansion for the inverse function of the function $x/\log x$; from the identity $x = y/\log y$, where $y = v(x)$, one deduces $\log x = \log y - \log\log y$; on putting $z = \log y$, $t = \log x$, one has $t = z - \log z$, and one is led to expand the inverse function $u(t)$ of $t - \log t$; for example, one has

$$u_2(t) = t + \log(t + \log t) = t + \log t + \frac{\log t}{t} - \frac{(\log t)^2}{2t^2} + o_1\left(\frac{\log t}{t^2}\right)$$

and moreover, by (11) (V, p. 256)

$$u(t) - u_2(t) \sim \frac{\log t}{t^2}$$

whence

$$u(t) = t + \log t + \frac{\log t}{t} - \frac{(\log t)^2}{2t^2} + \frac{\log t}{t^2} + o_2\left(\frac{\log t}{t^2}\right)$$

and on returning to the original problem one obtains the asymptotic expansion

$$v(x) = x\log x + x\log\log x + x\frac{\log\log x}{\log x} + o\left(x\frac{\log\log x}{\log x}\right).$$

Remark. Note that two equivalent (H) functions can have non-equivalent inverses, as the example of the two functions $\log x$ and $1 + \log x$ shows.

EXERCISES

§ 1.

1) Show that for x real tending to $+\infty$ the function

$$f(x) = (x\cos^2 x + \sin^2 x)\, e^{x^2}$$

is monotone, but is neither comparable to e^{x^2} nor weakly comparable to $x^{1/2}\, e^{x^2}$.

2) Let φ be a strictly positive function, defined and increasing for $x > 0$, and such that $\varphi \succ\!\succ 1$.

a) Show that if the function $\log\varphi(x)/\log x$ is increasing, then the relation $f \prec\!\prec g$ between functions > 0 implies that $\varphi \circ f \prec\!\prec \varphi \circ g$ if $g \succ\!\succ 1$.

b) Give an example where $\log\varphi(x)/\log x$ is decreasing, f and g are two functions > 0 such that $g \succ\!\succ 1$ and $f \sim g$, but $\varphi \circ f$ is not equivalent to $\varphi \circ g$.

¶ 3) *a*) Let (α_i, β_i) be n distinct pairs of real numbers $\geqslant 0$, distinct from $(0, 0)$, and such that $\inf\limits_i \alpha_i = \inf\limits_i \beta_i = 0$. Consider the equation

$$f(x, y) = \sum_{i=1}^{n} a_i\, x^{\alpha_i}\, y^{\beta_i} \left(1 + \varphi_i(x, y)\right) = 0$$

where the a_i are real numbers $\neq 0$ and the φ_i are continuous functions on a square $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$, tending to 0 as (x, y) tends to $(0, 0)$. Suppose that there exists a function g which is positive and continuous on an interval $[0, b]$, tending to 0 with x and such that $f(x, g(x)) = 0$ for all $x \in [0, b]$. For every real number $\mu > 0$ show that $g(x)/x^{\mu}$ tends to a limit, finite or infinite, as x tends to 0 (use V, p. 217, prop. 9, considering, for every number $t \geqslant 0$, the equation $f(x, tx^{\mu}) = 0$).

b) For $g(x)/x^{\mu}$ to tend to a finite limit $\neq 0$ it is necessary that μ should be such that there exist at least two distinct pairs (α_h, β_h) and (α_k, β_k) such that $\alpha_h + \mu\beta_h = \alpha_k + \mu\beta_k$ and such that, for every other pair (α_i, β_i), one has $\alpha_i + \mu\beta_i \geqslant \alpha_h + \mu\beta_h$. One thus obtains a finite number of possible values μ_j $(1 \leqslant j \leqslant r)$; the numbers $-1/\mu_j$ are the gradients of the affine lines in the plane $\mathbf{R}^2$ which contain at least two of the set of points (α_i, β_i) and such that all the other points of this set lie above the line considered ("*Newton polygon*").

c) Let μ_1 be the smallest of the numbers μ_j. Show that $g(x)/x^{\mu_1}$ tends to a *finite* limit (possibly zero). (Show first that one may always assume that if i and j are two distinct indices one cannot have both $\alpha_i \leqslant \alpha_j$ and $\beta_i \leqslant \beta_j$; deduce that one can assume $\alpha_1 = 0$, $\alpha_i > 0$ and $\beta_i < \beta_1$ for $i \neq 1$; now, putting $g(x) = t(x)\, x^{\mu_1}$, show that $t(x)$ cannot tend to $+\infty$ as x tends to 0.)

d) Deduce from *c*), by recursion on r, that there exists an index j such that $g(x)/x^{\mu_j}$ tends to a finite limit $\neq 0$ as x tends to 0.

§ 3.

1) Define an increasing function g, admitting a continuous derivative on a neighbourhood of $+\infty$, such that g and $1/x$ are comparable to order 1, but that x and $1/g$ are not comparable to order 1 (take $g'(x) = 1$ except on sufficiently small intervals with centres at the points $x = n$ (n an integer > 0) on which g' takes very large values).

2) Let f and g be two functions $\geqslant 0$ tending to $+\infty$ with x and comparable to order 1; if h is a differentiable function, $\geqslant 0$ and increasing as x tends to $+\infty$, show that hf and hg are comparable to order 1.

3) Give an example of two functions which are positive, decreasing, tend to 0 as x tends to $+\infty$, comparable to order 1 and such that $xf(x)$ and $xg(x)$ are not comparable to order 1 (take f and g equivalent and such that

$$\frac{f'(x)}{f(x)} + \frac{1}{x} \qquad \text{and} \qquad \frac{g'(x)}{g(x)} + \frac{1}{x}$$

are not comparable).

4) *a*) Show that $\dfrac{\sin x}{\sqrt{x}} \sim \dfrac{\sin x}{\sqrt{x}}\left(1 + \dfrac{\sin x}{\sqrt{x}}\right)$, but that the integral $\displaystyle\int_a^{+\infty} \frac{\sin t}{\sqrt{t}}\,dt$ converges while the integral $\displaystyle\int_a^{+\infty} \frac{\sin t}{\sqrt{t}}\left(1 + \frac{\sin t}{\sqrt{t}}\right)dt$ is infinite.

b) Show that the integrals $\displaystyle\int_a^{+\infty} \frac{\sin t}{t}\,dt$ and $\displaystyle\int_a^{+\infty} \frac{\sin^2 t}{t^2}\,dt$ are both convergent but that $\displaystyle\int_x^{+\infty} \frac{\sin^2 t}{t^2}\,dt$ is not negligible relative to $\displaystyle\int_x^{+\infty} \frac{\sin t}{t}\,dt$, even though $\sin^2 t/t^2 \lll \sin t/t$.

c) Consider the two continuous functions with values in $\mathbf{R}^2$, defined on $[1, +\infty[$:

$$\mathbf{f} : x \mapsto \left(\frac{1}{x^2}, \frac{\sin x}{x}\right), \qquad \mathbf{g} : x \mapsto \left(\frac{1}{x^2}, \frac{\sin x}{x}\left(1 + \frac{\sin x}{x}\right)\right).$$

Show that they do not vanish for any value of x, that $\mathbf{f} \sim \mathbf{g}$ as x tends to $+\infty$, that the integrals $\int_a^{+\infty} \mathbf{f}(t)\,dt$ and $\int_a^{+\infty} \mathbf{g}(t)\,dt$ converge, but that $\int_x^{+\infty} \mathbf{f}(t)\,dt$ is not equivalent to $\int_x^{+\infty} \mathbf{g}(t)\,dt$ as x tends to $+\infty$.

¶ 5) Let f be a *convex* real function defined on a neighbourhood of $+\infty$ such that $f(x) \ggg x$. One says that f is *regularly convex* on a neighbourhood of $+\infty$ if, for *every convex* function g defined on a neighbourhood of $+\infty$ and such that $f \sim g$ one also has $f'_d \sim g'_r$.

For every number $\alpha > 0$ and x sufficiently large let $k(\alpha, x)$ be the infimum of the numbers $\big(f(y) - (y - x)f'_d(y)\big)/f(x)$ as y runs through the set of numbers $\geqslant x$ such that $f'_d(y) \leqslant (1 + \alpha)f'_d(x)$. Similarly let $h(\alpha, x)$ be the infimum of the numbers

$$\big(f(z) + (x - z)f'_d(z)\big)/f(x)$$

as z runs through the set of numbers $\leqslant x$ such that $f'_d(z) \geqslant (1 - \alpha)f'_d(x)$. Let $\psi(\alpha) = \limsup\limits_{x \to +\infty} k(\alpha, x)$, $\varphi(\alpha) = \limsup\limits_{x \to +\infty} h(\alpha, x)$. Show that for f to be regularly convex on a neighbourhood of $+\infty$ it is necessary and sufficient that for all sufficiently small α one has $\psi(\alpha) < 1$ and $\varphi(\alpha) < 1$.

¶ 6) Let $\mathbf{f}$ be a continuous vector function on an interval $[x_0, +\infty[$ of $\mathbf{R}$ and such that for all $\lambda \geqslant 0$ the function $\mathbf{f}(x+\lambda) - \mathbf{f}(x)$ tends to 0 when x tends to $+\infty$.

a) Show that $\mathbf{f}(x+\lambda) - \mathbf{f}(x)$ tends *uniformly* to 0 with $1/x$ when λ belongs to any compact interval $\mathrm{K} = [a, b]$ of $[0, +\infty[$. (Argue by contradiction: if there exists a sequence (x_n) tending to $+\infty$ and a sequence of points (λ_n) of K such that

$$\|\mathbf{f}(x_n + \lambda_n) - \mathbf{f}(x_n)\| > \alpha > 0$$

for every n, then there exists a neighbourhood J_n of λ_n in K such that, for every $\lambda \in \mathrm{J}_n$, one has $\|\mathbf{f}(x_n + \lambda) - \mathbf{f}(x_n)\| > \alpha$. Construct by induction a *decreasing* sequence of closed intervals $\mathrm{I}_k \subset \mathrm{K}$ and a subsequence (x_{n_k}) of (x_n) such that $\left\|\mathbf{f}(x_{n_k} + \lambda) - \mathbf{f}(x_{n_k})\right\| \geqslant \alpha/3$ for *every* $\lambda \in \mathrm{I}_k$; remark for this that if δ_k is the length of I_k and q an integer such that $q\delta_k > b - a$, then $\|\mathbf{f}(x + \delta_k) - \mathbf{f}(x)\| \leqslant \alpha/3q$ once x is sufficiently large.)

b) Deduce from *a*) that $\int_x^{x+1} \mathbf{f}(t)\,dt - \mathbf{f}(x)$ tends to 0 as x tends to $+\infty$, and conclude that $\mathbf{f}(x) = o(x)$.

7) Let g be a strictly positive real function, continuous on an interval $[x_0, +\infty[$, and such that, for every $\mu > 0$, one has $g(\mu x) \sim g(x)$. Deduce from exerc. 6 that g is of order 0 relative to x.

§ 4.

1) If the series with general term $u_n \geqslant 0$ converges, so does the series with general term $\sqrt{u_n u_{n+1}}$. The converse is false in general; show that it holds if the series (u_n) is decreasing.

2) Let (p_n) be an increasing sequence of numbers > 0 tending to $+\infty$.

a) If the ratio p_n/p_{n-1} tends to 1 as n increases indefinitely show that

$$\sum_{k=1}^{n} \frac{p_k - p_{k-1}}{p_k} \sim \log p_n$$

$$\sum_{k=1}^{n} p_k^{\rho}(p_k - p_{k-1}) \sim \frac{1}{\rho+1}\, p_n^{\rho+1} \quad \text{for } \rho > -1$$

(apply prop. 2 of V, p. 237).

b) Without the hypothesis on the ratio p_n/p_{n-1}, show that the series with general term $(p_n - p_{n-1})/p_n$ always has an infinite sum (distinguish the two cases according to whether p_n/p_{n-1} tends to 1 or not).

c) Show that for every $\rho > 0$ the series with general term $(p_n - p_{n-1})/p_n\, p_{n-1}^{\rho}$ converges (compare to the series with general term $\dfrac{1}{p_{n-1}^{\rho}} - \dfrac{1}{p_n^{\rho}}$).

3) Show that, for every convergent series (u_n) with terms > 0, there exists a series (v_n) with infinite sum, and terms > 0, such that $\liminf\limits_{n\to\infty} v_n/u_n = 0$.

4) Let (u_n) be a decreasing sequence of numbers > 0; if there exists an integer k such that $ku_{kn} \geqslant u_n$ for all n from some point on, then the series with general term u_n has an infinite sum.

5) Let (u_n) be a series with terms > 0 from some point on. Show that if

$$\limsup_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right)^n < \frac{1}{e}$$

then the series converges, and if $\liminf\limits_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right)^n \geqslant \dfrac{1}{e}$ the series has an infinite sum.

6) Let (u_n) be a series of reals $\neq 0$ such that $\lim\limits_{n\to\infty} u_n = 0$. Show that if there exists a number r such that $0 < r < 1$ and if from some point on one has $-1 \leqslant u_{n+1}/u_n \leqslant r$, then the series (u_n) converges.

7) Let (u_n) be a convergent series with terms > 0.

a) Show that

$$\lim_{n\to\infty} \frac{u_1 + 2u_2 + \cdots + nu_n}{n} = 0.$$

b) Show that

$$\sum_{n=1}^{\infty} \frac{u_1 + 2u_2 + \cdots + nu_n}{n(n+1)} = \sum_{n=1}^{\infty} u_n$$

$\left(\text{write } \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}\right)$.

c) Deduce from *a*) and *b*) that

$$\lim_{n\to\infty} n(u_1 u_2 \ldots u_n)^{1/n} = 0$$

and that

$$\sum_{n=1}^{\infty} \frac{(n!\, u_1 u_2 \ldots u_n)^{1/n}}{n+1} < \sum_{n=1}^{\infty} u_n$$

(apply the inequality (8) of III, p. 93).

¶ 8) *a*) Let (z_n) be a sequence of complex numbers. Show that if the series with general terms $z_n, z_n^2, \ldots, z_n^{q-1}, |z_n|^q$ converge, then the infinite product with general factor $1 + z_n$ converges.

b) If z_n is *real* and if the series with general term z_n converges, then the infinite product with general factor $1 + z_n$ converges if the series with general term z_n^2 converges, and one has

$$\operatorname*{P}_{n=1}^{\infty} (1 + z_n) = 0 \qquad \text{if} \qquad \sum_{n=1}^{\infty} z_n^2 = +\infty.$$

c) For every integer $p \geqslant 2$ put

$$k_p = [\log\log p], \qquad h_p = \sum_{j=1}^{p-1} k_j, \qquad \omega_p = \frac{2\pi}{k_p}.$$

Define the sequence (z_n) of complex numbers in the following way: for $n = h_p + m$, $0 \leqslant m \leqslant k_p - 1$, put $z_n = e^{mi\omega_p}/\log p$. Show that for *every* integer $q > 0$ the series with general term z_n^q converges, but that the product with general factor $1 + z_n$ does not converge.

9) *a*) Show, with the help of Stirling's formula, that the maximum of the function $f_n(x) = \left| e^{-2x} - e^{-x} \sum_{k=0}^{n} (-1)^k \frac{x^k}{k!} \right|$ over the interval $[0, +\infty[$ tends to 0 with $1/n$.

b) Deduce, by induction on the integer p, that for every $\varepsilon > 0$ there exists a polynomial $g(x)$ such that $\left|e^{-px} - e^{-x}g(x)\right| \leqslant \varepsilon$ for every $x \geqslant 0$ (replace x by $px/2$ in *a*), and apply the induction hypothesis to $e^{-(p-1)x/2}$).

10) For every number $\alpha > 0$ derive the formula

$$1^{\alpha n} + 2^{\alpha n} + \cdots + n^{\alpha n} \sim \frac{n^{\alpha n}}{1 - e^{-\alpha}}$$

as n tends to $+\infty$.

¶ 11) For every number $\alpha > 0$ derive the formula

$$(1!)^{-\alpha/n} + (2!)^{-\alpha/n} + \cdots + (n!)^{-\alpha/n} \sim \frac{1}{\alpha} \frac{n}{\log n}$$

as n tends to $+\infty$ (compare each term $(p\,!)^{-\alpha/n}$ to $n^{-\alpha p/n}$).

Appendix.

1) Let $\mathfrak{K}$ be a Hardy field such that, for every function $f \in \mathfrak{K}$ not identically zero on a neighbourhood of $+\infty$ there exists a $\lambda > 0$ such that

$$\frac{1}{e_m(x^\lambda)} \prec\!\!\prec f(x) \prec\!\!\prec e_m(x^\lambda)$$

(m an integer *independent of* f).

a) Let $u_1, u_2, \ldots, u_p$ be p functions of the form $u_k = \log|z_k|$, where $z_k \in \mathfrak{K}$ is not zero on a neighbourhood of $+\infty$. Show that for every function g (not zero on a neighbourhood of $+\infty$) of the Hardy field $\mathfrak{K}(u_1, u_2, \ldots, u_p)$ obtained by adjoining the functions u_k $(1 \leqslant k \leqslant p)$ to $\mathfrak{K}$, there exists a number $\mu > 0$ such that

$$\frac{1}{e_m(x^\mu)} \prec\!\!\prec g(x) \prec\!\!\prec e_m(x^\mu)$$

(reduce to the case where g is a polynomial in the u_k with coefficients in $\mathfrak{K}$ and argue by induction on p, then, for $p = 1$, argue by induction on the degree of the polynomial g, proceeding as in lemma 2 of V, p. 248).

b) Let u_k $(1 \leqslant k \leqslant p)$ be p functions of the form $u_k = \exp(z_k)$ where $z_k \in \mathfrak{K}$. Show that for every function g of the Hardy field $\mathfrak{K}(u_1, u_2, \ldots, u_p)$ not identically zero on a neighbourhood of $+\infty$ there exists a number $\mu > 0$ such that

$$\frac{1}{e_{m+1}(x^\mu)} \prec\!\!\prec g(x) \prec\!\!\prec e_{m+1}(x^\mu)$$

(similar method).

c) Deduce that if f is an (H) function having a definition sequence with n terms, and not identically zero on a neighbourhood of $+\infty$, there exists a number $\lambda > 0$ such that

$$\frac{1}{e_n(x^\lambda)} \prec\!\prec f(x) \prec\!\prec e_n(x^\lambda).$$

2) *a*) Show that every (H) function having a definition sequence of just one term is equivalent to a function of one of the forms $x^p(\log x)^q$, or $x^p e^{g(x)}$, where p and q are rational integers and g is a polynomial in x (method of exerc. 1).

b) Deduce from *a*) that every function in the Hardy field $\mathbf{R}(x, u_1, \dots, u_p)$, where $u_1, \dots, u_p$ are (H) functions having a single-term definition sequence, is equivalent to a function of the form $x^p (\log x)^q e^{g(x)}$ where p and q are rational integers and g is a polynomial in x.

3) Let f and g be two (H) functions such that f/g is of order 0 relative to $l_m(x)$; show that if g is not of order 0 relative to $l_m(x)$ one has $f'/g' \sim f/g$ (compare $\log|f|$ and $\log|g|$).

4) Let $\mathfrak{K}$ be a Hardy field such that, for every function $f \in \mathfrak{K}$ not equivalent to a constant, and of order 0 relative to $l_{m-1}(x)$, there exist a constant k and a rational integer r such that $f(x) \sim k(l_m(x))^r$.

a) Let z be any function in $\mathfrak{K}$, not identically zero on a neighbourhood of $+\infty$. Show that every function g in the Hardy field $\mathfrak{K}(\log|z|)$ not equivalent to a constant, and of order 0 relative to $l_m(x)$, is equivalent to a function of the form $k(l_{m+1}(x))^r$ (k constant, r a rational integer). (First consider the case where g is a polynomial of degree p in $\log|z|$, with coefficients in $\mathfrak{K}$, and argue by induction on p, using exerc. 3; if g is a rational function of $\log|z|$, with coefficients in $\mathfrak{K}$, argue by induction on the degree of the numerator, using exerc. 3.)

b) Show that every function in $\mathfrak{K}(l_{m+1}(x))$ is equivalent to a function of the form $f(x)(l_{m+1}(x))^r$ where $f \in \mathfrak{K}$ and r is a rational integer.

c) Let z be any function in $\mathfrak{K}$. Show that every function g of the Hardy field $\mathfrak{K}(e^z)$ of order 0 relative to $l_m(x)$ is equivalent to a constant. (First consider the case where $g = u^q\,\mathrm{P}(u)$, where $u = e^z$, q is a rational integer, and $\mathrm{P}(u)$ is a polynomial in u, of degree p, with coefficients in $\mathfrak{K}$; then argue by induction on p, using *a*) and exerc. 3. Pass from there to the general case by using exerc. 3.)

d) Extend the result of *a*) to the field $\mathfrak{K}(u_1, u_2, \dots, u_s)$, where the u_k are of the form e^{z_k} or $\log|z_k|$, the z_k being functions in $\mathfrak{K}$ not identically zero on a neighbourhood of $+\infty$ (argue by induction on s).

5) *a*) Deduce from exerc. 4 that if f is an (H) function having a definition sequence of n terms, not equivalent to a constant, and of order 0 relative to $l_{n-1}(x)$, there exist a constant k and a rational integer r such that $f(x) \sim k(l_n(x))^r$.

b) Deduce from exerc. 4 that if f is any (H) function there exists an integer n such that the logarithmic criterion of order n will determine whether the integral $\int_a^{+\infty} f(t)\,dt$ converges or is infinite.

6) Compare among themselves the functions $e_p\Big((l_q(x))^\mu\Big)$ according to the values of the integers p and q and the real number μ, assumed > 0 and $\neq 1$.

¶ 7) Let f be an (H) function having a definition sequence of n terms. Show that if one has $e_p((l_{q+1}(x))^\beta) \prec\!\prec f(x) \prec\!\prec e_p((l_q(x))^\alpha)$ (resp. $e_p((l_q(x))^\beta) \prec\!\prec f(x) \prec\!\prec e_{p+1}((l_q(x))^\alpha)$)

for every $\beta > 0$ and every α such that $0 < \alpha < 1$, then necessarily $p+q+1 < n$ (argue by induction on n, using methods similar to those of exerc. 1 (V, p. 263) and 4 (V, p. 264)).

8) *a*) Let (f_n) be a sequence of increasing continuous functions belonging to $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ and such that $f_n \prec\!\!\prec f_{n+1}$ for every n. Show that there exists a function f that is continuous, increasing, belongs to $\mathcal{H}(\mathfrak{F}, \mathbf{R})$, and is such that $f \succ\!\!\succ f_n$ for every n (reduce to the case where $f_n \leqslant f_{n+1}$ and define f so that $f_n(x) \leqslant f(x) \leqslant f_{n+1}(x)$ for $n \leqslant x \leqslant n+1$).

b) Let (f_n) be a sequence of increasing continuous functions belonging to $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ and such that $1 \prec\!\!\prec f_{n+1} \prec\!\!\prec f_n$ for every n. Show that there exists a function f that is continuous, increasing, belongs to $\mathcal{H}(\mathfrak{F}, \mathbf{R})$, and is such that $1 \prec\!\!\prec f \prec\!\!\prec f_n$ for every n (reduce to the case where $f_{n+1} \leqslant f_n$ and show that one can define an increasing sequence (x_n) of real numbers and a continuous increasing function f such that $f_{n+1}(x) \leqslant f(x) \leqslant f_n(x)$ for $x_n \leqslant x \leqslant x_{n+1}$).

c) Let (f_n), (g_n) be two sequences of continuous increasing functions belonging to $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ such that $f_n \prec\!\!\prec f_{n+1}$, $g_m \succ\!\!\succ g_{m+1}$ and $f_n \prec\!\!\prec g_m$ for all m and n; show that there exists a function h, continuous, increasing, belonging to $\mathcal{H}(\mathfrak{F}, \mathbf{R})$, such that $f_n \prec\!\!\prec h \prec\!\!\prec g_m$ for all m and n (similar method).

In particular, show that there exists a continuous decreasing function f in $\mathcal{H}(\mathfrak{F}, \mathbf{R})$ such that *no* logarithmic criterion allows one to determine whether the integral $\int_a^{+\infty} f(t)\,dt$ converges or is infinite ("*theorems of Du Bois-Reymond*").

9) Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{e_n(x)}{e_n(n)}$ converges uniformly on every compact subset of $\mathbf{R}$, and that its sum $f(x)$ is such that $f(x) \succ\!\!\succ e_n(x)$ for every integer n.

10) Show that there exists an increasing function f, defined, continuous and > 0 for $x \geqslant 0$, such that $f(2x) = 2^{f(x)}$ for all $x \geqslant 0$; deduce from this that $f(x) \succ\!\!\succ e_n(x)$ for every integer n.

11) Let f be an increasing function, continuous and $\geqslant 0$, defined for $x \geqslant 0$, and such that $f(x) \succ\!\!\succ e_n(x)$ for every integer n; show that, if g is the inverse function of f, one has $1 \prec\!\!\prec g(x) \prec\!\!\prec l_n(x)$ for every integer n.

¶ 12) For every integer $n > 0$ let $n = \sum\limits_{k=0}^{\infty} \varepsilon_k 2^k$ be the dyadic expansion of n (ε_k is an integer, zero for all but a finite number of indices, $0 \leqslant \varepsilon_k \leqslant 1$). Put

$$\alpha(n) = \sum_{k=0}^{\infty} \varepsilon_k, \qquad \mathrm{A}_1(n) = \sum_{j=1}^{n} \alpha(j), \qquad \mathrm{A}_2(n) = \sum_{j=1}^{n} \alpha(\alpha(j)).$$

a) Prove the formula $\mathrm{A}_1(n) = \displaystyle\sum_{k=0}^{\infty} (k+2)\varepsilon_k 2^{k-1}$. Deduce the formula

$$\mathrm{A}_1(n) = \frac{n}{2}\,\frac{\log n}{\log 2} + o(n \log\log n)$$

as n grows indefinitely (decompose the sum which expresses $\mathrm{A}_1(n)$ into two parts, the index k varying from 0 to $\mu(n)$ in the first sum, from $\mu(n)$ to n_1 in the second, where n_1 is the largest of the numbers k such that $\varepsilon_k \neq 0$ and $\mu(n)$ is chosen suitably; bound the first sum using prop. 6 of V, p. 239, and then estimate the difference between the second sum and $n_1 n/2$).

b) Establish the relation

$$\sum_{j=0}^{2^m-1} f(\alpha(j)) = \sum_{k=0}^{m} \binom{m}{k} f(k)$$

f being an arbitrary function defined on $\mathbf{N}$ (for a given k consider the number of $j < 2^m$ such that $\alpha(j) = k$).

c) For $m = 2^r - 1$ show that $\alpha(m-j) + \alpha(j) = r$. Deduce from this and from *b*) that

$$\mathrm{A}_2(2^m) = r2^{m-1} \sim \frac{2^m \log\log 2^m}{2\log 2}.$$

d) For $m = 2^r$ show that

$$\mathrm{A}_2(2^m) = 2\mathrm{A}_2(2^{m-1}) + 2^{m-1} - \sum_{j=0}^{m-1} \binom{m-1}{j} \Delta(j+1)$$

where we have put $\alpha(n+1) - \alpha(n) = 1 - \Delta(n+1)$ for all n (use *b*) and the relation $\alpha(2^k + r) = \alpha(r) + 1$ for $r < 2^k$). Show that $\sum_{j=0}^{m-1} \binom{m-1}{j} \Delta(j+1) \leqslant r^{m-2}$ (remark that $\Delta(j+1) = 0$ if j is even, and $\Delta(j+1) \leqslant \log j/\log 2$ for all j). Deduce, with the help of *c*), that

$$\mathrm{A}_2(2^m) \geqslant \frac{3}{2} r2^{m-1} \geqslant \frac{3}{2}\,\frac{2^m \log\log 2^m}{2\log 2}.$$

Conclude from *c*) and this relation that $\mathrm{A}_2(n)$ *is not equivalent to any* (H) *function* as n tends to $+\infty$.

13) Let f be an (H) function such that $1 \prec\!\!\prec f(x) \prec\!\!\prec x$. Show that in the Taylor formula

$$\begin{aligned} f(x + f(x)) = f(x) &+ f(x)f'(x) + \frac{1}{2!}(f(x))^2 f''(x) + \cdots + \\ &+ \frac{1}{n!}(f(x))^n f^{(n)}(x) + \frac{1}{(n+1)!}(f(x))^{n+1} f^{(n+1)}(x + \theta f(x)) \end{aligned}$$

with $0 < \theta < 1$ (I, p. 42, exerc. 9) each term is negligible relative to the previous one. If f is of order < 1 relative to x, the last term in this sum tends to 0 with $1/x$ once n is sufficiently large.

14) Deduce from exerc. 13 that if f is an (H) function such that $\log f(e^x)$ is of order < 1 relative to x, the function $f(xf(x))$ is equivalent to a function of the form $e^{g(x)}$, where g is a rational function of x, $\log f(x)$, and of a certain number of derivatives of this last function.

¶ 15) Let f be an (H) function, convex on a neighbourhood of $+\infty$, such that $f(x) \succ\!\!\succ x$. For every $\alpha > 0$ let x_α be the point such that $f'(x_\alpha) = (1+\alpha)f'(x)$.

a) If f is of order $+\infty$ relative to x show that

$$x_\alpha - x \sim \log(1+\alpha)\,\frac{f(x)}{f'(x)}$$

(use prop. 2 (V, p. 251) and 4 (V, p. 255), applying the Taylor formula to $\log f'(x)$)). Show that $\Big(f(x_\alpha) - (x_\alpha - x)f'(x_\alpha)\Big)/f(x)$ tends to $(1+\alpha)\Big(1 - \log(1+\alpha)\Big)$ as x tends to $+\infty$.

b) If f is of order $r > 1$ relative to x show that

$$x_\alpha \sim (1+\alpha)^{1/(r-1)}x$$

(remark that if f_1 is of order 0 relative to x one has $f_1(kx) \sim f_1(x)$ for every constant k, by prop. 4 of V, p. 255). Deduce that $\Big(f(x_\alpha) - (x_\alpha - x)f'(x_\alpha)\Big)/f(x)$ tends to

$$r(1+\alpha) - (r-1)(1+\alpha)^{r/(r-1)}$$

as x tends to $+\infty$.

c) Suppose finally that f is of order 1 relative to x. Putting $f(x) = xf_1(x)$ show that

$$x_\alpha - x \sim \log(1+\alpha)\,\frac{f_1(x)}{f_1'(x)}$$

(consider the inverse function of f_1, which is of order $+\infty$ relative to x). Deduce that $\Big(f(x_\alpha) - (x_\alpha - x)f'(x_\alpha)\Big)/f(x)$ tends to $-\infty$ as x tends to $+\infty$.

Similarly let x'_α be the point such that $f'(x'_\alpha) = (1-\alpha)f'(x)$ (for $0 < \alpha < 1$). State the formulae analogous to the previous ones that express the principal part of $x - x'_\alpha$ and the limit of

$$\Big(f(x'_\alpha) + (x - x'_\alpha)f'(x'_\alpha)\Big)/f(x)$$

as x tends to $+\infty$.

Conclude from these results that f is a *regularly convex* function on a neighbourhood of $+\infty$ (V, p. 260, exerc. 5).

CHAPTER VI
Generalized Taylor expansions
Euler-Maclaurin summation formula

§ 1. GENERALIZED TAYLOR EXPANSIONS

1. COMPOSITION OPERATORS ON AN ALGEBRA OF POLYNOMIALS

Let K be a commutative field of *characteristic* 0, and K[X] the algebra of polynomials in one indeterminate over K (*Alg*., IV. 1); throughout this section by an *operator* on K[X] we shall mean a *linear map* U of the vector space K[X] (over K) into itself; since the monomials X^n ($n \geqslant 0$) form a basis for this space, U is determined by the polynomials $U(X^n)$; specifically, if $f(X) = \sum_{k=0}^{\infty} \lambda_k X^k$ with $\lambda_k \in K$, then $U(f) = \sum_{k=0}^{\infty} \lambda_k \, U(X^k)$.

If G is a commutative algebra over K, with an identity element, the G-module G[X] is obtained by extending the field of scalars K of the vector space K[X] to G; every operator U on K[X] extends in a unique manner to a linear map of the G-module G[X] to itself, which we again denote by U (*Alg*., II, p. 278); for each element $g(X) = \sum_{k=0}^{\infty} \gamma_k X^k$, with $\gamma_k \in G$, one has $U(g) = \sum_{k=0}^{\infty} \gamma_k U(X^k)$.

Consider in particular the case where G = K[Y]; then G[X] is the ring K[X,Y] of polynomials in two indeterminates over K; to avoid confusion we denote the extension of U to G[X] by U_X. For any polynomial $g(X, Y) = \sum_{k=0}^{\infty} \gamma_k(Y) X^k$ where $\gamma_k(Y) \in K[Y]$ one thus has $U_X(g) = \sum_{k=0}^{\infty} \gamma_k(Y) U(X^k)$. Since U_X is linear one sees that if one writes $g(X, Y) = \sum_{h=0}^{\infty} \beta_h(X) Y^h$ then one also has $U_X(g) = \sum_{h=0}^{\infty} U(\beta_h) Y^h$.

Under the canonical isomorphism of K[X] onto K[Y] which associates Y with X the operator U transforms to an operator on K[Y], which we will denote U_Y, to avoid confusion, so that $U_Y(f(Y))$ is the polynomial obtained by replacing X by Y in the polynomial $U(f(X)) = U_X(f(X))$. This operator U_Y can in turn be extended to an operator (again denoted by U_Y) on K[X,Y]: if $g(X, Y) = \sum_{h=0}^{\infty} \beta_h(X) Y^h$ then $U_Y\big(g(X, Y)\big) = \sum_{h=0}^{\infty} \beta_h(X) U_Y(Y^h)$.

As an example of these extensions we cite the *derivation* operator D on K[X] (*Alg*., IV. 6), which gives the partial derivation operators D_X and D_Y on K[X,Y].

For any polynomial $f \in K[X]$ we denote by $T_Y(f)$ the polynomial $f(X+Y)$ in K[X,Y]; the map T_Y is a K-linear map from K[X] into K[X,Y], called a *translation operator.*

DEFINITION 1. *One says that an operator U on* K[X] *is a composition operator if it commutes with the translation operator, that is, if $U_X T_Y = T_Y U$.*

In other words, if f is an arbitrary polynomial in K[X], and if $g = U(f)$, one must have $g(X+Y) = U_X(f(X+Y))$.

It follows immediately from this definition that for every polynomial $f(X) \in K[X]$ one has, in the above notation,

$$U_X\big(f(X+Y)\big) = U_Y\big(f(X+Y)\big). \tag{1}$$

Examples. 1) For every $\lambda \in K$ the operator which to every polynomial $f(X)$ associates the polynomial $f(X+\lambda)$ is a composition operator.

2) The *derivation* D on K[X] is a composition operator (*cf.* prop. 1).

Remark. Since K is an infinite field the operator U on K[X] is a composition operator if and only if for every polynomial $f \in K[X]$ and every element $\alpha \in K$ one has, putting $g = U(f)$, that $g(X+\alpha) = U(f(X+\alpha))$. (*Alg*., IV. 16, cor.).

It is clear that every linear combination of composition operators, with coefficients in K, is a composition operator; the same is true for the composition of two composition operators. In other words, the composition operators form a *subalgebra Γ* of the algebra of endomorphisms of the vector space K[X].

PROPOSITION 1. *For an operator U on* K[X] *to be a composition operator it is necessary and sufficient that it commute with the derivation* D *on* K[X].

Indeed, the Taylor formula shows that for every polynomial $f \in K[X]$ one has

$$U_X\big(f(X+Y)\big) = U_X\left(\sum_{k=0}^{\infty} \frac{1}{k!} Y^k D^k f(X)\right) = \sum_{k=0}^{\infty} \frac{1}{k!} Y^k U\big(D^k f(X)\big);$$

if one puts $g = U(f)$ then one has

$$g(X+Y) = \sum_{k=0}^{\infty} \frac{1}{k!} Y^k D^k g(X) = \sum_{k=0}^{\infty} \frac{1}{k!} Y^k D^k \big(U(f(X))\big);$$

for U to be a composition operator, one must then have $UD^k = D^k U$ for every integer $k \geqslant 1$, and in particular $UD = DU$. Conversely, if this relation holds, it implies $UD^k = D^k U$ for every integer $k \geqslant 1$, by induction on k; the Taylor formula then shows that $g(X+Y) = U_X\big(f(X+Y)\big)$.

For every polynomial $f \in \mathrm{K[X, Y]}$ we denote by $U_0(f)$ the term *independent of* X in the polynomial $U_{\mathrm{X}}(f)$; in particular, if $f \in \mathrm{K[X]}$, $U_0(f)$ is the *constant term* in $U(f)$, and U_0 is a *linear form* on K[X]. For every polynomial $f \in \mathrm{K[X]}$ let $g = U(f)$; by def. 1 of VI, p. 270

$$g(\mathrm{X}+\mathrm{Y}) = U_{\mathrm{X}}\big(f(\mathrm{X}+\mathrm{Y})\big) = U_{\mathrm{X}}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\mathrm{X}^k\,\mathrm{D}^k f(\mathrm{Y})\right) = \sum_{k=0}^{\infty}\frac{1}{k!}U(\mathrm{X}^k)\,\mathrm{D}^k f(\mathrm{Y})$$

and if, in this formula, one replaces X by 0, one obtains

$$g(\mathrm{Y}) = \sum_{k=0}^{\infty}\frac{1}{k!}U_0(\mathrm{X}^k)\,\mathrm{D}^k f(\mathrm{Y}).$$

Thus one sees that

$$U\big(f(\mathrm{X})\big) = \sum_{k=0}^{\infty}\frac{1}{k!}\,\mu_k\,\mathrm{D}^k f(\mathrm{X}) \tag{2}$$

where μ_k *is the constant term in the polynomial* $U(\mathrm{X}^k)$.

This formula shows that the μ_k determine the composition operator U completely; conversely, if (μ_n) is an *arbitrary* sequence of elements of K then the formula (2) defines an operator U which clearly commutes with D, so (VI, p. 270, prop. 1) is a composition operator. From now on we shall write (2) in the form

$$U = \sum_{k=0}^{\infty}\frac{1}{k!}\mu_k\mathrm{D}^k. \tag{3}$$

This formula can be interpreted in topological language as follows: if one considers the discrete topology on K[X], and the topology of *simple* convergence on the algebra End(K[X]) of endomorphisms of K[X] (*Gen. Top.*, X, p. 277), the series with general term $\frac{1}{k!}\mu_k\mathrm{D}^k$ is commutatively convergent in End(K[X]) and has sum U (*Gen. Top.*, III, p. 269).

The formula (3) shows that to every *formal series* $u(\mathrm{S}) = \sum_{k=0}^{\infty}\alpha_k\,\mathrm{S}^k$ in one indeterminate over K (*Alg.*, IV. 41) one can associate the composition operator $U = \sum_{k=0}^{\infty}\alpha_k\,\mathrm{D}^k$, which in future we shall denote by $u(\mathrm{D})$. This remark can be clarified in the following manner:

THEOREM 1. *The map which to every formal series* $u(\mathrm{S}) = \sum_{k=0}^{\infty}\alpha_k\,\mathrm{S}^k$ *in one indeterminate over* K *associates the composition operator* $u(\mathrm{D}) = \sum_{k=0}^{\infty}\alpha_k\,\mathrm{D}^k$ *on* K[X] *is an isomorphism of the algebra* K[[S]] *of formal series onto the algebra* Γ *of composition operators.*

One verifies immediately that this map is an homomorphism. It remains to see that it is injective, in other words, that the relation $\sum_{k=0}^{\infty} \alpha_k D^k = 0$ implies $\alpha_k = 0$ for every k; now $h!\,\alpha_h$ is the constant term in the polynomial obtained by applying $\sum_{k=0}^{\infty} \alpha_k D^k$ to X^h, whence the theorem.

COROLLARY. *The algebra Γ of composition operators in* K[X] *is commutative.*

Example. If U is the operator which to each polynomial $f(X)$ associates $f(X+\lambda)$ (where $\lambda \in K$), one has $U_0(X^k) = \lambda^k$, and so $U = \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda D)^k$. By analogy with the series expansion of e^x (III, p. 105) we write e^S or $\exp(S)$ for the formal series $\sum_{n=0}^{\infty} \frac{1}{n!} S^n$ in the ring K[[S]]; so one can write $U = e^{\lambda D}$. Replacing the field K by the field of rational fractions K(Y) in this argument one sees similarly that the *translation operator* T_Y can be written e^{YD}.

Furthermore, in the ring K[[S, T]] of formal series in two indeterminates over K one has

$$
\begin{aligned}
(\exp S)(\exp T) &= \sum_{p,q} \frac{S^p}{p\,!}\,\frac{T^q}{q\,!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\binom{n}{0}S^n + \binom{n}{1}S^{n-1}T + \cdots + \binom{n}{n}T^n\right) \qquad (4)\\
&= \exp(S+T)
\end{aligned}
$$

and in particular

$$(\exp S)(\exp(-S)) = 1 \qquad (5)$$

which justifies the notation we have introduced.

Scholium. The isomorphism of the algebra K[[S]] of formal series and the algebra Γ of composition operators on K[X] sometimes allows one to prove propositions on formal series most simply, by proving them for the corresponding composition operators (*cf.* VI, p. 274, prop. 6).

2. APPELL POLYNOMIALS ATTACHED TO A COMPOSITION OPERATOR

Given a composition operator $U = \sum_{k=0}^{\infty} \alpha_k D^k \neq 0$ let p be the least of the integers k such that $\alpha_k \neq 0$; we shall say that p is the *order* of the operator U.

PROPOSITION 2. *Every composition operator of order* 0 *is invertible in the algebra* Γ *of composition operators on* K[X].

Indeed, a formal series $\sum_{k=0}^{\infty} \alpha_k \mathrm{S}^k$ such that $\alpha_0 \neq 0$ is *invertible* in the ring K[[S]] (*Alg.*, IV. 41); the proposition thus follows from th. 1 of VI, p. 271.

PROPOSITION 3. *Let* U *be a composition operator of order* p; *then* $U(f) = 0$ *for every polynomial* f *of degree* $< p$; *for every polynomial* $f \neq 0$ *of degree* $n \geqslant p$, $U(f)$ *is a polynomial* $\neq 0$ *of degree* $n - p$.

This is an immediate consequence of formula (2) of VI, p. 271 and of the definition of the order of U.

It is clear that every operator U of order p can be written in a unique way as $U = \mathrm{D}^p\, V = V\, \mathrm{D}^p$, where V is an operator of order 0 (and so invertible).

DEFINITION 2. *Let* $U = \mathrm{D}^p\, V$ *be a composition operator of order* p *on* K[X]. *The polynomial* $u_n(\mathrm{X}) = V^{-1}(\mathrm{X}^n)$ *is called the Appell polynomial of index* n *attached to the operator* U.

If $V^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!} \beta_k\, \mathrm{D}^k$ (with $\beta_0 \neq 0$) one thus has

$$u_n(\mathrm{X}) = \sum_{k=0}^{n} \binom{n}{k} \beta_k\, \mathrm{X}^{n-k}. \tag{6}$$

One verifies that u_n is a polynomial of *degree n* (prop. 3); further

$$u_n(0) = \beta_n.$$

PROPOSITION 4. *The Appell polynomials attached to* U *satisfy the relations*

$$\frac{du_n}{d\mathrm{X}} = n\, u_{n-1} \tag{7}$$

$$u_n(\mathrm{X} + \mathrm{Y}) = \sum_{k=0}^{n} \binom{n}{k} u_{n-k}(\mathrm{X})\, \mathrm{Y}^k \tag{8}$$

$$U\big(u_n(\mathrm{X})\big) = \frac{n!}{(n-p)!}\, \mathrm{X}^{n-p}. \tag{9}$$

These formulae are in fact respectively equivalent to the following relations (bearing def. 2 in mind):

$$\mathrm{D}V^{-1} = V^{-1}\mathrm{D} \tag{10}$$

$$\big(\exp(\mathrm{Y}\mathrm{D}_{\mathrm{X}})\big)V_{\mathrm{X}}^{-1} = V_{\mathrm{X}}^{-1}\exp(\mathrm{Y}\mathrm{D}_{\mathrm{X}}) \tag{11}$$

$$UV^{-1} = \mathrm{D}^p. \tag{12}$$

PROPOSITION 5. *For every polynomial* $f \in \mathrm{K[X]}$ *and every composition operator* U *of order* p *one has*

$$f^{(p)}(\mathrm{X}+\mathrm{Y}) = \sum_{k=0}^{\infty} \frac{1}{k!}\, U\big(f^{(k)}(\mathrm{X})\big)\, u_k(\mathrm{Y}) \tag{13}$$

(*generalized Taylor expansion*).

Indeed, if one puts $U = \mathrm{D}^pV = V\mathrm{D}^p$ one has (VI, p. 270, formula (1))

$$V_{\mathrm{X}}^{-1}\big(f(\mathrm{X}+\mathrm{Y})\big) = V_{\mathrm{Y}}^{-1}\big(f(\mathrm{X}+\mathrm{Y})\big) = \sum_{k=0}^{\infty}\frac{1}{k!}\, f^{(k)}(\mathrm{X})\, u_k(\mathrm{Y}) \tag{14}$$

by the Taylor formula and by def. 2 of VI, p. 273; it suffices to apply the operator U_{X} to the first and last terms of formula (14) to obtain (13).

3. GENERATING SERIES FOR THE APPELL POLYNOMIALS

Let E be the ring of *formal series* in an indeterminate S, with coefficients in the ring of polynomials K[X] (*Alg*., IV. 41); in other words, the ring of formal series $g(\mathrm{X}, \mathrm{S}) = \sum_{n=0}^{\infty} \alpha_n(\mathrm{X})\,\mathrm{S}^n$ where the α_n belong to K[X]. For every operator U on K[X] one defines a map U_{X} of E to itself by putting $U_{\mathrm{X}}\big(g(\mathrm{X}, \mathrm{S})\big) = \sum_{n=0}^{\infty} U(\alpha_n)\mathrm{S}^n$. It is clear that E is a module over the ring K[[S]] of formal series in S with coefficients in K; by the linearity of U on K[X] one verifies immediately that for every element $\theta \in \mathrm{K[[S]]}$ and every $g \in \mathrm{E}$ one has $U_{\mathrm{X}}(\theta g) = \theta U_{\mathrm{X}}(g)$; in other words, U_{X} is a *linear* map of the module E into itself.

PROPOSITION 6. *Let* $U = \mathrm{D}^pV = u(\mathrm{D})$ *be a composition operator of order* p *on* K[X], $u(\mathrm{S})$ *being a formal power series of order* p *in* K[[S]]. *Then*

$$U_{\mathrm{X}}\big(\exp(\mathrm{XS})\big) = u(\mathrm{S})\exp(\mathrm{XS}) \tag{15}$$

$$\frac{\mathrm{S}^p}{u(\mathrm{S})}\exp(\mathrm{XS}) = \sum_{n=0}^{\infty}\frac{1}{n!}\, u_n(\mathrm{X})\,\mathrm{S}^n \tag{16}$$

u_n *being the Appell polynomial of index* n *attached to* U.

By the scholium to th. 1 (VI, p. 271), to establish (15) it is enough to show that for every polynomial $f(\mathrm{Y}) \in \mathrm{K}[\mathrm{Y}]$ one has

$$U_{\mathrm{X}}\big(\exp(\mathrm{X}\mathrm{D}_{\mathrm{Y}})\,(f(\mathrm{Y}))\big) = u(\mathrm{D}_{\mathrm{Y}})\big(\exp(\mathrm{X}\mathrm{D}_{\mathrm{Y}})(f(\mathrm{Y}))\big). \tag{17}$$

Now the first term in (17) is $U_{\mathrm{X}}\big(f(\mathrm{X}+\mathrm{Y})\big)$, and, since $U = u(\mathrm{D})$, the second term in (17) is $U_{\mathrm{Y}}\big(f(\mathrm{X}+\mathrm{Y})\big)$, so the identity (17) reduces to (1) (VI, p. 270).

It now suffices to apply (15) to the composition operator $V^{-1} = \mathrm{D}^p/u(\mathrm{D})$ to obtain (16), since, by definition, one has

$$V^{-1}\big(\exp(\mathrm{XS})\big) = \sum_{n=0}^{\infty} \frac{1}{n!}\, u_n(\mathrm{X})\,\mathrm{S}^n.$$

Note that (16) can also be obtained by multiplying the formal series $\mathrm{S}^p/u(\mathrm{S})$ and $\exp(\mathrm{XS})$, taking account of (6).

One says that the formal series (16) is the *generating series* of the Appell polynomials attached to U.

4. BERNOULLI POLYNOMIALS

Consider the composition operator U defined by

$$U\big(f(\mathrm{X})\big) = f(\mathrm{X}+1) - f(\mathrm{X});$$

one can write $U = e^{\mathrm{D}} - 1$ (VI, p. 270, *Example* 1); this is an operator *of order* 1, and if one puts $U = \mathrm{D}V$ one has $V^{-1} = \dfrac{\mathrm{D}}{e^{\mathrm{D}} - 1}$. The Appell polynomial of degree n corresponding to the operator U is called the *Bernoulli polynomial* of degree n and is denoted by $\mathrm{B}_n(\mathrm{X})$; if one puts $b_n = \mathrm{B}_n(0)$ one has the formulae

$$\mathrm{B}_n(\mathrm{X}) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}\,\mathrm{X}^k \tag{18}$$

$$\frac{\mathrm{S}\,e^{\mathrm{XS}}}{e^{\mathrm{S}} - 1} = \sum_{n=0}^{\infty} \frac{1}{n!}\,\mathrm{B}_n(\mathrm{X})\,\mathrm{S}^n \tag{19}$$

and in particular

$$\frac{\mathrm{S}}{e^{\mathrm{S}} - 1} = \sum_{n=0}^{\infty} \frac{1}{n!}\,b_n\,\mathrm{S}^n \tag{20}$$

The formulae (7) and (9) of VI, p. 273, give, for the Bernoulli polynomials, the relations

$$\frac{d\mathrm{B}_n}{d\mathrm{X}} = n\mathrm{B}_{n-1}(X) \tag{21}$$

$$\mathrm{B}_n(\mathrm{X}+1) - \mathrm{B}_n(\mathrm{X}) = n\,\mathrm{X}^{n-1}. \tag{22}$$

In particular, one has $B_n(1) - B_n(0) = 0$ for $n > 1$, which, taking account of (18), gives the induction relation

$$\sum_{m=0}^{n-1} \binom{n}{m} b_m = 0 \qquad (\text{for } n > 1) \tag{23}$$

which enables one to calculate the b_n step-by-step. These numbers are clearly *rational*; since one can write

$$\frac{S}{e^S - 1} = -\frac{S}{2} + \frac{S}{2}\,\frac{e^S + 1}{e^S - 1}$$

and since (VI, p. 272, formula (5))

$$\frac{e^{-S} + 1}{e^{-S} - 1} = -\frac{e^S + 1}{e^S - 1}$$

one sees that in the formal series for $\dfrac{S}{2}\,\dfrac{e^S + 1}{e^S - 1}$ all the terms of *odd* degree have coefficient zero; so one has

$$b_0 = 1, \qquad b_1 = -\tfrac{1}{2}, \qquad b_{2n-1} = 0 \quad \text{for } n > 1. \tag{24}$$

The rational numbers b_{2n} ($n \geqslant 1$) are called the *Bernoulli numbers*; we shall see (VI, p. 288) that b_{2n} has the sign of $(-1)^{n-1}$. The formula (23) gives, for the first values of n,

$$b_2 = \frac{1}{6}, \qquad b_4 = -\frac{1}{30}, \qquad b_6 = \frac{1}{42}, \qquad b_8 = -\frac{1}{30},$$

$$b_{10} = \frac{5}{66}, \qquad b_{12} = -\frac{691}{2730}, \qquad b_{14} = \frac{7}{6},$$

$$b_{16} = -\frac{3617}{510}, \qquad b_{18} = \frac{43867}{798}, \qquad b_{20} = -\frac{174611}{330}, \qquad b_{22} = \frac{854513}{138},$$

$$b_{24} = -\frac{236364091}{2730}, \qquad b_{26} = \frac{8553103}{6}, \qquad b_{28} = -\frac{23749461029}{870}.$$

Note that the numerators 691, 3617, 43867 are prime; the prime factorisations of the others are

$$\begin{aligned} 174611 &= 283 \times 617 \\ 854513 &= 11 \times 131 \times 593 \\ 236364091 &= 103 \times 2294797 \\ 8553103 &= 13 \times 657931 \\ 23749461029 &= 7 \times 9349 \times 362903. \end{aligned}$$

From these we deduce expressions for the first Bernoulli polynomials

$$B_0(X) = 1, \qquad B_1(X) = X - \frac{1}{2}, \qquad B_2(X) = X^2 - X + \frac{1}{6},$$

$$B_3(X) = X^3 - \frac{3}{2}X^2 + \frac{1}{2}X, \qquad B_4(X) = X^4 - 2X^3 + X^2 - \frac{1}{30}.$$

5. COMPOSITION OPERATORS ON FUNCTIONS OF A REAL VARIABLE

Let I be an interval in $\mathbf{R}$ containing the interval $\mathbf{R}_+ = [0, +\infty[$; let E be a vector space over the field $\mathbf{C}$, formed of functions of a real variable with complex values, defined on I. We suppose that for every $a \geqslant 0$ and every function $f \in \mathrm{E}$ the function $x \mapsto f(x+a)$ belongs to E; further, we assume that E contains the restrictions to I of the *polynomials with complex coefficients* and the *exponentials* $e^{\lambda x}$, where λ is an arbitrary *complex* number. Any linear map U of E into the space of all maps from I into the field $\mathbf{C}$ of complex numbers will be called an *operator* on E; if $f \in \mathrm{E}$ and $g = U(f)$ it will be convenient to use the notation

$$g(x) = U_x^\xi\big(f(\xi)\big)$$

where ξ is a *dummy* variable in the functional symbolism of the right-hand term (*cf.* II, p. 58). For every $a \geqslant 0$ the operator which to any function $f \in \mathrm{E}$ associates the restriction to I of the function $x \mapsto f(x+a)$ is called the *translation operator by* a.

DEFINITION 3. *One says that an operator U on* E *is a composition operator if, for every $a \geqslant 0$, it is permutable with the translation operator by a.*

In the notation introduced above, this definition becomes the identity

$$U_{x+a}^\xi\big(f(\xi)\big) = U_x^\xi\big(f(\xi + a)\big) \tag{25}$$

in x and a ($x \in \mathrm{I}$, $a \geqslant 0$). One can exchange the rôles of x and a in this identity if $x \geqslant 0$, then put $a = 0$; one thus obtains for $x \geqslant 0$

$$U_x^\xi\big(f(\xi)\big) = U_0^\xi\big(f(\xi + x)\big) \tag{26}$$

where U_0 is the *linear form* on E which to each function $f \in \mathrm{E}$ associates the value $g(0)$ of $g = U(f)$.

If f is a polynomial, one has $f(\xi + x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(\xi)\, x^k$, and formula (26) shows that $U(f)$ is also a polynomial; restricted to the set of polynomials in x, with coefficients in $\mathbf{C}$, (a set which one can identify with the algebra $\mathbf{C}[\mathrm{X}]$), the operator U is then a composition operator in the sense of def. 1 of VI, p. 270, and all the results of the preceding sections can be applied to it.

We again write u_n for the Appell polynomials attached to the operator U. To the generalized Taylor expansion of a polynomial (VI, p. 274, formula (13)) there corresponds, for more general functions, the following result:

THEOREM 2. *Let f be a function admitting a continuous $(n+1)^{th}$ derivative on* I, *and belonging, with all its derivatives $f^{(m)}$ for $1 \leqslant m \leqslant n$, to* E. *If U is a composition operator of order $p \leqslant n$ on* E *one has, for $x \geqslant 0$ and $h \geqslant 0$*

$$f^{(p)}(x+h) = \sum_{m=0}^{n} \frac{1}{m!}\, u_m(x)\, U_h^{\xi}\left(f^{(m)}(\xi)\right) \;+\; \mathrm{R}_n(x,h) \tag{27}$$

with

$$\mathrm{R}_n(x,h) = -U_h^{\xi}\left(\int_0^{\xi-x-h} \frac{1}{n!}\, u_n(x+\eta)\, f^{(n+1)}(\xi-\eta)\, d\eta\right) \tag{28}$$

(*generalized Taylor expansion*).

Consider the integral $\displaystyle\int_0^{\xi-x-h} \frac{1}{n!}\, u_n(x+\eta)\, f^{(n+1)}(\xi-\eta)\, d\eta$, defined for all $\xi \in \mathrm{I}$, and apply to it the formula for integration by parts of order $n+1$ (II, p. 59, formula (11)); taking account of the relations

$$u_n^{(k)} = n(n-1)\ldots(n-k+1)u_{n-k}$$

derived from (7) (VI, p. 273) by recursion, one obtains

$$\begin{aligned}&\int_0^{\xi-x-h} \frac{1}{n!}\, u_n(x+\eta)\, f^{(n+1)}(\xi-\eta)\, d\eta \\ &\qquad = \sum_{m=0}^{n} \frac{1}{m!}\, u_m(x) f^{(m)}(\xi) - \sum_{m=0}^{n} \frac{1}{m!}\, u_m(\xi-h) f^{(m)}(x+h).\end{aligned} \tag{29}$$

We apply the operator U to the two sides of (29), considered as functions of ξ, then take the value of the function so obtained for the value h of the variable ξ; remarking that, by the formulae (26) (VI, p. 277) and (9) (VI, p. 272), one has

$$U_h^{\xi}\left(u_m(\xi-h)\right) = U_0^{\xi}\left(u_m(\xi)\right) = \begin{cases} 0 & \text{for} \quad m \neq p \\ p\,! & \text{for} \quad m = p \end{cases}$$

one finally obtains (27).

6. INDICATRIX OF A COMPOSITION OPERATOR

With the same hypotheses as in n° 5, the formula (26) of (VI, p. 277) applied to the function $e^{\lambda x}$, gives

$$U_x^{\xi}(e^{\lambda\xi}) = U_0^{\xi}(e^{\lambda x}\, e^{\lambda\xi}) = e^{\lambda x} U_0^{\xi}(e^{\lambda\xi}) = u(\lambda)e^{\lambda x} \tag{30}$$

on putting $u(\lambda) = U_0^{\xi}(e^{\lambda\xi})$. One says that the function $u(\lambda)$, defined on $\mathbf{C}$, with complex values, is the *indicatrix* of the composition operator U. Note that if the restriction of U to the ring $\mathbf{C}[\mathrm{X}]$ of polynomials is equal to the series

$$\mathrm{D}^p \sum_{n=0}^{\infty} \alpha_n \mathrm{D}^n$$

(VI, p. 271, th. 1) (which we have denoted $u(\mathrm{D})$ in VI, p. 271), the series of *complex terms* with general term $\alpha_n \lambda^{n+p}$ *is not necessarily convergent* for $\lambda \neq 0$, and that even if it converges for certain values of λ, the sum *is not necessarily equal to the indicatrix* $u(\lambda)$ *of* U (VI, p. 291, exerc. 2). We shall say that the composition operator U is *regular* if there exists a neighbourhood of 0 in $\mathbf{C}$ such that the series with general term $\alpha_n \lambda^{n+p}$ is *absolutely convergent* and has sum *equal to the indicatrix* $u(\lambda)$ on this neighbourhood[1]. Let us apply formula (27) of VI, p. 278, to the function $e^{\lambda x}$, making $h = 0$; since $\mathrm{D}^m(e^{\lambda x}) = \lambda^m e^{\lambda x}$ one has $U_0^{\xi}\left(\mathrm{D}^m(e^{\lambda x})\right) = \lambda^m u(\lambda)$; it then follows that for every complex λ such that $u(\lambda) \neq 0$

$$\frac{\lambda^p e^{\lambda x}}{u(\lambda)} = \sum_{m=0}^{n} u_m(x) \frac{\lambda^m}{m!} - \frac{\lambda^{n+1}}{u(\lambda)} U_0^{\xi} \left(\int_0^{\xi - x} \frac{1}{n!} u_n(x + \eta) e^{\lambda(\xi - \eta)} d\eta \right) \tag{31}$$

and in particular, for $x = 0$

$$\frac{\lambda^p}{u(\lambda)} = \sum_{m=0}^{n} \beta_m \frac{\lambda^m}{m!} - \frac{\lambda^{n+1}}{u(\lambda)} U_0^{\xi} \left(\int_0^{\xi} \frac{1}{n!} u_n(\eta) e^{\lambda(\xi - \eta)} d\eta \right) \tag{32}$$

with $\beta_m = u_m(0)$.

If U is a *regular* operator, for all $\lambda \in \mathbf{C}$ such that the entire series $u(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^{n+p}$ and $\sum_{n=0}^{\infty} \beta_n \frac{\lambda^n}{n!}$ are absolutely convergent [2], it follows from formula (16) and the formula for the product of two absolutely convergent series (*Gen. Top.*, VIII, p. 115, prop. 1) that one has

$$\frac{\lambda^p}{u(\lambda)} = \sum_{n=0}^{\infty} \beta_n \frac{\lambda^n}{n!}. \tag{33}$$

Similarly, since the Taylor expansion of $e^{\lambda x}$ is absolutely convergent for every $\lambda \in \mathbf{C}$ and every $x \in \mathbf{C}$ (III, p. 106) one also has (formulae (6) (VI, p. 273) and (16) (VI, p. 274)), for all the values considered and for all $x \in \mathbf{C}$

$$\frac{\lambda^p e^{\lambda x}}{u(\lambda)} = \sum_{n=0}^{\infty} u_n(x) \frac{\lambda^n}{n!}. \tag{34}$$

Remark. One can use formula (33) (resp. (34)) to calculate the β_n (resp. the $u_n(x)$) by using the following lemma on entire series:

[1] Later we shall study series whose general terms are of the form $c_n z^n$ ($c_n \in \mathbf{C}$, $z \in \mathbf{C}$), which one calls *entire series*; in particular we shall see that when such a series is absolutely convergent for $z = z_0$ it is *normally convergent* for $|z| \leqslant |z_0|$.

[2] It follows from the theory of entire series that when one of these series is absolutely convergent in a neighbourhood V of 0, the other is absolutely convergent in a neighbourhood W ⊂ V of 0.

Lemma. *If two entire series* $\sum_{n=0}^{\infty} c_n\lambda^n$, $\sum_{n=0}^{\infty} d_n\lambda^n$ *are absolutely convergent for all* λ *in a neighbourhood of* 0, *and if* $\sum_{n=0}^{\infty} c_n\lambda^n = \sum_{n=0}^{\infty} d_n\lambda^n$ *for these values of* λ, *then* $c_n = d_n$ *for all integers* $n \geqslant 0$ [3].

If, by any procedure, one can find an entire series that converges to $\lambda^p/u(\lambda)$ on a neighbourhood of 0, then the coefficients of this series are necessarily equal to the $\beta_n/n!$. It is this procedure that we shall apply in the examples that follow.

Examples. 1) If U is the identity map one has $u(\lambda) = 1$ and the operator U is clearly regular; since $u_n(x) = x^n$ the formula (27) of VI, p. 278, can be written, putting $t = \xi - \eta$

$$f(x+h) = \sum_{n=0}^{\infty} \frac{1}{m!} f^{(m)}(h)\, x^m + \int_h^{x+h} f^{(n+1)}(t)\, \frac{(x+h-t)^n}{n!}\, dt$$

that is, it reduces to the Taylor formula (II, p. 62).

2) Let us take for U the composition operator which, to every function f defined on $\mathbf{R}_+$, associates the function $x \mapsto f(x+1) - f(x)$; then

$$U_x^{\xi}\big(f(\xi)\big) = f(x+1) - f(x);$$

we have seen (VI, p. 275) that the restriction of U to $\mathbf{C}[\mathrm{X}]$ is equal to $e^{\mathrm{D}} - 1$. Since, on the other hand, $u(\lambda) = e^{\lambda} - 1$, the operator U is *regular*; we shall see (VI, p. 288) how to determine the Bernoulli numbers b_n by calculating an expansion of $\dfrac{\lambda}{e^{\lambda} - 1}$ as a convergent entire series. On applying formula (27) of VI, p. 278 to a *primitive* of a function f admitting a continuous n^{th} derivative on $\mathbf{R}_+$, we obtain

$$\begin{aligned} f(x+h) = {} & \int_h^{h+1} f(t)\, dt \\ & + \sum_{m=1}^{n} \frac{1}{m!} \mathrm{B}_m(x) \left(f^{(m-1)}(h+1) - f^{(m-1)}(h)\right) + \mathrm{R}_n(x,h) \end{aligned} \tag{35}$$

with

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[3] This lemma is a particular case of a general result that we will prove later; here is the proof. If an entire series $\sum_{n=0}^{\infty} c_n\lambda^n$ is absolutely convergent for $\lambda = \lambda_0$ then for every integer $k \geqslant 0$ the series $\sum_{n=0}^{\infty} c_{n+k}\lambda^n$ is normally convergent for $|\lambda| \leqslant |\lambda_0|$, so is continuous on this disc (*Gen. Top.*, X, p. 283); one concludes that $\sum_{n=k+1}^{\infty} c_n\lambda^n = o(\lambda^k)$ on a neighbourhood of 0. The lemma then follows from the uniqueness of the coefficients of an asymptotic expansion of a function in terms of the λ^n (V, p. 223).

$$\begin{aligned}\mathrm{R}_n(x,h) = -\int_0^{1-x} \frac{\mathrm{B}_n(x+\eta)}{n!}\, f^{(n)}(h+1-\eta)\,d\eta \\ + \int_0^{-x} \frac{\mathrm{B}_n(x+\eta)}{n!}\, f^{(n)}(h-\eta)\,d\eta.\end{aligned} \tag{36}$$

*3) Let E be the vector space of functions f defined and continuous on $\mathbf{R}$, and such that the integral $\int_{-\infty}^{+\infty} f(x+\xi)\,e^{-\xi^2/2}\,d\xi$ converges for all $x \geqslant 0$. The operator U defined by

$$U_x^\xi\Big(f(\xi)\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2/2}\, f(x+\xi)\,d\xi$$

is then defined on E and is clearly a composition operator. The space E contains all the exponentials $e^{\lambda x}$ (λ arbitrary complex), and one has

$$\begin{aligned}u(\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(\xi^2/2)+\lambda\xi}\,d\xi \\ &= \frac{1}{\sqrt{2\pi}}\, e^{\lambda^2/2} \int_{-\infty}^{+\infty} e^{-(\xi-\lambda)^2/2}\,d\xi = e^{\lambda^2/2}\end{aligned}$$

(*cf.* III, p. 120, exerc. 24, and VII, p. 313, formula (22)). One has $n!\,\alpha_n = U_0^\xi(\xi^n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2/2}\,\xi^n\,d\xi$. For every integer n one can write

$$\sum_{k=0}^{n} \int_{-\infty}^{+\infty} \frac{|\lambda\xi|^k}{k!} e^{-\xi^2/2}\,d\xi \leqslant 2 \int_0^{+\infty} e^{-(\xi^2/2)+|\lambda|\xi}\,d\xi.$$

The series $\displaystyle\sum_{n=0}^{\infty} e^{-\xi^2/2}\,\frac{(\lambda\xi)^n}{n!}$ can therefore be integrated term-by-term over $\mathbf{R}$ (II, p. 72, cor. 1), which proves that the series $\displaystyle\sum_{n=0}^{\infty} \alpha_n\lambda^n$ converges absolutely for every $\lambda \in \mathbf{C}$, and has a sum equal to $u(\lambda) = e^{\lambda^2/2} = \displaystyle\sum_{n=0}^{\infty} \frac{\lambda^{2n}}{2^n n!}$; thus the operator U is *regular*. Applying the lemma mentioned above shows that $\alpha_{2n} = 1/2^n n!$, $\alpha_{2n+1} = 0$ for every $n \geqslant 0$; the operator U is thus of order 0. One has

$$\frac{1}{u(\lambda)} = e^{-\lambda^2/2} = \sum_{n=0}^{\infty} \frac{(-1)^n\,\lambda^{2n}}{2^n n!},$$

the series being absolutely convergent for every $\lambda \in \mathbf{C}$; another application of the lemma shows that $\beta_{2n} = \dfrac{(-1)^n}{2^n}\,\dfrac{(2n)!}{n!}$, $\beta_{2n+1} = 0$; further, the series $\displaystyle\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,u_n(x)$ is absolutely convergent for every $\lambda \in \mathbf{C}$ and every $x \in \mathbf{R}$, and one has

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,u_n(x) = \exp\left(-\frac{\lambda^2}{2} + \lambda x\right) = \exp\left(\frac{x^2}{2}\right)\,\exp\left(-\frac{1}{2}\,(\lambda - x)^2\right).$$

On applying the Taylor formula to the function $\exp(-x^2/2)$ one then obtains the following expression for the polynomials $u_n(x)$:

$$u_n(x) = (-1)^n\,e^{x^2/2}\,\frac{d^n}{dx^n}(e^{-x^2/2}).$$

This polynomial is called the *Hermite polynomial* of degree n, and is often denoted by $\mathrm{H}_n(x)$. The formulae (7), (8) and (9) of VI, p. 273, here give

$$\frac{d\mathrm{H}_n}{dx} = n\mathrm{H}_{n-1}(x)$$

$$\mathrm{H}_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{H}_{n-k}(x)\, y^k$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2/2}\, \mathrm{H}_n(x+\xi)\, d\xi = x^n$$

and the formula (27) of VI, p. 278, becomes, for $h = 0$

$$\sqrt{2\pi}\, f(x) = \sum_{m=0}^{n} \left(\int_{-\infty}^{+\infty} e^{-\xi^2/2} f^{(m)}(\xi)\, d\xi \right) \frac{\mathrm{H}_m(x)}{m!} - \int_{-\infty}^{+\infty} d\xi \int_0^{\xi} \frac{\mathrm{H}_n(x+\eta)}{n!}\, e^{-(\xi+x)^2/2}\, f^{(n+1)}(x+\xi-\eta)\, d\eta. \;*$$

7. THE EULER-MACLAURIN SUMMATION FORMULA

In the formula (35) of VI, p. 280, let us replace x by 0 and h by x; since $\mathrm{B}_m(0) = b_m$ it follows from the relations (24) of VI, p. 276, that for each integer $p > 0$ one can write

$$f(x) = \int_x^{x+1} f(t)\, dt - \tfrac{1}{2}\big(f(x+1) - f(x)\big) + \sum_{k=1}^{p} \frac{b_{2k}}{(2k)!} \big(f^{(2k-1)}(x+1) - f^{(2k-1)}(x)\big) + \mathrm{R}_p(x) \tag{37}$$

with

$$\mathrm{R}_p(x) = -\frac{1}{(2p+1)!} \int_0^1 \mathrm{B}_{2p+1}(t)\, f^{(2p+1)}(x+1-t)\, dt. \tag{38}$$

In this formula let us successively replace x by $x+1, x+2, \ldots, x+n$, and combine the formulae obtained, one by one; we obtain

$$\left\{ \begin{aligned} & f(x) + f(x+1) + \cdots + f(x+n) \\ & \quad = \int_x^{x+n+1} f(t)\, dt - \frac{1}{2}\big(f(x+n+1) - f(x)\big) \\ & \qquad + \sum_{k=1}^{p} \frac{b_{2k}}{(2k)!}\big(f^{(2k-1)}(x+n+1) - f^{(2k-1)}(x)\big) + \mathrm{T}_p(x,n) \end{aligned} \right. \tag{39}$$

with

$$\mathrm{T}_p(x,n) = -\frac{1}{(2p+1)!} \int_0^1 \mathrm{B}_{2p+1}(t) \left(\sum_{k=0}^{n} f^{(2p+1)}(x+k+1-t) \right) dt. \tag{40}$$

The remainder $\mathrm{T}_p(x, n)$ in this formula can be rewritten in the following way: denote by $\overline{\mathrm{B}}_{2p+1}(t)$ the *periodic* function with period 1 which is equal to $\mathrm{B}_{2p+1}(t)$ on the interval $[0, 1[$. Then

$$\int_0^1 \mathrm{B}_{2p+1}(t)\, f^{(2p+1)}(x+k+1-t)\, dt = \int_k^{k+1} \overline{\mathrm{B}}_{2p+1}(1-s)\, f^{(2p+1)}(x+s)\, ds$$

and consequently

$$\mathrm{T}_p(x, n) = -\frac{1}{(2p+1)!} \int_0^{n+1} \overline{\mathrm{B}}_{2p+1}(1-s) f^{(2p+1)}(x+s)\, ds. \tag{41}$$

The formula (39) is called the *Euler-Maclaurin summation formula*; it is applicable to every complex function having a continuous $(2p+1)^{th}$ derivative on an interval $[x_0, +\infty[$, for every $x \geqslant x_0$. We shall see (VI, p. 288) how to estimate the *remainder* $\mathrm{T}_p(x, n)$ in this formula.

§ 2. EULERIAN EXPANSIONS OF THE TRIGONOMETRIC FUNCTIONS AND BERNOULLI NUMBERS

1. EULERIAN EXPANSION OF $\cot z$

By formula (20) of VI, p. 275, the numbers $b_n/n!$ are the coefficients in the expansion of $\mathrm{S}/(e^{\mathrm{S}}-1)$ as a *formal* series; we shall show in this section that the function $z/(e^z-1)$ is equal to the sum of an absolutely convergent entire series on a neighbourhood of 0 in $\mathbf{C}$; it will follow from the lemma of VI, p. 280, that the coefficients of this series are the numbers $b_n/n!$, from which we shall deduce estimates for the Bernoulli numbers b_n.

In the first place we note that

$$\frac{z}{e^z-1} = -\frac{z}{2} + \frac{z}{2}\,\frac{e^z+1}{e^z-1} = -\frac{z}{2} + \frac{iz}{2} \cot \frac{iz}{2}. \tag{1}$$

We shall obtain below a series expansion for $\cot z$, valid for every z not an integer multiple of π.

PROPOSITION 1. *For every complex number z and every integer n one has*

$$\sin nz = 2^{n-1} \sin z \sin\left(z+\frac{\pi}{n}\right) \sin\left(z+\frac{2\pi}{n}\right) \ldots \sin\left(z+\frac{(n-1)\pi}{n}\right). \tag{2}$$

Indeed, one can write

$$\begin{aligned}\sin nz &= \frac{e^{niz} - e^{-niz}}{2i} = \frac{e^{-niz}(e^{2niz} - 1)}{2i} \\ &= \frac{e^{-niz}(e^{2iz} - 1)(e^{2iz} - e^{-2i\pi/n}) \ldots (e^{2iz} - e^{-2(n-1)i\pi/n})}{2i} \\ &= \mathrm{A} \sin z \, \sin\left(z + \frac{\pi}{n}\right) \, \sin\left(z + \frac{2\pi}{n}\right) \ldots \, \sin\left(z + \frac{(n-1)\pi}{n}\right)\end{aligned}$$

with

$$\mathrm{A} = (2i)^{n-1} \, e^{-\frac{i\pi}{n}(1+2+\cdots+(n-1))} = (2i)^{n-1} \, e^{-i(n-1)\frac{\pi}{2}} = 2^{n-1}.$$

COROLLARY 1. *For every integer n one has*

$$\sin\frac{\pi}{n} \, \sin\frac{2\pi}{n} \, \cdots \, \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}. \tag{3}$$

It suffices to divide both sides of (2) by $\sin z$ and let z tend to 0.

COROLLARY 2. *For every odd integer $n = 2m + 1$, and every complex number z such that nz is not an integral multiple of π, one has*

$$\cot nz = (-1)^m \, \cot z \, \cot\left(z + \frac{\pi}{n}\right) \, \ldots \, \cot\left(z + \frac{(n-1)\pi}{n}\right). \tag{4}$$

Indeed, $\sin n\left(z + \frac{\pi}{2}\right) = \sin\left(nz + \frac{\pi}{2} + m\pi\right) = (-1)^m \cos nz$, whence, replacing z by $z + \frac{\pi}{2}$ in (2),

$$\cos nz = (-1)^m \, 2^{n-1} \, \cos z \, \cos\left(z + \frac{\pi}{n}\right) \ldots \, \cos\left(z + \frac{(n-1)\pi}{n}\right) \tag{5}$$

and the formulae (2) and (5) give (4) on division term-by-term when $\sin nz \neq 0$.

In all that follows we shall always assume that $n = 2m + 1$ is an odd integer; the formula (4) can also be written

$$\cot nz = (-1)^m \prod_{k=-m}^{m} \cot\left(z - \frac{k\pi}{n}\right).$$

For, one has

$$\cot\left(z - \frac{k\pi}{n}\right) = \frac{1 + \tan z \tan\dfrac{k\pi}{n}}{\tan z - \tan\dfrac{k\pi}{n}}$$

for $\tan z$ finite; $\cot nz$ is thus a rational function in $u = \tan z$, whose numerator is of degree $n-1$, and whose denominator, of degree n, has the n simple roots $\tan k\pi/n$; decomposing this into simple fractions yields

$$\cot nz = \sum_{k=-m}^{m} \frac{A_k}{u - \tan\frac{k\pi}{n}} \tag{6}$$

where

$$A_k = \lim_{z\to k\pi/n} \cot nz \left(\tan z - \tan\frac{k\pi}{n}\right) = \lim_{z\to k\pi/n} \frac{\cos nz}{\sin nz} \frac{\sin\left(z - \dfrac{k\pi}{n}\right)}{\cos z \cos\dfrac{k\pi}{n}}$$
$$= \lim_{h\to 0} \frac{\cos nh}{\cos\dfrac{k\pi}{n}\cos\left(h + \dfrac{k\pi}{n}\right)} \frac{\sin h}{\sin nh} = \frac{1}{n\cos^2\dfrac{k\pi}{n}}$$

whence, on isolating the term in (6) corresponding to $k = 0$ and combining the terms corresponding to opposite values of k, and replacing z by z/n,

$$\cot z = \frac{1}{n\tan\dfrac{z}{n}} + \sum_{k=1}^{m} \frac{2n\tan\dfrac{z}{n}}{\cos^2\dfrac{k\pi}{n}\left(n\tan\dfrac{z}{n}\right)^2 - \left(n\sin\dfrac{k\pi}{n}\right)^2} \tag{7}$$

valid for every complex number z not an integral multiple of $\pi/2$. One can write this formula in the form

$$\cot z = \frac{1}{n\tan\dfrac{z}{n}} + \sum_{k=1}^{\infty} v_k(n, z)$$

with $v_k(n, z) = 0$ for $k > m$ and

$$v_k(n, z) = \frac{2n\tan\dfrac{z}{n}}{\cos^2\dfrac{k\pi}{n}\left(n\tan\dfrac{z}{n}\right)^2 - \left(n\sin\dfrac{k\pi}{n}\right)^2}$$

for $1 \leqslant k \leqslant m$. We shall see that for every z contained in a *compact* subset K of $\mathbf{C}$, not containing any integral multiple of π, and for every sufficiently large odd n, the series with general term $v_k(n, z)$ is *normally convergent*. Indeed, as n tends to $+\infty$, $\tan\dfrac{z}{n}$ tends to $\dfrac{z}{n}$ uniformly on K, so there exists a number $M > 0$ such that $\left|n\tan\dfrac{z}{n}\right| \leqslant M$ for every sufficiently large m and every $z \in K$. On the other hand, for $0 \leqslant x \leqslant \pi/2$ one has $\sin x/x \geqslant 1 - \dfrac{x^2}{6} \geqslant \dfrac{1}{2}$, so for $1 \leqslant k \leqslant m$ one has $n\sin\dfrac{k\pi}{n} \geqslant k\pi/2$; consequently, when m is sufficiently large, for every integer k such that $k\pi/2 > \mathrm{M}$ one has $|v_k(n,z)| \leqslant \dfrac{8\mathrm{M}}{k^2\pi^2 - 4\mathrm{M}^2}$, which proves our assertion. For k fixed, $v_k(n,z)$ tends (uniformly on K) to $\dfrac{2z}{z^2 - k^2\pi^2}$ as n tends to $+\infty$. Consequently:

THEOREM 1. *For every complex number z not an integral multiple of π one has*

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2} \tag{8}$$

the series on the right being normally convergent on every compact subset $\mathrm{K} \subset \mathbf{C}$ *not containing any integer multiple of π* (Eulerian expansion of $\cot z$).

2. EULERIAN EXPANSION OF sin *z*

For every *odd* integer $n = 2m + 1$ and complex z the formula (2) of VI, p. 283, can be written

$$\begin{aligned}\sin nz &= (-1)^m\, 2^{n-1} \prod_{k=-m}^{m} \sin\left(z - \frac{k\pi}{n}\right)\\ &= (-1)^m\, 2^{n-1}\ \sin z \prod_{k=1}^{m} \sin\left(z - \frac{k\pi}{n}\right) \sin\left(z + \frac{k\pi}{n}\right).\end{aligned}$$

Now, one has $\sin\left(z - \dfrac{k\pi}{n}\right) \sin\left(z + \dfrac{k\pi}{n}\right) = \sin^2 z - \sin^2 \dfrac{k\pi}{n}$, and, by (3) (VI, p. 284), $\prod\limits_{k=1}^{m} \sin^2 \dfrac{k\pi}{n} = \dfrac{n}{2^{n-1}}$, whence, on replacing z by z/n,

$$\sin z = n \sin \frac{z}{n} \prod_{k=1}^{m} \left(1 - \frac{\sin^2 \dfrac{z}{n}}{\sin^2 \dfrac{k\pi}{n}}\right). \tag{9}$$

One can write this formula as $\sin z = n \sin \dfrac{z}{n} \prod\limits_{k=1}^{m} \big(1 - w_k(n,z)\big)$, with $w_k(n,z) = 0$ for $k > m$ and $w_k(n,z) = \dfrac{\sin^2 \dfrac{z}{n}}{\sin^2 \dfrac{k\pi}{n}}$ for $1 \leqslant k \leqslant m$. We shall see that for every z contained in a compact subset K of $\mathbf{C}$, and for n odd, the series with general term $w_k(n,z)$ is *normally convergent*. Indeed, as n tends to $+\infty$, $n \sin \dfrac{z}{n}$ tends uniformly to z on K, so there exists a number $\mathrm{M} > 0$ such that $\left|n \sin \dfrac{z}{n}\right| \leqslant \mathrm{M}$ for every integer m and every $z \in \mathrm{K}$. We saw, moreover, in the proof of th. 1 of VI, p. 286, that for

$1 \leqslant k \leqslant m$ one has $n \sin \dfrac{k\pi}{n} \geqslant \dfrac{k\pi}{2}$; thus for every integer k such that $k\pi/2 \geqslant \mathrm{M}$ one has $|w_k(n, z)| \leqslant 4\mathrm{M}^2/k^2\pi^2$, which proves our assertion. Since, for every fixed k, $w_k(n, z)$ tends (uniformly on K) to $z^2/k^2\pi^2$ as n tends to $+\infty$, one sees that:

THEOREM 2. *For every complex number z one has*

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) \tag{10}$$

the infinite product on the right being absolutely and uniformly convergent on every compact subset of $\mathbf{C}$ (Eulerian expansion of $\sin z$).

3. APPLICATION TO THE BERNOULLI NUMBERS

Theorem 1 of VI, p. 286, shows that, for $0 \leqslant x < \pi$, the series with general term $\dfrac{2x}{n^2\pi^2 - x^2} \geqslant 0$ converges. On the other hand, one can write, for every complex number z such that $|z| < \pi$,

$$\frac{2z}{n^2\pi^2 - z^2} = \frac{2z}{n^2\pi^2} \sum_{k=0}^{\infty} \frac{z^{2k}}{n^{2k}\pi^{2k}}$$

the series on the right being absolutely convergent. We shall deduce from this that the *"double" series*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{-2z^{2k-1}}{n^{2k}\pi^{2k}} \tag{11}$$

is absolutely convergent in the open disc $|z| < \pi$, normally convergent on every compact set contained in this disc, and has sum $\cot z - \dfrac{1}{z}$. Indeed, for $|z| \leqslant a < \pi$ the absolute value of the general term of (11) is at most equal to $2a^{2k-1}/n^{2k}\pi^{2k}$, and the sum of any finite number of the terms $2a^{2k-1}/n^{2k}\pi^{2k}$ is less than the finite number $\displaystyle\sum_{n=1}^{\infty} \frac{2a}{n^2\pi^2 - a^2}$; on summing first over k, then over n, one sees that the sum of the series (11) is equal to $\displaystyle\sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$, which proves our statement.

If now one sums the series (11) first over n and then over k one obtains the identity (for $|z| < \pi$)

$$\cot z - \frac{1}{z} = -2 \sum_{k=1}^{\infty} \frac{\mathrm{S}_{2k}}{\pi^{2k}} z^{2k-1} \tag{12}$$

where we have put $\mathrm{S}_k = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^k}$. By (1) (VI, p. 283) one thus has, for $|z| < 2\pi$

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \mathrm{S}_{2n}}{2^{2n-1}\pi^{2n}} z^{2n} \tag{13}$$

whence we obtain the formula

$$b_{2n} = (-1)^{n-1} (2n)! \frac{2\mathrm{S}_{2n}}{(2\pi)^{2n}} \qquad \text{for } n \geqslant 1, \tag{14}$$

a formula which shows in particular that the numbers S_{2n}/π^{2n} are *rational*. Clearly $\mathrm{S}_{k+1} \leqslant \mathrm{S}_k$ so, for every integer $k \geqslant 2$, we have $\mathrm{S}_k \leqslant \mathrm{S}_2 = \pi^2/6 \leqslant 2$; from (14) one deduces the following inequalities for the Bernoulli numbers

$$\frac{2\,(2n)!}{(2\pi)^{2n}} \leqslant |b_{2n}| \leqslant 4\frac{(2n)!}{(2\pi)^{2n}} \qquad \text{for } n \geqslant 1. \tag{15}$$

From these inequalities one can deduce an estimate for the Bernoulli polynomial $\mathrm{B}_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_k\, x^{n-k}$; in particular, for $0 \leqslant x \leqslant 1$ one has

$$|\mathrm{B}_n(x)| \leqslant 4 \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(2\pi)^k} = 4\frac{n!}{(2\pi)^n} \sum_{k=0}^{n} \frac{(2\pi)^k}{k!} \leqslant 4e^{2\pi} \frac{n!}{(2\pi)^n}. \tag{16}$$

§ 3. BOUNDS FOR THE REMAINDER IN THE EULER-MACLAURIN SUMMATION FORMULA

1. BOUNDS FOR THE REMAINDER IN THE EULER-MACLAURIN SUMMATION FORMULA

The estimate obtained in (16) for the Bernoulli polynomials on the interval $[0, 1]$ allows one to estimate the remainder $\mathrm{T}_p(x, n)$ in the Euler-Maclaurin summation formula (VI, p. 282, formula (39)) easily:

$$\left\{\begin{aligned} & f(x) + f(x+1) + \cdots + f(x+n) \\ & \qquad = \int_x^{x+n+1} f(t)\,dt - \frac{1}{2}\big(f(x+n+1) - f(x)\big) \\ & \qquad\qquad + \sum_{k=1}^{p} \frac{b_{2k}}{(2k)!}\big(f^{(2k-1)}(x+n+1) - f^{(2k-1)}(x)\big) + \mathrm{T}_p(x, n). \end{aligned}\right. \tag{1}$$

Indeed, one has (VI, p. 283, formula (41))

$$\mathrm{T}_p(x, n) = -\frac{1}{(2p+1)!} \int_0^{n+1} \overline{\mathrm{B}}_{2p+1}(1-s) f^{(2p+1)}(x+s)\,ds \tag{2}$$

where $\overline{\mathrm{B}}_{2p+1}(t)$ is the periodic function of period 1 equal to $\mathrm{B}_{2p+1}(t)$ on the interval $[0, 1[$. The formula (16) of VI, p. 288, shows that

$$\left|\overline{\mathrm{B}}_{2p+1}(t)\right| \leqslant 4e^{2\pi}\,\frac{(2p+1)!}{(2\pi)^{2p+1}} \tag{3}$$

for all $t \in \mathbf{R}$, and applying the mean value formula gives the estimate

$$\left|\mathrm{T}_p(x,n)\right| \leqslant \frac{4e^{2\pi}}{(2\pi)^{2p+1}} \int_x^{x+n+1} \left|f^{(2p+1)}(t)\right|\,dt \tag{4}$$

for $\mathrm{T}_p(x,n)$.

2. APPLICATION TO ASYMPTOTIC EXPANSIONS

The Euler-Maclaurin formula allows one to give a more complete solution (in the most important cases) to the problem treated in V, p. 238 to 242, that of obtaining an asymptotic expansion of the partial sum $s_n = \sum_{m=0}^{n} g(m)$ (resp. of the remainder $r_n = \sum_{m=n+1}^{\infty} g(m)$), where g is a scalar function, > 0, monotone, defined on $[0, +\infty[$. We shall restrict ourselves to the case where g is an (H) *function* (V, p. 252), *of order* 0 relative to e^x; in other words, one has the relation $g' \prec\!\!\prec g$; from this relation one deduces $g^{(k+1)} \prec\!\!\prec g^{(k)}$ for every integer $k > 0$ such that none of the derivatives $g^{(h)}$ of order $h \leqslant k$ is equivalent to a constant (V, p. 232, prop. 7). Let p be an integer such that none of the derivatives $g^{(h)}$ of order $h \leqslant 2p$ is equivalent to a constant. First we assume that the series with general term $g(n)$ has infinite sum, and distinguish several cases:

1° $\left|g^{(2p-1)}(n)\right|$ tends to $+\infty$ with n; we have the same, by the hypothesis, for $\left|g^{(2k-1)}(n)\right|$ for $1 \leqslant k \leqslant p$; further, since $g^{(2p+1)}$ is monotone on a neighbourhood of $+\infty$, the formula (4) of VI, p. 289, gives $\mathrm{T}_p(0,n) = O\big(g^{(2p)}(n+1)\big) = o\big(g^{(2p-1)}(n+1)\big)$; the Euler-Maclaurin formula, applied for $x = 0$, shows that

$$\begin{aligned} s_n = \sum_{m=0}^{n} g(m) = \int_0^{n+1} g(t)\,dt - \frac{1}{2}g(n+1) \\ + \sum_{k=1}^{p} \frac{b_{2k}}{(2k)!}\, g^{(2k-1)}(n+1) + o\big(g^{(2p-1)}(n+1)\big) \end{aligned}$$

each of the terms of this sum being negligible relative to the preceding one; on expanding each of them relative to a comparison scale $\mathcal{E}$ one will then have an asymptotic expansion for s_n.

2° Now suppose that for an index q such that $1 \leqslant q \leqslant p$ we have $\left|g^{(2q-1)}(n)\right|$ tending to $+\infty$ with n, but that $g^{(2k-1)}(n)$ tends to 0 for $k > q$. Since $g^{(2p+1)}$ is monotone on a neighbourhood of $+\infty$ the integral $\int_0^{\infty} \left|g^{(2p+1)}(u)\right|\,du$ converges, and one can then write

$$s_n = \sum_{m=0}^{n} g(m) = \int_0^{n+1} g(t)\,dt - \frac{1}{2} g(n+1) + \sum_{k=1}^{q} \frac{b_{2k}}{(2k)!}\, g^{(2k-1)}(n+1) + \mathrm{C}$$
$$+ \sum_{k=q+1}^{p} \frac{b_{2k}}{(2k)!}\, g^{(2k-1)}(n+1) + o\big(g^{(2p-1)}(n+1)\big)$$

where C is a constant: indeed

$$\int_{n+1}^{\infty} \left|g^{(2p+1)}(u)\right| \, du = O\big(g^{(2p)}(n+1)\big) = o\big(g^{(2p-1)}(n+1)\big).$$

The same formula is valid when $g(n)$ itself tends to 0. Finally, when the series with general term $g(n)$ converges one has, for the remainder $r_n = \sum\limits_{m=n+1}^{\infty} g(m)$, the expansion

$$r_n = \sum_{m=n+1}^{\infty} g(m) = \int_{n+1}^{\infty} g(t)\,dt + \frac{1}{2}\, g(n+1)$$
$$- \sum_{k=1}^{p} \frac{b_{2k}}{(2k)!}\, g^{(2k-1)}(n+1) + o\big(g^{(2p-1)}(n+1)\big).$$

EXERCISES

§ 1.

1) Let K be a field of characteristic 0, let U and V two composition operators in K[X], and $W = VU = UV$; let (u_n), (v_n) (w_n) be the sequences of Appell polynomials corresponding to U, V, W respectively. Show that if p is the order of U then

$$w_n^{(p)}(\mathrm{X}) = \sum_{k=0}^{n} \binom{n}{k} v_{n-k}(\mathrm{X}) V_0(u_k)$$

$$w_n(\mathrm{X}+\mathrm{Y}) = \sum_{k=0}^{n} \binom{n}{k} u_k(\mathrm{X}) v_{n-k}(\mathrm{Y}).$$

2) Let E be the vector space over $\mathbf{C}$ generated by the functions x^n $(n \in \mathbf{N})$, $e^{\lambda x}$ $(\lambda \in \mathbf{C},\ \lambda \neq 0)$ and $|x+\mu|$ $(\mu \in \mathbf{R})$.

a) Show that the preceding functions form a basis for E.

b) Let U be the composition operator defined on E by the conditions: $U_0^{\xi}(\xi^n) = (n!)^2$, $U(e^{\lambda x}) = e^{\lambda x}$ for $\lambda \neq 0$, $U(|x+\mu|) = |x+\mu|$ for $\mu \in \mathbf{R}$. The indicatrix $u(\lambda)$ is the constant 1 but the series with general term $n!\,\lambda^n$ does not converge for any value $\lambda \neq 0$.

c) Let V be the composition operator defined on E by the conditions $V(x^n) = x^n$, $V(|x+\mu|) = |x+\mu|$, $V(e^{\lambda x}) = 0$ for $\lambda \neq 0$; the indicatrix $v(\lambda)$ is equal to 1 for $\lambda = 0$, to 0 for $\lambda \neq 0$, so is different from the sum of the series $\sum_{n=0}^{\infty} \alpha_n \lambda^n$ where $\alpha_n = V_0^{\xi}(\xi^n)/n!$.

d) Let W be the composition operator defined on E by

$$W(x^n) = x^n, \qquad W(e^{\lambda x}) = e^{\lambda x}, \qquad W(|x+\mu|) = e^{x+\mu};$$

show that $VW \neq WV$.

3) Let K be a field of characteristic 0. One says that an endomorphism U of the algebra K[X,Y] of polynomials in two indeterminates X, Y over K is a composition operator if, for every polynomial $f \in \mathrm{K}[\mathrm{X},\mathrm{Y}]$, one has, putting $g = U(f)$, that $g(\mathrm{X}+\mathrm{S}, \mathrm{Y}+\mathrm{T}) = U_{\mathrm{X},\mathrm{Y}}(f(\mathrm{X}+\mathrm{S}, \mathrm{Y}+\mathrm{T}))$, where S and T are two other indeterminates.

a) Generalize prop. 1 and th. 1 to these operators. Derive a new proof of the formula $e^{\mathrm{S}+\mathrm{T}} = e^{\mathrm{S}} e^{\mathrm{T}}$ from this.

b) For composition operators of the form $\mathrm{D}_{\mathrm{X}}^{p}\,\mathrm{D}_{\mathrm{Y}}^{q}\,u(\mathrm{D}_{\mathrm{X}},\mathrm{D}_{\mathrm{Y}})$ where the constant term of the formal series u is not zero, define the Appell polynomials u_{mn} and generalize props. 4, 5 and 6 of VI, p. 273 and 274.

c) Consider in particular the composition operator U defined by $U(f(\mathrm{X},\mathrm{Y})) = f(\mathrm{X}+1,\mathrm{Y}+1) - f(\mathrm{X},\mathrm{Y}+1) - f(\mathrm{X}+1,\mathrm{Y}) + f(\mathrm{X},\mathrm{Y})$; the Appell polynomials corresponding to this operator are called Bernoulli polynomials and are denoted by $\mathrm{B}_{m,n}$. Show that $\mathrm{B}_{m,n}(\mathrm{X},\mathrm{Y}) = \mathrm{B}_m(\mathrm{X})\mathrm{B}_n(\mathrm{Y})$.

§ 2.

1) Establish the formulae

$$\tan z = \sum_{n=1}^{\infty} (-1)^{n-1}\, 2^{2n}\, (2^{2n}-1)\, b_{2n}\, \frac{z^{2n-1}}{(2n)!}$$

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^{n-1}\, 2\, (2^{2n-1}-1)\, b_{2n}\, \frac{z^{2n-1}}{(2n)!}$$

where the series on the right-hand sides converge absolutely, the first for $|z| < \dfrac{\pi}{2}$ and the second for $|z| < \pi$ (express $\tan z$ and $1/\sin 2z$ as linear combinations of $\cot z$ and $\cot 2z$). Deduce that the numbers $\dfrac{2^{2n}(2^{2n}-1)}{2n}\, b_{2n}$ are *integers*. (Use the following lemma: if, in two absolutely convergent series $\sum_{n=0}^{\infty} \alpha_n \dfrac{z^n}{n!}$, $\sum_{n=0}^{\infty} \beta_n \dfrac{z^n}{n!}$ the coefficients α_n and β_n are integers, then, in their product written in the form $\sum_{n=0}^{\infty} \gamma_n \dfrac{z^n}{n!}$, the γ_n are integers.)

2) Establish the formula

$$(n-1)\mathrm{B}_n(\mathrm{X}) = n(\mathrm{X}-1)\mathrm{B}_{n-1}(\mathrm{X}) - \sum_{k=0}^{n} \binom{n}{k} b_k\, \mathrm{B}_{n-k}(\mathrm{X})$$

(differentiate the series $\mathrm{S}e^{\mathrm{SX}}/(e^{\mathrm{S}}-1)$ with respect to S). Deduce the formula

$$(2n+1)b_{2n} = -\sum_{k=1}^{n-1} \binom{2n}{2k} b_{2k}\, b_{2n-2k}$$

for the Bernoulli numbers.

3) Show that, for every integer $p > 1$,

$$\mathrm{B}_n\left(\frac{x}{p}\right) + \mathrm{B}_n\left(\frac{x+1}{p}\right) + \cdots + \mathrm{B}_n\left(\frac{x+p-1}{p}\right) = \frac{1}{p^{n-1}}\, \mathrm{B}_n(\mathrm{X}).$$

4) *a*) Prove the relation

$$\mathrm{B}_n(1-\mathrm{X}) = (-1)^n \mathrm{B}_n(\mathrm{X})$$

(use the fact that $b_{2n-1} = 0$ for $n > 1$, and the relation

$$\mathrm{B}_n(1-\mathrm{X}) - \mathrm{B}_n(-\mathrm{X}) = (-1)^n\, n\, \mathrm{X}^{n-1}.)$$

b) Show that

$$\mathrm{B}_n\left(\tfrac{1}{2}\right) = b_n\left(\frac{1}{2^{n-1}} - 1\right)$$

(use exerc. 3).

c) Show that, for n even, $\mathrm{B}_n(\mathrm{X})$ has two roots in the interval $[0, 1]$ of $\mathbf{R}$, and that for n odd > 1, $\mathrm{B}_n(\mathrm{X})$ has a simple root at the points 0, $\frac{1}{2}$ and 1 and does not vanish at any other point of $[0, 1]$ (use *b*) and the relation $\mathrm{B}'_n = n\mathrm{B}_{n-1}$).

d) Deduce from *c*) that, for n even, the maximum of $|\mathrm{B}_n(x)|$ on the interval $[0, 1]$ is $|b_n|$, and that for n odd, if a_n is the maximum of $|\mathrm{B}_n(x)|$ on $[0, 1]$, then

$$\frac{4}{n+1}\,|b_{n+1}|\left(1 - \frac{1}{2^n}\right) \leqslant a_n \leqslant \frac{1}{2} n\,|b_{n-1}|$$

(use the mean value theorem).

5) If one puts $\mathrm{S}_n(x) = \dfrac{1}{n+1}(\mathrm{B}_{n+1}(x) - \mathrm{B}_{n+1}(0))$ then, for every integer $a > 0$ one has

$$\mathrm{S}_n(a) = 1^n + 2^n + \cdots + (a-1)^n.$$

a) Show that for every integer $n \geqslant 0$ and every integer $a > 0$ one has $2\,\mathrm{S}_{2n+1}(a) \equiv 0 \pmod{a}$ (consider the sum $k^{2n+1} + (a-k)^{2n+1}$).

b) If r and s are any two integers show that

$$\mathrm{S}_n(rs) \equiv s\mathrm{S}_n(r) + nr\mathrm{S}_{n-1}(r)\mathrm{S}_1(s) \qquad (\text{mod. } r^2).$$

c) Let p be a prime number. Show that if n is divisible by $p-1$ one has $\mathrm{S}_n(p) \equiv -1$ (mod. p), and if n is not divisible by $p-1$ then $\mathrm{S}_n(p) \equiv 0$ (mod. p) (if p does not divide the integer g remark that $\mathrm{S}_n(p) \equiv g^n\mathrm{S}_n(p)$ (mod. p)).

6) *a*) The rational numbers b_n having been defined by the formula (20) of VI, p. 275, one denotes by d_n the denominator > 0 of b_n written as an irreducible fraction. Show that no prime factor of d_n can be $> n+1$ (use the induction formula (23) of VI, p. 276).

b) Show that for every integer $p > 0$ and every integer $n > 0$

$$\mathrm{S}_n(p) = b_n p + \binom{n}{1}\frac{p}{2}b_{n-1}p + \cdots + \binom{n}{r}\frac{p^r}{r+1}b_{n-r}p + \cdots + \frac{p^{n+1}}{n+1}.$$

c) Deduce from *b*) by recursion on n that, for every prime number p the denominator of $\mathrm{S}_n(p) - b_n p$ written as an irreducible fraction, is not divisible by p (observe that p^{r+1} cannot divide $r+1$).

d) Deduce from *c*) that the number

$$b_n - \sum_p \frac{\mathrm{S}_n(p)}{p}$$

where p runs through the set of prime numbers $p \leqslant n+1$ and n is even, is an integer. Conclude that

$$b_{2n} + \sum_p \frac{1}{p}$$

where p runs through the set of prime numbers such that $p-1$ divides $2n$, is an integer (*Clausen-von Staudt theorem*; use exerc. 5 c).

* 7) We accept that for every integer $a > 0$ there are infinitely many prime numbers in the set of integers $1 + ma$ (m running through the set of integers $\geqslant 1$; this is a particular case of Dirichlet's theorem on arithmetic progressions).

a) Let n be an integer $\geqslant 1$, and let $s \geqslant 1$ be an integer such that $q = 1 + (2n+1)!\,s$ is prime; show that if p is a prime number such that $p-1$ divides $2nq$ then $p-1$ must divide $2n$ (in the opposite case one would have $p - 1 = qd$ with d an integer, and p would be divisible by $d+1$).

d) Deduce from *a*) that for every integer $n > 0$ there exist infinitely many integers $m > n$ such that $b_{2m} - b_{2n}$ is an *integer*.$_*$

8) Show that, for every prime number $p > 3$, $\mathrm{S}_{2n}(p^k) - p^k b_{2n}$, written as an irreducible fraction, has a numerator divisible by p^{2k} (argue as in exerc. 6).

9) To say that a rational number r is a *p-adic integer* (*Gen. Top.*, III, p. 322, exerc. 23) for p a prime number, means that when r is expressed as an irreducible fraction its denominator is not divisible by p; one writes $r \equiv 0 \pmod{p}$ to express the fact that r/p is a p-adic integer, and $r \equiv r' \pmod{p}$ is equivalent by definition to $r - r' \equiv 0 \pmod{p}$.

a) Let m be a rational integer; the function $\mathrm{F}_m(z) = \dfrac{m}{e^{mz}-1} - \dfrac{1}{e^z - 1}$ is analytic for $|z|$ sufficiently small, so can be written as an entire series convergent on a neighbourhood of 0

$$\mathrm{F}_m(z) = \sum_{n=1}^{\infty} (m^n - 1)\,\frac{b_n}{n}\,\frac{z^{n-1}}{(n-1)!}.$$

Show that also on a neighbourhood of 0

$$\mathrm{F}_m(z) = \sum_{n=0}^{\infty} c_n\,(e^z - 1)^n$$

where the c_n are p-adic integers; deduce that the numbers $a_n = (m^n - 1)\,\dfrac{b_n}{n}$ are p-adic integers; further, one has $a_{n+p-1} \equiv a_n \pmod{p}$.

b) Deduce from *a*) that if $p - 1$ does not divide n, then b_n/n is a p-adic integer and that

$$\frac{b_{n+p-1}}{n+p-1} \equiv \frac{b_n}{n} \qquad \pmod{p}$$

(*Kummer's congruences*). (Take for m an integer whose class mod. p is a generator of the multiplicative group of the invertible elements of $\mathbf{Z}/p\mathbf{Z}$ (*Alg.*, VII. 12).)

10) With the notation of exerc. 9 show that the numbers $m^n a_n = m^n(m^n - 1)b_n/n$ are *integers* (write $\mathrm{F}_m(z)$ as the quotient of two polynomials in e^z). Deduce that (for n even $\geqslant 2$) the denominator δ_n of b_n/n in irreducible form is such that, for *every* integer m prime to n one has $m^n \equiv 1$ (mod. δ_n). Further, if an integer d is such that $m^n \equiv 1$ (mod. d) for every integer m prime to n, then d divides δ_n (use the Clausen-von Staudt theorem and the structure of the multiplicative group of $\mathbf{Z}/d\mathbf{Z}$ (*Alg.*, VII. 12)).

11) *a*) Let p be a prime number $\neq 2$. Then the following properties are equivalent:

α) $b_n/n \not\equiv 0$ (mod. p) for every even n such that $p-1$ does not divide n.

β) $b_n \not\equiv 0$ (mod. p) for $n = 2, 4, 6, \ldots, p-3$.
(Use exerc. 9*b*).)

b) One says that p is a *regular* prime number if it is equal to 2 or satisfies the equivalent conditions of *a*); otherwise p is called *irregular*. The prime numbers $2, 3, 5, 7, 11$ are regular; 691 is irregular.
Let I be a finite set of irregular primes. Let n be an integer $\geqslant 2$ multiple of $\prod_{q\in \mathrm{I}} (q-1)$ and such that $|b_n/n| > 1$, (*cf.* VI, p. 288, formula (15)). Let p be a prime factor of the numerator of $|b_n/n|$ in irreducible form. Show that $p-1$ does not divide n and that consequently p is irregular. Deduce that the set of irregular prime numbers is *infinite*.

12) For every polynomial Q with real coefficients such that $\mathrm{Q}(0) = \mathrm{Q}(1)$ one writes $\tilde{\mathrm{Q}}$ for the continuous function on $\mathbf{R}$ such that $\tilde{\mathrm{Q}}(x) = \mathrm{Q}(x)$ for $0 \leqslant x \leqslant 1$ and $\tilde{\mathrm{Q}}(x+1) = \tilde{\mathrm{Q}}(x)$ for every $x \in \mathbf{R}$.

a) For every integer $m \geqslant 2$ the Bernoulli polynomial B_m is the unique polynomial Q with real coefficients, of degree m, monic, and such that $\int_0^1 \mathrm{Q}(x)\,dx = 0$, $\mathrm{Q}(1) = \mathrm{Q}(0)$ and such that the function $\tilde{\mathrm{Q}}$ is $m-2$ times differentiable on $\mathbf{R}$ and has a continuous $(m-2)^{nd}$ derivative. (Argue by recursion on m.)

b) Show that for $m \geqslant 1$

$$\int_0^1 \mathrm{B}_m(x)\, e^{-2\pi i n x}\, dx = \begin{cases} 0 & \text{if} \quad n = 0 \\ -\dfrac{m!}{(2\pi i n)^m} & \text{if} \quad n \neq 0 \quad \text{on } \mathbf{Z}. \end{cases}$$

* *c*) Deduce from *b*) that, for $0 \leqslant x \leqslant 1$ and $m \geqslant 2$ one has

$$\mathrm{B}_m(x) = -\frac{m!}{(2\pi i)^m} \sum_{n\in\mathbf{Z}-\{0\}} \frac{e^{2\pi i n x}}{n^m}$$

the series being normally convergent.*

13) Let f be an integer $\geqslant 1$, and χ a function defined on $\mathbf{Z}$, with complex values, such that $\chi(x+f) = \chi(x)$ for every $x \in \mathbf{Z}$. If one puts

$$\hat{\chi}(y) = \frac{1}{f} \sum_{x=0}^{f-1} \chi(x)\, e^{-2\pi i x y/f} \qquad \text{for } y \in \mathbf{Z},$$

one has

$$\chi(x) = \sum_{y=0}^{f-1} \hat{\chi}(y)\, e^{-2\pi i x y/f} \qquad \text{for } x \in \mathbf{Z}.$$

Deduce from exerc. 12 that, for every integer $m \geqslant 2$,

$$\sum_{n \in \mathbf{Z}\setminus\{0\}} \frac{\chi(n)}{n^m} = -\frac{(2\pi i)^m}{m!} \sum_{y=0}^{f-1} \hat{\chi}(y)\, \mathrm{B}_m\left(\frac{y}{f}\right).$$

For example, one has

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots + \frac{(-1)^n}{(2n+1)^3} + \cdots = \frac{\pi^3}{32}.$$

§ 3.

1) Show that if $f^{(2p+1)}(t)$ is monotone on the interval $[x, x+n+1]$ then the remainder $\mathrm{T}_p(x,n)$ in the Euler-Maclaurin formula VI, p. 288 (formula (2)) has the same sign as the term

$$\frac{1}{(2p+2)!}\, b_{2p+2}\, (f^{(2p+1)}(x+n+1) - f^{(2p+1)}(x))$$

and has absolute value at most equal to that of this term.

If one assumes that $f^{(2p+2)}$ is continuous (but of any sign), show that one has

$$\left|\mathrm{T}_p(x,n)\right| \leqslant \frac{1}{(2p+2)!} \left|b_{2p+2}\right| \left(\left|f^{(2p+1)}(x+n+1) - f^{(2p+1)}(x)\right| + \int_x^{x+n+1} \left|f^{(2p+2)}(t)\right| dt \right).$$

2) Establish the formula

$$\frac{1}{2} f(x) - f(x+1) + f(x+2) - \cdots - f(x+2n-1) + \frac{1}{2} f(x+2n)$$
$$= \sum_{k=1}^{p} \frac{b_{2k}}{(2k)!} (2^{2k}-1)(f^{(2k-1)}(x+2n) - f^{(2k-1)}(x)) + \mathrm{V}_p(x,n)$$

with

$$\left|\mathrm{V}_p(x,n)\right| \leqslant \frac{4e^{2\pi}(2^{2p+1}+1)}{(2\pi)^{2p+1}} \int_x^{x+2n} \left|f^{(2p+1)}(t)\right| dt$$

(apply the Euler-Maclaurin formula to $g(x) = f(2x)$).

3) Let f be a continuous function with $2p+1$ continuous derivatives on the interval $[0, 1]$. Establish the formula

$$\int_0^1 f(t)\,dt = \frac{1}{n}\left(\frac{1}{2}f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) + \frac{1}{2}f(1)\right)$$
$$- \sum_{k=1}^{2p} \frac{b_{2k}}{(2k)!}\,\frac{1}{n^{2k}}(f^{(2k-1)}(1) - f^{(2k-1)}(0)) + \mathrm{R}_p(n)$$

with

$$\left|\mathrm{R}_p(n\right| \leqslant \frac{4e^{2\pi}}{(2\pi)^{2p+1}}\,\frac{1}{n^{2p+1}}\int_0^1 \left|f^{(2p+1)}(t)\right|\,dt.$$

HISTORICAL NOTE
(*Chapters V and VI*)

(N.B. Roman numerals refer to the bibliography to be found at the end of this note.)

The distinction between the "infinitely small" (or "infinitely large") of different orders appeared implicitly in the first writings on the Differential Calculus, and for example in those of Fermat; it became fully articulated with Newton and Leibniz, with the theory of "differences of higher order"; and no time was lost before observing that in the most simple cases, the limit (or "true value") of an expression of the form $f(x)/g(x)$ at a point where f and g both tend to 0 is given by the Taylor expansion of these functions on a neighbourhood of the point considered ("l'Hôpital's rule", probably due to Johann Bernoulli).

Apart from in the elementary case, the principal problem of "asymptotic evaluation" which presented itself to the mathematicians from the end of the XVIIth century was the calculation, exact or approximate, of sums of the form $\sum_{k=1}^{n} f(k)$ when n is very large; such a calculation was truly needed as much for interpolation and for the numerical evaluation of the sum of a series, as in the Calculus of Probabilities, where the "functions of large numbers" such as $n!$ or $\binom{a}{n}$ play a preponderant rôle. Already Newton, to obtain approximate values for $\sum_{k=1}^{n} \frac{1}{a+k}$ when n is large, indicated a method which reduces (in this particular case) to the calculation of the first terms in the Euler-Maclaurin formula (I). Towards the end of the century, Jakob Bernoulli, in the course of his research in the Calculus of Probabilities, set himself to determine the sums $\mathrm{S}_k(n) = \sum_{p=1}^{n-1} p^k$, polynomials in n of which he discovered the general law of formation (without giving a proof) [4], so introducing for the first time, in the expression for the coefficients of these polynomials, the numbers which bear his name, and the induction relation which allows one to calculate them ((II), p. 97). In 1730, Stirling obtained an asymptotic expansion for $\sum_{k=1}^{n} \log(x+ka)$, as n increases indefinitely, by a procedure for calculating the coefficients recursively.

[4] They are the primitives of the "Bernoulli polynomials" $\mathrm{B}_k(x)$.

The decisive works of Euler on series and on related questions date from 1730 to 1745. Putting $S(n) = \sum_{k=1}^{n} f(k)$, he applied the Taylor formula to the function $S(n)$, which gave him

$$f(n) = S(n) - S(n-1) = \frac{dS}{dn} - \frac{1}{2!}\frac{d^2S}{dn^2} + \frac{1}{3!}\frac{d^3S}{dn^3} - \cdots,$$

an equation which he "inverted" by the method of undetermined coefficients, looking for a solution of the form

$$S(n) = \alpha \int f(n)\,dn + \beta\, f(n) + \gamma\, \frac{df}{dn} + \delta \frac{d^2 f}{dn^2} + \cdots;$$

he thus obtained, term-by-term,

$$S(n) = \int f(n)\,dn + \frac{f(n)}{2} + \frac{1}{12}\frac{df}{dn} - \frac{1}{720}\frac{d^3 f}{dn^3} + \frac{1}{30\,240}\frac{d^5 f}{dn^5} - \cdots$$

without in the first place being able to determine the law of formation of the coefficients (III *a* and *d*). But about 1735, in analogy with the decomposition of a polynomial into linear factors, he did not hesitate to write the formula

$$\begin{aligned} 1 - \frac{\sin s}{\sin \alpha} = \left(1 - \frac{s}{\alpha}\right)&\left(1 - \frac{s}{\pi - \alpha}\right) \\ &\left(1 - \frac{s}{-\pi - \alpha}\right)\left(1 - \frac{s}{2\pi - \alpha}\right)\left(1 - \frac{s}{-2\pi - \alpha}\right)\cdots \end{aligned}$$

and on equating the coefficients of the expansions of both sides as entire series he obtained in particular (for $\alpha = \pi/2$) the famous expressions for the series $\sum_{k=1}^{\infty} \frac{1}{n^{2k}}$ in terms of powers of π (III *b*) [5]. Several years later he perceived at last that the coefficients of these powers of π are given by the same equations as those of his summation formula, and recognised their connection with the numbers introduced by Bernoulli, and with the coefficients of the series expansion of $z/(e^z - 1)$ (III *g*).

Independently of Euler, Maclaurin had arrived about the same time at the same summation formula, by a slightly less hazardous way, close to that which we have followed in the text; he effectively iterated the "Taylor" formula which expresses $f(x)$ in terms of the differences $f^{(2k+1)}(x+1) - f^{(2k+1)}(x)$, a formula which he obtained by "inverting" the Taylor expansions of these differences by the method of

[5] In 1743 Euler, in response to various contemporary critics, gave a somewhat more plausible derivation of the "Eulerian expansions" of the trigonometric functions; for example, the expansion of $\sin x$ as an infinite product is derived from the expression $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ and the fact that e^{ix} is the limit of the polynomial $\left(1 + \frac{ix}{n}\right)^n$ (III *e*).

undetermined coefficients (IV); but he did not perceive the law of formation of the coefficients, discovered by Euler.

But Maclaurin, like Euler and all the mathematicians of his time, presented all these formulae as *series* expansions, whose convergence was not even studied. Not that the notion of a convergent series was totally neglected at this period; one had known since Jakob Bernoulli that the harmonic series is divergent, and Euler had even made this result clear by evaluating the sum of the first n terms of this series with the help of his summation formula (III c and d); it was also Euler who remarked that the ratio of two consecutive Bernoulli numbers increases indefinitely, and consequently that an entire series having these numbers as coefficients cannot converge ((III f), p. 357) [6]. But the tendency towards formal calculus was the stronger, and Euler's extraordinary intuition even so did not prevent him from descending into the absurd, as when, for example, he wrote $0 = \sum_{n=-\infty}^{+\infty} x^n$ ((III f), p. 362) [7].

We have already seen (Historical Note to chap. IV) how the mathematicians at the beginning of the XIXth century, weary of this unbridled and unfounded formalism, brought Analysis back to the ways of rigour. Once the concept of a convergent series was made precise, the need appeared for simple criteria for showing the convergence of integrals and series by comparison with known integrals or series; Cauchy gave a number of such criteria in his *Analyse algébrique* (Va), while Abel, in a posthumous memoir (VI), obtained the logarithmic convergence criteria. Cauchy, on the other hand (V b), elucidated the paradox of such series as that of Stirling, obtained by applying the Euler-Maclaurin formula (and often called "semiconvergent series"); he showed that if (in view of Euler's remark on the Bernoulli numbers) the general term $u_k(n)$ of such a series, for a *fixed* value of n, increases indefinitely with k, nonetheless for a *fixed* value of k the partial sum $s_k(n) = \sum_{h=1}^{k} u_h(n)$ gives an asymptotic expansion (as n tends to $+\infty$) of the function "represented" by the series, correspondingly more precise as k is larger.

In the majority of the calculations of Classical Analysis it is possible to obtain a general law of formation for the asymptotic expansions of a function having a number of *arbitrarily large* terms; this fact has contributed to creating a lasting confusion (at least in the language) between series and asymptotic expansions; so much so that H. Poincaré, when he took the trouble, in 1886 (VIII), to codify the elementary rules of asymptotic expansions (following the integer powers of $1/x$ on a neighbourhood of $+\infty$), still employed the vocabulary of the theory of series. It was only with the appearance of asymptotic expansions coming from analytic number theory that a

[6] Since the series which Euler considered here was introduced with a view to numerical calculation, he took only the sum of the terms which start decreasing and from the index where the terms begin to increase he replaced them by a remainder whose origin he did not indicate (the remainder in the Euler-Maclaurin formula in its general form did not appear until Cauchy).

[7] It is ironic that this formula follows one page after a passage where Euler warned against the unconsidered use of divergent series!

clear distinction between the concept of an asymptotic expansion and a series was finally established, by reason of the fact that, in the majority of the problems which this theory treats, one cannot obtain more than a very small number of terms explicitly (most often only one) of the expansion sought.

These problems have also familiarised mathematicians with the use of comparison scales other than those of the (real or integer) powers of the variable. This extension is due above all to the works of P. du Bois-Reymond (VII) who, first, treated systematically the comparison of functions in a neighbourhood of a point, and, in very original works, recognised the "nonarchimedean" character of comparison scales, at the same time as he studied in a general manner integration and differentiation of comparison relations, and deduced a host of interesting consequences (VII *b*). His proofs sometimes lack clarity and rigour, and it is to G.H. Hardy (IX) that a correct presentation of du Bois-Reymond's results is due: his principal contribution consisted in recognising and proving the existence of a set of "elementary functions", the (H) functions, on which the usual operations of Analysis (notably differentiation) are applicable to the comparison relations [8].

[8] It is not our remit to develop in these chapters the methods which allow one to obtain asymptotic expansions of functions belonging to certain particular categories, as for example certain types of integrals depending on a parameter, which appear quite frequently in Analysis; on this point (and in particular on the important methods of Laplace and Darboux) the reader may consult the book of Hardy (IX) already mentioned, which contains a very complete bibliography.

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CHAPTER VII
The Gamma function

§ 1. THE GAMMA FUNCTION IN THE REAL DOMAIN

1. DEFINITION OF THE GAMMA FUNCTION

We have defined (*Set Theory*, III, p. 179) the function $n!$ for every integer $n \geqslant 0$, as equal to the product $\prod\limits_{0 \leqslant k < n} (n-k)$; so $0! = 1$ and $(n+1)! = (n+1)\,n!$ for $n \geqslant 0$. We set $\Gamma(n) = (n-1)!$ for each integer $n \geqslant 1$; we propose to define, on the set of real numbers $x > 0$, a *continuous function* $\Gamma(x)$ *extending* the function Γ defined on the set of integers $\geqslant 1$.

It is clear that there are infinitely many such functions; since $\Gamma(n+1) = n\Gamma(n)$ for every integer $n \geqslant 1$ we shall restrict ourselves to considering, among the continuous functions that extend Γ, those which satisfy the equation

$$f(x+1) = xf(x) \tag{1}$$

for every $x > 0$.

For a solution of this equation to be an extension of $\Gamma(n)$ it is necessary and sufficient that it also satisfies $f(1) = 1$.

If f satisfies (1) then, by recursion on n,

$$f(x+n) = x(x+1)(x+2)\dots(x+n-1)\,f(x) \tag{2}$$

for every integer $n > 1$ and for all $x > 0$. This relation shows, in particular, that the values of f on an interval $]n, n+1]$ (n an integer $\geqslant 1$) are determined by its values on the interval $]0, 1]$. Conversely, let φ be a continuous function on $]0, 1]$ satisfying only the conditions $\varphi(1) = 1$, $\lim\limits_{x \to 0} x\varphi(x) = 1$; for every integer $n \geqslant 1$ let us define f on the interval $]n, n+1]$ by the relation

$$f(x) = (x-1)(x-2)\dots(x-n)\,\varphi(x-n);$$

it is clear that f is continuous on $]0, +\infty[$, satisfies the equation (1), and extends $\Gamma(n)$.

If f is a continuous solution of (1) and takes values > 0 on $]0, 1]$ it takes values > 0 on $]0, +\infty[$, by (2); the function $g(x) = \log f(x)$ is then defined and continuous on $]0, +\infty[$ and satisfies the equation

$$g(x+1) - g(x) = \log x \tag{3}$$

on this interval.

If g_1 is a second continuous solution of (3) on $]0, +\infty[$, and if $h = g_1 - g$, then one has $h(x+1) - h(x) = 0$ for every $x > 0$; in other words, h is a continuous *periodic* function of period 1, defined on $]0, +\infty[$; conversely, for every h of this nature, $g + h$ is a continuous solution of (3).

PROPOSITION 1. *There exists one and only one convex function* g *defined on* $]0, +\infty[$ *that satisfies the equation* (3) *and takes the value* 0 *for* $x = 1$.

First we show that if there is a function g satisfying the conditions stated then it is *well-determined* on the interval $]0, 1]$, and consequently on the interval $]0, +\infty[$. Indeed, for every integer $n > 1$ the gradient of the line joining the point $(n, g(n))$ to the point $(x, g(x))$ is an increasing function of x, since g is convex (I, p. 27, prop. 5); one thus must have, for $0 < x \leqslant 1$,

$$\frac{g(n-1) - g(n)}{(n-1) - n} \leqslant \frac{g(n+x) - g(n)}{(n+x) - n} \leqslant \frac{g(n+1) - g(n)}{(n+1) - n}$$

that is, by (3),

$$x \log(n-1) \leqslant g(x+n) - g(n) \leqslant x \log n. \tag{4}$$

Now, by (3),

$$g(x+n) - g(n) = g(x) + \log x + \sum_{k=1}^{n-1} (\log(x+k) - \log k).$$

Moreover, one can write $\log n = \sum_{k=2}^{n} \log \frac{k}{k-1}$ so the inequality (4) can be written

$$x \sum_{k=2}^{n-1} \log \frac{k}{k-1} \leqslant g(x) + \log x + \sum_{k=2}^{n} (\log(x+k-1) - \log(k-1))$$
$$\leqslant x \sum_{k=2}^{n} \log \frac{k}{k-1}.$$

Let us put, for every $n \geqslant 2$,

$$u_n(x) = x \log \frac{n}{n-1} - \log(x+n-1) + \log(n-1) \tag{5}$$

and

$$g_n(x) = -\log x + \sum_{k=2}^{n} u_k(x).$$

For $0 < x \leqslant 1$ one then has

$$g_n(x) - x \log \frac{n}{n-1} \leqslant g(x) \leqslant g_n(x). \tag{6}$$

Since $\log \dfrac{n}{n-1}$ tends to 0 as n tends to $+\infty$ one deduces from (6) that if a solution g exists then it is necessarily equal, on $]0, 1]$, to the *limit* of the $g_n(x)$.

Now one deduces immediately from (5) that for x fixed and > 0 one has

$$u_n(x) = -x \log \left(1 - \frac{1}{n}\right) - \log \left(1 + \frac{x-1}{n}\right) + \log \left(1 - \frac{1}{n}\right) \sim \frac{x(x-1)}{2n^2}$$

as n tends to $+\infty$, which proves that the series with general term $u_n(x)$ converges for every $x > 0$. Each of the functions $u_n(x)$ being convex on $]0, +\infty[$, as is $-\log x$, the function $g(x) = -\log x + \sum_{n=2}^{\infty} u_n(x)$ is convex on this interval (I, p. 27, prop. 2 and prop. 4); finally, one has $u_n(1) = 0$, whence $g(1) = 0$, and

$$u_n(x+1) = u_{n+1}(x) + x \left(\log \frac{n}{n-1} - \log \frac{n+1}{n}\right)$$

whence

$$g(x+1) = -\log(x+1) + x \log 2 + \sum_{n=3}^{\infty} u_n(x) = \log x + g(x);$$

in other words, g satisfies equation (3) of VII, p. 306.

DEFINITION 1. *We denote by $\Gamma(x)$ the function > 0 which is defined on the interval $]0, +\infty[$, that satisfies the equation*

$$\Gamma(x+1) = x\,\Gamma(x), \tag{7}$$

and is such that $\Gamma(1) = 1$ and that $\log \Gamma(x)$ is convex on $]0, +\infty[$.

2. PROPERTIES OF THE GAMMA FUNCTION

PROPOSITION 2. *For every $x > 0$ one has*

$$\Gamma(x) = \lim_{n \to \infty} \frac{n^x\, n!}{x(x+1)\ldots(x+n)} \tag{8}$$

(*Gauss' formula*), *and*

$$\Gamma(x) = e^{-\gamma x} \frac{1}{x} \prod_{n=1}^{\infty} \frac{e^{x/n}}{1 + \frac{x}{n}} \tag{9}$$

where γ *denotes Euler's constant, and the infinite product on the right hand side of* (9) *is absolutely and uniformly convergent on every compact interval of* $\mathbf{R}$ *not containing any integer* < 0 (*Weierstrass' formula*).

The function $\Gamma(x)$ *is indefinitely differentiable on* $]0, +\infty[$ *and one has*

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{x+n}\right) \tag{10}$$

and

$$\mathrm{D}^k(\log \Gamma(x)) = \sum_{n=0}^{\infty} \frac{(-1)^k (k-1)!}{(x+n)^k} \qquad \text{for} \quad k \geqslant 2, \tag{11}$$

the series on the right-hand sides of (10) *and* (11) *being absolutely and uniformly convergent on every compact interval not containing any integer* $\leqslant 0$.

Indeed, the proof of prop. 1 of VII, p. 306, shows that

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x (n-1)!}{x(x+1)\dots(x+n-1)}$$

whence Gauss' formula, since $\dfrac{n}{x+n}$ tends to 1 as n tends to $+\infty$. One can also write

$$\log \frac{n}{n-1} = \frac{1}{n-1} + \left(\log \frac{n}{n-1} - \frac{1}{n-1}\right),$$

so (in the notation of prop. 1)

$$\exp(u_n(x)) = e^{x\left(\log \frac{n}{n-1} - \frac{1}{n-1}\right)} \frac{e^{x/n-1}}{1 + \dfrac{x}{n-1}}$$

and the series with general term $\log \dfrac{n}{n-1} - \dfrac{1}{n-1}$ is absolutely convergent and has sum $-\gamma$, where γ denotes Euler's constant (V, p. 242), whence we obtain Weierstrass' formula.

For $|x| \leqslant a$ one has $\left|1/(x+n)^k\right| \leqslant 1/(n-a)^k$ once $n > a$, so the series on the right-hand side in formula (11) is absolutely and uniformly convergent on every compact interval of $\mathbf{R}$ not containing any integer $\leqslant 0$, for any integer $k \geqslant 2$; the same argument applies to the right-hand side of (10), since $\left|\dfrac{1}{n} - \dfrac{1}{x+n}\right| \leqslant \dfrac{a}{n(n-a)}$ for $|x| \leqslant a$ and $n > a$. Since these series are obtained by differentiating the series

$$\log \Gamma(x) = -\gamma x - \log x + \sum_{n=1}^{\infty} \left(\frac{x}{n} - \log\left(1 + \frac{x}{n}\right)\right)$$

term-by-term, and this converges for every $x > 0$, the series with general term $\dfrac{x}{n} - \log\left(1 + \dfrac{x}{n}\right)$ is absolutely and uniformly convergent on every compact interval contained in $[0, +\infty[$, and one has the relations (10) and (11) of VII, p. 308, for every $x > 0$ (II, p. 52, th. 1). Moreover, for every $x \in \mathbf{R}$, the expression $\frac{x}{n} - \log\left(1 + \frac{x}{n}\right)$ is defined once n is large enough, so th. 1 of II, p. 52, again shows that the infinite product in the right-hand side of (9) (VII, p. 307) is absolutely and uniformly convergent on every compact interval containing no integer $\leqslant 0$.

The function $\Gamma(x)$, defined for $x > 0$, can be extended to the whole set of points x different from the integers $\leqslant 0$ so as to satisfy equation (7) of VII, p. 307, on this set: it suffices, for $-(n+1) < x < -n$, to put

$$\Gamma(x) = \frac{1}{x(x+1)\dots(x+n)}\,\Gamma(x+n+1).$$

By prop. 2 of VII, p. 307, the formulae (8), (9), (10) and (11) of VII, p. 307 and 308, with $\mathrm{D}^k(\log|\Gamma(x)|)$ replacing $\mathrm{D}^k(\log\Gamma(x))$ in (11), remain valid on this set. Formula (9) (VII, p. 307) shows that $\Gamma(x) \sim 1/x$ as x tends to 0, whence, by (7) of VII, p. 307,

$$\Gamma(x) \sim \frac{(-1)^n}{n!\,(x+n)}$$

as x tends to $-n$ (n an integer $\geqslant 0$). The function $1/\Gamma(x)$ can then be extended by continuity to all of $\mathbf{R}$, assigning it the value 0 at integers $\leqslant 0$; then, for *all* $x \in \mathbf{R}$

$$\frac{1}{\Gamma(x)} = \lim_{n\to\infty} \frac{x(x+1)\dots(x+n)}{n^x\,n!} \tag{12}$$

and

$$\frac{1}{\Gamma(x)} = e^{\gamma x}\,x \prod_{n=1}^{\infty}\left(1 + \frac{x}{n}\right)e^{-x/n} \tag{13}$$

and one shows as in prop. 2 of VII, p. 307, that the infinite product on the right of (13) is absolutely and uniformly convergent on every compact interval of $\mathbf{R}$.

Since $\Gamma(x) > 0$ for $x > 0$, equation (7) of VII, p. 307, shows that $\Gamma(x) < 0$ for $-(2n-1) < x < -(2n-2)$ and $\Gamma(x) > 0$ for

$$-2n < x < -(2n-1)$$

(n an integer $\geqslant 1$); also $\Gamma(x)$ has right limit $+\infty$ at the points $-2n$ and $-\infty$ at the points $-(2n+1)$, and has left limit $-\infty$ at the points $-2n$ and $+\infty$ at the points $-(2n+1)$ (for all $n \in \mathbf{N}$). Formula (11) of VII, p. 308, shows that, for $k = 2$, the right-hand side is always $\geqslant 0$ when it is defined, so

$$\Gamma''(x)\,\Gamma(x) - \left(\Gamma'(x)\right)^2 \geqslant 0,$$

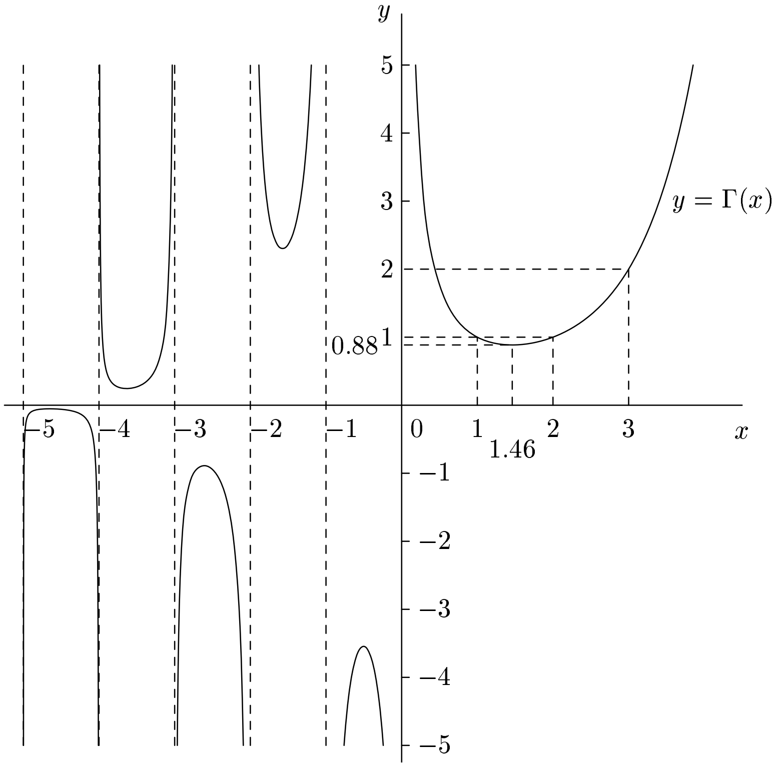

Fig. 1

and consequently $\Gamma''(x)$ has the same sign as $\Gamma(x)$; thus Γ is *convex* for $x > 0$ and for $-(2n+2) < x < -(2n+1)$, and *concave* for $-(2n+1) < x - 2n$ $(n \in \mathbf{N})$; one deduces from this that, on the intervals where Γ is convex, $\Gamma'(x)$ increases from $-\infty$ to $+\infty$, and on the intervals where Γ is concave, $\Gamma'(x)$ decreases from $+\infty$ to $-\infty$. Whence the graph of Γ (fig. 1).

3. THE EULER INTEGRALS

For brevity we shall say that a function f defined on an interval $\mathrm{I} \subset \mathbf{R}$, and > 0 on this interval, is *logarithmically convex* on I if $\log f$ is convex on I. The definition of $\Gamma(x)$ shows that this function is logarithmically convex on $]0, +\infty[$.

It is clear that the *product* of two logarithmically convex functions on I is also logarithmically convex on I. Further:

Lemma 1. Let f and g be two functions > 0 and twice differentiable on an open interval I. *If f and g are logarithmically convex functions on* I, *then $f + g$ is logarithmically convex on* I.

The relation $\mathrm{D}^2\big(\log f(x)\big) \geqslant 0$ can be written $f(x)f''(x) - \big(f'(x)\big)^2 \geqslant 0$. We are reduced to showing that the relations $a > 0, a' > 0, ac - b^2 \geqslant 0, a'c' - b'^2 \geqslant 0$ imply $(a + a')(c + c') - (b + b')^2 \geqslant 0$; now, when $a > 0$, the relation $ac - b^2 \geqslant 0$ is equivalent to the fact that the quadratic form $ax^2 + 2bxy + cy^2$ is $\geqslant 0$ on $\mathbf{R}^2$, and it is clear that if

$$ax^2 + 2bxy + cy^2 \geqslant 0 \qquad \text{and} \qquad a'x^2 + 2b'xy + c'y^2 \geqslant 0$$

on $\mathbf{R}^2$ then also $(a + a')x^2 + 2(b + b')xy + (c + c')y^2 \geqslant 0$ on $\mathbf{R}^2$.

Lemma 2. Let f be a finite real function, > 0, defined and continuous on the product $\mathrm{I} \times \mathrm{J}$ of two open intervals of $\mathbf{R}$, and such that, for every $t \in \mathrm{J}$ the function $x \mapsto f(x, t)$ is logarithmically convex and twice differentiable on I. Under these hypotheses, if the integral $g(x) = \int_{\mathrm{J}} f(x, t)\, dt$ converges for every $x \in \mathrm{I}$, then g is logarithmically convex on I.

First we show that for every compact interval $\mathrm{K} \subset \mathrm{J}$ the function $g_{\mathrm{K}}(x) = \int_{\mathrm{K}} f(x, t)\, dt$ is logarithmically convex. Indeed, if $\mathrm{K} = [a, b]$, the sequence of functions

$$g_n(x) = \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(x, a + k\frac{b-a}{n}\right)$$

converges simply to $g_{\mathrm{K}}(x)$ on I (II, p. 57, prop. 5) hence $\log g_n$ converges simply to $\log g_{\mathrm{K}}$; by lemma 1 of VII, p. 310, $\log g_n$ is convex on I, so (I, p. 27, prop. 4) it is the same for $\log g_{\mathrm{K}}$.

On the other hand, g is the pointwise limit of the g_{K} along the directed set of compact subintervals of I (II, p. 64), so $\log g$ is the pointwise limit of the $\log g_{\mathrm{K}}$; these last functions being convex on I, so is $\log g$ (I, p. 27, prop. 4).

One can show easily that lemmas 1 and 2 remain valid even when one does not assume that the functions are twice differentiable (VII, p. 327, exerc. 5).

Lemma 3. Let φ be a continuous function and > 0 on an open interval J contained in $[0, +\infty[$. If I is an open interval such that the integral $g(x) = \int_{\mathrm{J}} t^{x-1} \varphi(t)\, dt$ converges for all $x \in \mathrm{I}$, then g is logarithmically convex on I.

Indeed, $\log t^{x-1} = (x - 1) \log t$ is a function of x which is convex and twice differentiable for all $t > 0$, so lemma 2 applies.

PROPOSITION 3. *For all $x > 0$*

$$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}\, dt \tag{14}$$

(*second Euler integral*).

Now the function $g(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$ is defined for all $x > 0$ (V, p. 229); lemma 3 of VII, p. 311, then shows that it is *logarithmically convex* on $]0, +\infty[$. Moreover, on integrating by parts, one has

$$g(x+1) = \int_0^\infty e^{-t}\, t^x\, dt = -e^{-t}\, t^x \Big|_0^\infty + x \int_0^\infty e^{-t}\, t^{x-1}\, dt = x\, g(x).$$

In other words, g is a solution of equation (1) of VII, p. 305; finally,

$$g(1) = \int_0^\infty e^{-t}\, dt = 1;$$

the proposition therefore follows from prop. 1 of VII, p. 306.

By the change of variable $e^{-t} = u$ one deduces from (14) (VII, p. 311) the formula

$$\Gamma(x) = \int_0^1 \left(\log \frac{1}{t}\right)^{x-1} dt. \tag{15}$$

Similarly, from the change of variable $u = t^x$ we obtain

$$x\,\Gamma(x) = \int_0^\infty e^{-t^{1/x}}\, dt$$

or again, taking account of (7) (VII, p. 3),

$$\Gamma\left(1 + \frac{1}{x}\right) = \int_0^\infty e^{-t^x}\, dt \tag{16}$$

and in particular, for $x = 2$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t^2}\, dt. \tag{17}$$

PROPOSITION 4. *For $x > 0$ and $y > 0$ the integral*

$$\mathbf{B}(x, y) = \int_0^1 t^{x-1}(1-y)^{y-1}\, dt$$

(*first Euler integral*) *has the value*

$$\mathbf{B}(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{18}$$

Indeed this integral converges for $x > 0$ and $y > 0$ (V, p. 229). By lemma 3 of VII, p. 311, the function $x \mapsto \mathbf{B}(x, y)$ is *logarithmically convex* for $x > 0$. Moreover,

$$\mathbf{B}(x+1, y) = \int_0^1 (1-t)^{x+y-1} \left(\frac{t}{1-t}\right)^x dt$$

whence, on integrating by parts,

$$
\begin{aligned}
\mathbf{B}(x+1, y) = &- \frac{(1-t)^{x+y}}{x+y} \left(\frac{t}{1-t} \right)^x \Bigg|_0^1 \\
&+ \frac{x}{x+y} \int_0^1 (1-t)^{x+y} \left(\frac{t}{1-t} \right)^{x-1} \frac{dt}{(1-t)^2} = \frac{x}{x+y} \mathbf{B}(x, y).
\end{aligned}
$$

It follows that $f(x) = \mathbf{B}(x, y)\Gamma(x+y)$ satisfies the identity (1) of VII, p. 305. Moreover, this function is logarithmically convex, being the product of two logarithmically convex functions. Finally, one has $f(1) = \mathbf{B}(1, y)\Gamma(y+1)$, and $\mathbf{B}(1, y) = \int_0^1 (1-t)^{y-1}\, dt = 1/y$, whence $f(1) = \dfrac{1}{y}\Gamma(y+1) = \Gamma(y)$. The function $f(x)/\Gamma(y)$ is thus equal to $\Gamma(x)$ by prop. 1 of VII, p. 306, which proves (18).

By the change of variable $t = \dfrac{u}{u+1}$ the formula (18) becomes

$$
\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\, dt = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \tag{19}
$$

and by the change of variable $t = \sin^2 \varphi$

$$
\int_0^{\pi/2} \sin^{2x-1} \varphi \, \cos^{2y-1} \varphi \, d\varphi = \frac{1}{2} \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{20}
$$

If one puts $x = y = \frac{1}{2}$ in this last formula it follows that

$$
\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{21}
$$

whence, by (17),

$$
\int_0^\infty e^{-t^2}\, dt = \frac{1}{2}\sqrt{\pi}. \tag{22}
$$

From the relation (7) of VII, p. 307, one has the asymptotic expansion

$$
\begin{aligned}
\Gamma(x) &= \frac{1}{x}\, \Gamma(x+1) \\
&= \frac{1}{x} + \Gamma'(1) + \frac{1}{2!}\Gamma''(1)\, x + \cdots + \frac{1}{n!}\Gamma^{(n)}(1)\, x^{n-1} + O(x^n)
\end{aligned} \tag{23}
$$

for $\Gamma(x)$, on a neighbourhood of 0.

Similarly, for all y fixed and > 0 one can write

$$
\begin{aligned}
\frac{1}{\Gamma(x+y)} = &\frac{1}{\Gamma(y)} + \mathrm{D}\left(\frac{1}{\Gamma(y)} \right) x \\
&+ \frac{1}{2!}\, \mathrm{D}^2 \left(\frac{1}{\Gamma(y)} \right) x^2 + \cdots + \frac{1}{n!}\, \mathrm{D}^n \left(\frac{1}{\Gamma(y)} \right) x^n + O_1(x^{n+1})
\end{aligned}
$$

and the formula (18) then gives, for y fixed, the asymptotic expansion

$$\begin{aligned}\mathbf{B}(x, y) = &\frac{1}{x} + \left(\Gamma'(1) - \frac{\Gamma'(y)}{\Gamma(y)}\right) \\ &+ \left(\frac{\Gamma''(1)}{2} - \Gamma'(1)\frac{\Gamma'(y)}{\Gamma(y)} + \frac{2\Gamma'^2(y) - \Gamma(y)\Gamma''(y)}{2\Gamma^2(y)}\right) x + O(x^2)\end{aligned} \tag{24}$$

on a neighbourhood of $x = 0$.

Moreover, for $x > 0$ and $y > 0$ one has

$$\begin{aligned}\mathbf{B}(x, y) &= \int_0^1 \left(t^{x-1} + t^x \frac{(1-t)^{y-1} - 1}{t}\right) dt \\ &= \frac{1}{x} + \int_0^1 t^x \frac{(1-t)^{y-1} - 1}{t}\, dt.\end{aligned} \tag{25}$$

The function $\varphi(t) = \dfrac{(1-t)^{y-1} - 1}{t}$ is continuous on the compact interval $[0,\ 1]$; since

$$t^x = e^{x \log t} = 1 + x \log t + \frac{x^2}{2!}(\log t)^2 + \cdots + \frac{x^n}{n!}(\log t)^n + r_n(x, t)$$

with $|r_n(x, t)| \leqslant \dfrac{x^{n+1}}{(n+1)!}\,|\log t|^{n+1}$ (since $\log t \leqslant 0$ and $x > 0$), formula (25) gives the asymptotic expansion

$$\begin{aligned}\mathbf{B}(x, y) = \frac{1}{x} + \int_0^1 \varphi(t)\, dt + x \int_0^1 \varphi(t)\, \log t\, dt + \cdots \\ + \frac{x^n}{n!} \int_0^1 \varphi(t)\,(\log t)^n\, dt + O_2(x^{n+1})\end{aligned}$$

for $\mathbf{B}(x, y)$ on a neighbourhood of 0.

For $n = 1$ the identification of this expansion with (24) gives in particular

$$\Gamma'(1) - \frac{\Gamma'(y)}{\Gamma(y)} = \int_0^1 \frac{(1-t)^{y-1} - 1}{t}\, dt.$$

Furthermore, the formula (10) gives $\Gamma'(1) = \Gamma'(1)/\Gamma(1) = -\gamma$, so (*Gauss' integral*)

$$\frac{\Gamma'(x)}{\Gamma(x)} + \gamma = \int_0^1 \frac{1 - (1-t)^{x-1}}{t}\, dt. \tag{26}$$

§ 2. THE GAMMA FUNCTION IN THE COMPLEX DOMAIN

1. EXTENDING THE GAMMA FUNCTION TO C

Let us return to Weierstrass' formula which gives the expression

$$\frac{1}{\Gamma(x)} = x\, e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n} \tag{1}$$

for $1/\Gamma(x)$ for all real x, and consider the infinite product with general term $\left(1 + \frac{z}{n}\right) e^{-z/n}$ for arbitrary complex z. One can write $e^{-z/n} = 1 - \frac{z}{n} + h(z)$, with $|h(z)| \leqslant \frac{|z|^2}{2n^2} e^{|z/n|}$ (III, p. 106, formula (8)), whence

$$\left(1 + \frac{z}{n}\right) e^{-z/n} = 1 + v_n(z)$$

with $|v_n(z)| \leqslant \frac{|z|^2}{n^2}\left(1 + \frac{e^{|z|}}{2}(1 + |z|)\right)$; thus the infinite product under consideration is *absolutely and uniformly convergent* on every compact subset of **C**; further, its value is zero only at the points $z = -n$ (*Gen. Top.*, IX, p. 214, corollary). In view of the formula (1) of VII, p. 315, for every complex z one puts

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}. \tag{2}$$

The function $\Gamma(z)$ is thus defined for every point $z \in \mathbf{C}$ apart from the points $-n$ ($n \in \mathbf{N}$); it is continuous on this set, and $(z + n)\Gamma(z) \sim \frac{(-1)^n}{n!}$ on a neighbourhood of $-n$. Formula (2) shows that one has $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ for every z different from a negative integer.

The argument that allows one to pass from Gauss' formula (VII, p. 307, formula (8)) to Weierstrass' formula, on reversing the steps, applies also to complex z and shows that, for $z \neq -n$ ($n \in \mathbf{N}$), one has

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z\, n!}{z(z+1)\dots(z+n)} \tag{3}$$

on agreeing to put $n^z = e^{z \log n}$. Since

$$\frac{n^{z+1}\, n!}{(z+1)(z+2)\dots(z+n+1)} = z\, \frac{n}{n+1+z}\, \frac{n^z\, n!}{z(z+1)\dots(z+n)}$$

one again has, on passing to the limit, the fundamental functional equation

$$\Gamma(z+1) = z\,\Gamma(z) \tag{4}$$

for every $z \neq -n$ ($n \in \mathbf{N}$).

Let p be an arbitrary integer > 0, and K_p the open disc $|z| < p$; for every $z \in \mathrm{K}_p$, and every integer $n > p$, $1 + \frac{z}{n}$ is not a negative real number, so $\log\left(1 + \frac{z}{n}\right)$ is defined, and it follows from the above that the series with general term $\log\left(1 + \frac{z}{n}\right) - \frac{z}{n}$ $(n > p)$ is *normally convergent* on K_p; the same holds for the series obtained by differentiating the general term a finite number of times, since one has

$$\left|\frac{1}{n} - \frac{1}{z+n}\right| \leqslant \frac{p}{n(n-p)} \qquad \text{and} \qquad \left|\frac{1}{(z+n)^k}\right| \leqslant \frac{1}{(n-p)^k} \qquad (k > 1)$$

for $z \in \mathrm{K}_p$ and $n > p$. One then sees (*cf.* II, p. 59, *Remark* 3) that $\Gamma(z)$ is *indefinitely differentiable* at all the points $z \in \mathbf{C}$ apart from the points $-n$, and at these points one has

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right) \tag{5}$$

$$\mathrm{D}^{k-1}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty} \frac{(-1)^k\,(k-1)!}{(z+n)^k} \qquad \text{for} \quad k \geqslant 2, \tag{6}$$

the right-hand sides of (5) and (6) being *normally convergent* on every compact subset of $\mathbf{C}$ that does not contain any integer $\leqslant 0$. Further, one can write

$$\log \Gamma(z) \equiv -\gamma z - \log z + \sum_{n=1}^{\infty}\left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) \qquad (\mathrm{mod.}\ 2\pi i) \tag{7}$$

agreeing that when a logarithm in this formula is that of a real negative number it has one or the other limit values (differing by $2\pi i$) of $\log z$ at this point; the series on the right-hand side of (7) is then normally convergent on every compact subset of $\mathbf{C}$ not containing any integer $\leqslant 0$.

2. THE COMPLEMENTS' RELATION AND THE LEGENDRE-GAUSS MULTIPLICATION FORMULA

One derives immediately from formula (2) of VII, p. 315, that, for every $z \in \mathbf{C}$,

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

Now the Euler expansion of $\sin z$ (VI, p. 287, th. 2) shows that

$$z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) = \frac{1}{\pi}\sin \pi z;$$

taking account of the functional equation (4) of VII, p. 315, one then sees that:

PROPOSITION 1. *For every complex z one has*

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{1}{\pi}\sin\pi z \tag{8}$$

(*the complements' relation*).

COROLLARY — *For every real t one has*

$$|\Gamma(it)| = \sqrt{\frac{\pi}{t\sinh\pi t}} \qquad (t \neq 0) \tag{9}$$

$$\left|\Gamma(\tfrac{1}{2}+it)\right| = \sqrt{\frac{\pi}{\cosh\pi t}}. \tag{10}$$

Indeed, one deduces from (8) that $\Gamma(it)\Gamma(-it) = \dfrac{i\pi}{t\sin\pi it} = \dfrac{\pi}{t\sinh\pi t}$, and one has $\Gamma(-it) = \overline{\Gamma(it)}$; similarly, (8) gives

$$\Gamma\left(\tfrac{1}{2}+it\right)\Gamma\left(\tfrac{1}{2}-it\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}+\pi it\right)} = \frac{\pi}{\cos\pi it} = \frac{\pi}{\cosh\pi t},$$

and one has

$$\Gamma\left(\tfrac{1}{2}-it\right) = \overline{\Gamma\left(\tfrac{1}{2}+it\right)}.$$

Now let p be any integer > 0 and consider the product

$$f(z) = \Gamma\left(\frac{z+1}{p}\right)\Gamma\left(\frac{z+2}{p}\right)\ldots\Gamma\left(\frac{z+p}{p}\right).$$

By (3) (VII, p. 315), for every $z \neq -n$ ($n \in \mathbf{N}$), $f(z)$ is the limit of the product

$$\frac{n^{(z+1)/p}\, n!}{\left(\frac{z+1}{p}\right)\left(\frac{z+1}{p}+1\right)\cdots\left(\frac{z+1}{p}+n\right)}\cdot\frac{n^{(z+2)/p}\, n!}{\left(\frac{z+2}{p}\right)\left(\frac{z+2}{p}+1\right)\cdots\left(\frac{z+2}{p}+n\right)}\cdots$$

$$\cdots\frac{n^{(z+p)/p}\, n!}{\left(\frac{z+p}{p}\right)\left(\frac{z+p}{p}+1\right)\cdots\left(\frac{z+p}{p}+n\right)}$$

$$= \frac{n^{z+(p+1)/2}\, p^{(n+1)p}\,(n!)^p}{(z+1)(z+2)\ldots(z+(n+1)p)}$$

and in particular $f(0)$ is the limit of the product

$$\frac{n^{(p+1)/2}\, p^{(n+1)p}\,(n!)^p}{((n+1)p)!}$$

from which it follows that $f(z)/f(0)$ is the limit of

$$\frac{n^z\,((n+1)p)!}{(z+1)(z+2)\dots(z+(n+1)p)} = z\,p^{-z}\left(\frac{n}{n+1}\right)^z \cdot \frac{((n+1)p)^z\,((n+1)p)!}{z(z+1)(z+2)\dots(z+(n+1)p)}$$

which, by (3) (VII, p. 315), gives

$$f(z) = f(0)\,z\,p^{-z}\,\Gamma(z). \tag{11}$$

But one can write

$$f(0) = \prod_{k=1}^{p-1}\Gamma\left(\frac{k}{p}\right) = \prod_{k=1}^{p-1}\Gamma\left(1-\frac{k}{p}\right) = \sqrt{\prod_{k=1}^{p-1}\Gamma\left(\frac{k}{p}\right)\Gamma\left(1-\frac{k}{p}\right)}$$

since $f(0) > 0$; the complements' relation then gives

$$f(0) = \sqrt{\pi^{p-1}\Big/\prod_{k=1}^{p-1}\sin\frac{k\pi}{p}}$$

and since the product on the right-hand side is equal to $p/2^{p-1}$ (VI, p. 284, cor. 1) one sees finally that:

PROPOSITION 2. *For every complex number z not an integer $\leqslant 0$ and for every integer $p > 0$ one has*

$$\Gamma\left(\frac{z}{p}\right)\Gamma\left(\frac{z+1}{p}\right)\dots\Gamma\left(\frac{z+p-1}{p}\right) = (2\pi)^{(p-1)/2}\,p^{\frac{1}{2}-z}\,\Gamma(z) \tag{12}$$

(*Legendre-Gauss multiplication formula*).

PROPOSITION 3. *For every real $x > 0$ one has*

$$\int_x^{x+1}\log\Gamma(t)\,dt = x(\log x - 1) + \tfrac{1}{2}\log 2\pi \tag{13}$$

(*Raabe's integral*).

First we establish formula (13) for $x = 0$. Since $\log\Gamma(x) \sim \log\dfrac{1}{x}$ as x tends to 0, the integral $\int_0^1 \log\Gamma(x)\,dx$ converges. Further, the function $\log\Gamma(x)$ decreases on $]0, 1]$ (VII, p. 310); for every $\alpha > 0$ one thus has

$$\frac{1}{n}\sum_{k=1}^{q}\log\Gamma\left(\frac{k}{n}\right) \leqslant \int_0^{\alpha}\log\Gamma(x)\,dx,$$

q being the largest integer such that $q/n \leqslant \alpha$. Since $\displaystyle\int_0^\alpha \log\Gamma(x)\,dx$ tends to 0 with α and also $\displaystyle\frac{1}{n}\sum_{k=q+1}^{n} \log\Gamma\left(\frac{k}{n}\right)$ tends to $\displaystyle\int_\alpha^1 \log\Gamma(x)\,dx$ as n tends to $+\infty$ (II, p. 57, prop. 5) one has

$$\int_0^1 \log\Gamma(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\Gamma\left(\frac{k}{n}\right).$$

But, by (12), the right-hand side of this formula is the limit of

$$\frac{n-1}{2n}\log 2\pi - \frac{1}{2}\frac{\log n}{n},$$

whence

$$\int_0^1 \log\Gamma(x)\,dx = \tfrac{1}{2}\log 2\pi. \tag{14}$$

Next we remark that from the identity

$$\log\Gamma(x+1) = \log\Gamma(x) + \log x$$

one deduces, on integrating, that for $x > 0$

$$\int_0^x \log\Gamma(t+1)\,dt = \int_0^x \log\Gamma(t)\,dt + \int_0^x \log t\,dt.$$

But the integral on the left-hand side is also equal to $\int_1^{x+1} \log\Gamma(t)\,dt$. Thus, by (14),

$$\int_x^{x+1} \log\Gamma(t)\,dt = \int_0^x \log t\,dt + \tfrac{1}{2}\log 2\pi = x\,(\log x - 1) + \tfrac{1}{2}\log 2\pi.$$

3. STIRLING'S EXPANSION

Let x and y be two complex numbers not lying on the real negative half-axis; by formula (3) of VII, p. 315, with the conventions of VII, p. 316, concerning the logarithms, $\log\Gamma(x) - \log\Gamma(y)$ is congruent modulo $2\pi i$ to the limit of the expression

$$(x-y)\log n + \sum_{k=0}^{n}\big(\log(y+k) - \log(x+k)\big). \tag{15}$$

Let us put $f(t) = \log(y+t) - \log(x+t)$; we apply the Euler-Maclaurin summation formula (VI, p. 288) to the function f:

$$\begin{aligned} f(0) + f(1) + \cdots + f(n) = {} & \int_0^{n+1} f(t)\,dt - \tfrac{1}{2}\big(f(n+1) - f(0)\big) \\ & + \sum_{k=1}^{p}\frac{b_{2k}}{(2k)!}\big(f^{(2k-1)}(n+1) - f^{(2k-1)}(0)\big) + \mathrm{T}_p(n) \end{aligned}$$

with

$$\left|\mathrm{T}_p(n)\right| \leqslant \frac{4\,e^{2\pi}}{(2\pi)^{2p+1}} \int_0^{n+1} \left|f^{(2p+1)}(u)\right|\,du. \tag{16}$$

Since

$$f^{(m)}(t) = (-1)^{m-1}\,(m-1)!\left(\frac{1}{(y+t)^m} - \frac{1}{(x+t)^m}\right),$$

$f^{(2k-1)}(n+1)$ tends to 0 as n tends to $+\infty$, for all $k \geqslant 1$; it is the same for

$$f(n+1) = \log\left(1+\frac{y}{n+1}\right) - \log\left(1+\frac{x}{n+1}\right).$$

Moreover, one has

$$\int_0^{n+1} \log(x+t)\,dt = (x+n+1)\big(\log(x+n+1)-1\big) - x(\log x - 1);$$

as n tends to $+\infty$, one has the asymptotic expansion

$$(x+n)\big(\log(x+n)-1\big) = n\log n - n + x\log n + O\left(\frac{1}{n}\right).$$

Substituting in the expression (15) one sees finally that, as n tends to $+\infty$, $\mathrm{T}_p(n)$ has a limit $\mathrm{R}_p(x, y)$ and that one can write

$$\log\Gamma(x) - g(x) \equiv \log\Gamma(y) - g(y) + \mathrm{R}_p(x, y) \qquad (\mathrm{mod.}\ 2\pi i)$$

on putting

$$g(x) = x\log x - x - \tfrac{1}{2}\log x + \sum_{k=1}^{p} \frac{b_{2k}}{2k(2k-1)}\,\frac{1}{x^{2k-1}}. \tag{17}$$

We shall now evaluate a bound for $\mathrm{R}_p(x, y)$ with the help of (16), assuming that x and y are both in the subset $\mathrm{H_A}$ of $\mathbf{C}$ defined by the relation "$\mathcal{R}(z) \geqslant \mathrm{A}$ or $|\mathcal{I}(z)| \geqslant \mathrm{A}$", where A is an arbitrary number > 0 (fig. 2). To this end we remark that if $x = s + it$ with $s > \mathrm{A}$ one has $|x+u| \geqslant \mathrm{A} + u$ for every $u > 0$, and consequently

$$\int_0^{n+1} \frac{du}{|x+u|^{2p+1}} \leqslant \int_0^{\infty} \frac{du}{(\mathrm{A}+u)^{2p+1}} = \frac{1}{2p\mathrm{A}^{2p}}.$$

Similarly, if $|t| \geqslant \mathrm{A}$ one has $|x+u| = |s+u+it| \geqslant \sqrt{\mathrm{A}^2+(s+u)^2}$ for all real u, whence

$$\int_0^{n+1} \frac{du}{|x+u|^{2p+1}} \leqslant \int_{-\infty}^{+\infty} \frac{du}{(\mathrm{A}^2+u^2)^{p+1/2}} = \frac{2}{\mathrm{A}^{2p}} \int_0^{\infty} \frac{dv}{(1+v^2)^{p+1/2}}.$$

Thus one sees that, when x and y are in $\mathrm{H_A}$, one has

$$\left|\mathrm{R}_p(x, y)\right| \leqslant \frac{\mathrm{C}_p}{\mathrm{A}^{2p}}$$

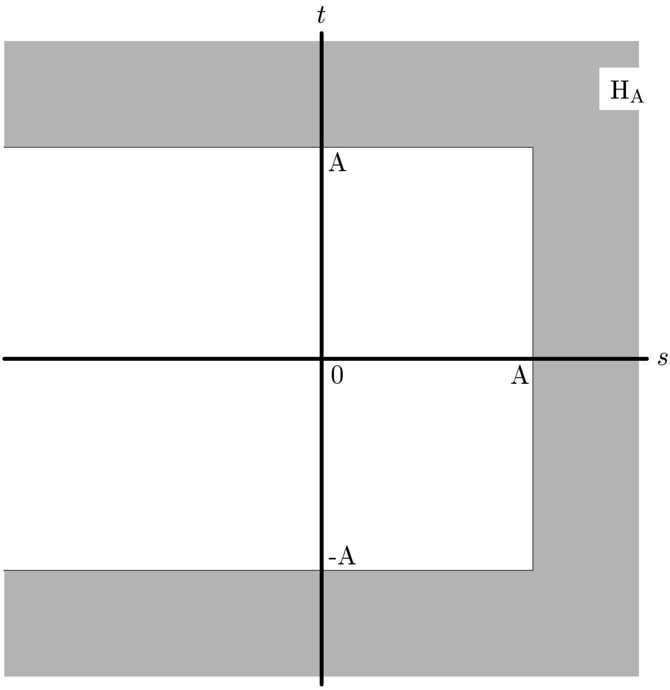

Fig. 2

where C_p depends only on p. Now let $\mathfrak{F}$ be the filter having the sets H_A as basis; the Cauchy criterion shows that, along the filter $\mathfrak{F}$, the function $\log \Gamma(z) - g(z)$ *has a finite limit* δ (modulo $2\pi i$) and that, if one puts $\omega(z) = \max(\mathcal{R}(z), |\mathcal{I}(z)|)$, one has

$$\log \Gamma(z) - g(z) - \delta \equiv O\left(\frac{1}{(\omega(z))^{2p}}\right) \qquad (\text{mod. } 2\pi i). \tag{18}$$

For x real and > 0 one has $\Gamma(x) > 0$, and $g(x)$ is real, so one can assume that δ is *real* and one has

$$\log \Gamma(x) = g(x) + \delta + O\left(\frac{1}{x^{2p}}\right).$$

Now we shall deduce the value of the constant δ; by prop. 2 of VII, p. 318, applied for $p = 2$, one has, for real x tending to $+\infty$

$$\begin{aligned}\frac{x-1}{2}\log\frac{x}{2} - \frac{x}{2} + \frac{x}{2}\log\frac{x+1}{2} - \frac{x+1}{2} + 2\delta \\ = x\log x - x - \tfrac{1}{2}\log x + (\tfrac{1}{2} - x)\log 2 + \tfrac{1}{2}\log 2\pi + \delta + o(1)\end{aligned}$$

from which one deduces easily that $\delta = \frac{1}{2}\log 2\pi$. Finally one has the following result:

PROPOSITION 4. *Along the filter* $\mathfrak{F}$ *one has* (*for every integer* $p \geqslant 1$) *the asymptotic expansion*

$$\log \Gamma(z) \equiv z \log z - z - \tfrac{1}{2} \log z + \tfrac{1}{2} \log 2\pi + \sum_{k=1}^{p} \frac{b_{2k}}{2k(2k-1)} \frac{1}{z^{2k-1}} + O\left(\frac{1}{(\omega(z))^{2p}}\right) \pmod{2\pi i} \tag{19}$$

(*Stirling's expansion*).

COROLLARY. *Along the filter* $\mathfrak{F}$ *one has*

$$\Gamma(z) \sim \sqrt{2\pi}\ \exp(z \log z - z - \tfrac{1}{2} \log z). \tag{20}$$

In particular, for x real tending to $+\infty$ the formula (20) can be written

$$\Gamma(x) \sim \sqrt{2\pi}\, x^{x-1/2} e^{-x}, \tag{21}$$

whence, as the integer n tends to $+\infty$,

$$n! \sim \sqrt{2\pi}\, n^{n+1/2}\, e^{-n}$$

(*cf.* V, p. 244).

One can deduce numerous formulae from this. For example, for every complex number α and every integer n one has, as n tends to $+\infty$,

$$\frac{\Gamma(n+\alpha)}{\Gamma(n)} \sim n^{\alpha} \quad (= e^{\alpha \log n}). \tag{22}$$

Similarly, for every complex number a not an integer $\leqslant 0$ one has

$$a(a+1)(a+2)\ldots(a+n) = \frac{\Gamma(n+a+1)}{\Gamma(a)} \sim \frac{\sqrt{2\pi}}{\Gamma(a)}\, n^{n+a+1/2}\, e^{-n} \tag{23}$$

and for every complex number a not an integer $\geqslant 0$

$$\binom{a}{n} = \frac{(-1)^n}{\Gamma(-a)} \frac{\Gamma(n-a)}{\Gamma(n+1)} \sim \frac{(-1)^n}{\Gamma(-a)}\, n^{-a-1}. \tag{24}$$

Finally, for every real constant $k > 1$ one has

$$\binom{kn}{n} = \frac{\Gamma(kn+1)}{\Gamma(n+1)\,\Gamma((k-1)n+1)} \sim \sqrt{\frac{k}{2\pi(k-1)n}} \left(\frac{k^k}{(k-1)^{k-1}}\right)^{n}. \tag{25}$$

The same argument leads to the following analogous proposition:

PROPOSITION 5. *Along the filter* $\mathfrak{F}$ *one has* (*for every integer* $p \geqslant 1$), *the asymptotic expansion*

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} - \sum_{k=1}^{p} \frac{b_{2k}}{2k} \frac{1}{z^{2k}} + O\left(\frac{1}{(\omega(z))^{2p+1}}\right). \tag{26}$$

Instead of prop. 2 of VII, p. 318, one uses the formula

$$\int_x^{x+1} \frac{\Gamma'(t)}{\Gamma(t)}\, dt = \log \Gamma(x+1) - \log \Gamma(x) = \log x$$

in order to determine the constant.

EXERCISES

§ 1.

¶ 1) Let g be a regulated function, and > 0 on $]0, +\infty[$.

a) Let u, v be two increasing functions on $]0, +\infty[$ such that $u(x+1) - u(x) = v(x+1) - v(x) = g(x)$ for all $x > 0$. Show that if $w = u - v$ one has

$$\sup_{0<x\leqslant y\leqslant 1} |w(y) - w(x)| \leqslant \inf_{x>0} g(x).$$

(Remark that, for $a \leqslant x \leqslant y \leqslant a+1$ one has $u(y) - u(x) \leqslant g(a)$.) In particular, if $\inf_{x>0} g(x) = 0$ there exists at most one increasing solution of the equation $u(x+1) - u(x) = g(x)$ taking a given value at a given point.

b) Assume that g is decreasing on $]0, +\infty[$. Show that the series

$$\varphi(x) = -g(x) + \sum_{n=1}^{\infty} (g(n) - g(x+n))$$

is absolutely and uniformly convergent on every compact interval contained in $]0, +\infty[$; if $\lambda = \lim_{x\to+\infty} g(x)$ then the function $u(x) = \varphi(x) + \lambda x$ is an increasing solution of the equation $u(x+1) - u(x) = g(x)$. Show that for every increasing solution v of this equation one has $v(y) - v(x) \geqslant \varphi(y) - \varphi(x)$ for $0 < x < y$. What is the upper (resp. lower) envelope of the set of increasing solutions of the equation $u(x+1) - u(x) = g(x)$ taking a given value at a given point? Show that this set reduces to a single element if and only if $\lambda = 0$.

c) Show that if $g(x)$ is increasing and > 0 on $[0, +\infty[$ there are infinitely many increasing solutions of the equation $u(x+1) - u(x) = g(x)$ which take a given value at a given point.

d) Let $\psi(x)$ be the function defined on $]0, +\infty[$ by the conditions: $\psi(x) = 0$ for $0 \leqslant x < 1$, $\psi(x) = 1$ for $1 \leqslant x < 2$, $\psi(x) = n$ for $n - 1 + \dfrac{1}{n-1} \leqslant x < n + \dfrac{1}{n}$ $(n \geqslant 2)$; let $g(x) = \psi(x+1) - \psi(x)$. Show that ψ is the unique increasing solution of the equation

$$u(x+1) - u(x) = g(x)$$

such that $u(1) = 1$.

¶2) Let g be a continuous increasing function on $]0, +\infty[$.

a) Show that if $\liminf_{x\to+\infty} g(x)/x = 0$ there exists at most one convex solution of the equation $u(x+1) - u(x) = g(x)$ taking a given value at a given point (remark that for all $h > 0$

the function $v(x) = u(x+h) - u(x)$ is an increasing function satisfying the equation $v(x+1) - v(x) = g(x+h) - g(x)$, and apply exerc. 1 *a*)).

b) Show that if g is concave on $]0, +\infty[$ there exists a convex solution of the equation $u(x+1) - u(x) = g(x)$; there exists a unique convex solution of this equation taking a given value at a given point if and only if $\lim_{x\to+\infty} g(x)/x = 0$ (*cf.* exerc. 1 *b*)).

c) Suppose from here on that g is increasing and concave on $]0, +\infty[$, and that $\lim_{x\to+\infty} g(x)/x = 0$. For every pair of numbers $h > 0$, $k > 0$ and every finite real function f defined on $]0, +\infty[$ one puts

$$\Delta\Big(f(x); h, k\Big) = \frac{f(x+h) - f(x)}{h} - \frac{f(x) - f(x-k)}{k}$$

for $x > k$. Show that if u is a convex solution of the equation $u(x+1) - u(x) = g(x)$, one has $\lim_{x\to+\infty} \Delta(u(x); h, k) = 0$ for all $h > 0$ and $k > 0$ (use the expression for u'_r from exerc. 1 *b*) of VII, p. 325).

d) With the notation of *c*) show that there exists a constant α such that one has $u(x) = v(x) + \alpha$, with

$$v(x) = \int_1^x g(t)\,dt - \tfrac{1}{2}g(x) + \mathrm{R}(x)$$

where we have put

$$\mathrm{R}(x) = \sum_{n=0}^{\infty} h(x+n), \qquad h(x) = \int_x^{x+1} g(t)\,dt - \tfrac{1}{2}(g(x+1) + g(x));$$

further, one has

$$0 \leqslant \mathrm{R}(x) \leqslant \tfrac{1}{2}(g(x+\tfrac{1}{2}) - g(x)).$$

(Remark that $v(x+1) - v(x) = g(x)$ and that $\lim_{x\to+\infty} \Delta(v(x); h, k) = 0$ irrespective of $h > 0$ and $k > 0$.)

3) *a*) Let g be a function defined and admitting a continuous k^{th} derivative on $]0, +\infty[$, such that $g^{(k)}$ is decreasing on this interval and that $\lim_{x\to+\infty} g^{(k)}(x) = 0$. Show that there exists one and only one solution u of the equation $u(x+1) - u(x) = g(x)$ which admits an increasing k^{th} derivative on $]0, +\infty[$ and takes a given value at a given point (use exerc. 1 *b*)).

b) Let α be an arbitrary real number. Let S_α be the function defined on $]0, +\infty[$ satisfying the relation

$$\mathrm{S}_\alpha(x+1) - \mathrm{S}_\alpha(x) = x^\alpha$$

such that $\mathrm{S}_\alpha(1) = 0$, and, further, that the derivative of S_α of order equal to the integer part of $(\alpha+1)^+$ is increasing (it is a unique function by *a*)). Show that

$$\mathrm{S}'_\alpha(x) - \mathrm{S}'_\alpha(1) = \alpha\,\mathrm{S}_{\alpha-1}(x),$$

and

$$\mathrm{S}_\alpha\left(\frac{x}{p}\right) + \mathrm{S}_\alpha\left(\frac{x+1}{p}\right) + \cdots + \mathrm{S}_\alpha\left(\frac{x+p-1}{p}\right) = \frac{1}{p^\alpha}\,\mathrm{S}_\alpha(x) + \mathrm{C}_\alpha$$

for every integer $p \geqslant 1$, where C_α is a constant, which, when $\alpha \neq -1$, is equal to

$$\left(\frac{1}{p^\alpha} - p\right)\frac{\mathrm{S}'_{\alpha+1}(1)}{\alpha+1}$$

(*cf.* VII, p.318, prop. 2 and VI, p. 291, exerc. 3).

4) Show that the function $f(x) = \dfrac{1}{\sqrt{2}} \dfrac{\Gamma(x/2)}{\Gamma\left(\dfrac{x+1}{2}\right)}$ is the unique convex solution of the equation $u(x+1) = 1/x\, u(x)$ (remark that this equation implies $u(x+2) = \dfrac{x}{x+1}\, u(x)$ and apply exerc. 1 *a*)).

5) Generalize lemmas 1 and 2 of VII, p. 310 and 311, to arbitrary logarithmically convex functions (*cf.* I, p. 45, exerc. 2). Show that every logarithmically convex function is convex.

6) Let $\psi(x) = \Gamma'(x)/\Gamma(x)$; for every integer $q > 1$ and every integer k such that $1 \leqslant k \leqslant q-1$ establish the formulae

$$\sum_{p=1}^{q} \psi\left(\frac{p}{q}\right) \exp\left(\frac{2pk\pi i}{q}\right) = -q \sum_{n=1}^{\infty} \frac{1}{n} \exp\left(\frac{2nk\pi i}{q}\right) = q \log\left(1 - \exp\frac{2k\pi i}{q}\right).$$

(Use formula (9) of VII, p. 307.)

§ 2.

¶ 1) *a*) Let g be a continuous real function for $x \geqslant 0$. Show that if g satisfies the two identities

$$\sum_{k=0}^{p-1} g\left(\frac{x+k}{p}\right) = g(x)$$

$$\sum_{h=0}^{q-1} g\left(\frac{x+h}{q}\right) = g(x)$$

it also satisfies the identity

$$\sum_{j=0}^{pq-1} g\left(\frac{x+j}{pq}\right) = g(x).$$

b) Deduce from this that if a function g has a continuous derivative for $x \geqslant 0$ and satisfies the identity

$$\sum_{k=0}^{p-1} g\left(\frac{x+k}{p}\right) = g(x)$$

then it is of the form $a(x - \frac{1}{2})$, where a is a constant (remark that g satisfies the analogous identity where p is replaced by p^n; let n tend to $+\infty$ and deduce that $g'(x) = \int_0^1 g'(t)\,dt$).

c) Conclude from *b*) that the Γ function is the only function having a continuous derivative for $x > 0$, which satisfies equation (1) of VII, p. 305, and the multiplication formula (12) of VII, p. 318, for a value of p.

2) For every integer $k > 1$ put $S_k = \sum_{n=1}^{\infty} n^{-k}$. Show that, for $-1 < x \leqslant 1$ one has

$$\log \Gamma(1+x) = -\gamma x + \sum_{k=2}^{\infty} (-1)^k \frac{S_k}{k} x^k$$

the series on the right being uniformly convergent on every compact subset of $]-1, 1]$, and absolutely convergent for $|x| < 1$.

3) Let s be a fixed real number; show that when t tends to $+\infty$ or to $-\infty$ one has

$$|\Gamma(s+it)| \sim \sqrt{2\pi}\ |t|^{s-1/2}\ e^{-(\pi/2)|t|}$$

and

$$\frac{\Gamma'(s+it)}{\Gamma(s+it)} \sim \log|t|\,.$$

4) Let t be a fixed real number $\neq 0$; show that as s tends to $+\infty$

$$|\Gamma(-s+it)| \sim \sqrt{\frac{\pi}{2}} \left(s^{s+1/2}\, e^{-s} \sqrt{\sinh^2 \pi t + \sin^2 \pi s} \right)^{-1}$$

(use the complements' relation).

5) Let x_n be the root of the equation $\Gamma'(x) = 0$ in the interval $]-n, -n+1[$. Show that

$$x_n = -n + \frac{1}{\log n} + O\left(\frac{1}{(\log n)^2}\right)$$

(use the complements' relation and formula (5) of VII, p. 316). Deduce that

$$\Gamma(x_n) \sim \frac{(-1)^n}{\sqrt{2\pi}}\, n^{-n-1/2}\, e^{n+1}\, \log n.$$

6) Let V_n be the Vandermonde determinant $V(1, 2, \ldots, n)$ (*Alg.*, III, p. 532). Show that

$$\log V_n = \frac{n^2}{2}\log n - \frac{3n^2}{4} + \left(\frac{1}{2}\log 2\pi - \frac{1}{4}\right) n - \frac{1}{12}\log n + k + O\left(\frac{1}{n}\right)$$

where k is a constant.

HISTORICAL NOTE

(N.B. Roman numerals refer to the bibliography to be found at the end of this note.)

The idea of "interpolating" a sequence (u_n) by the values of an integral depending on a real parameter λ and equal to u_n for $\lambda = n$, goes back to Wallis (III, p. 55). It was this idea that principally guided Euler when, in 1730 ((I), v. XIV, p. 1-24) he set himself to interpolate the sequence of factorials. He began by remarking that $n!$ is equal to the infinite product $\prod_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^n \frac{k}{k+n}$, which product is defined for every value of n (an integer or not), and that in particular, for the value $n = \frac{1}{2}$ it takes the value $\frac{1}{2}\sqrt{\pi}$ by Wallis' formula. The analogy between this result and Wallis' led him then to revisit the integral

$$\int_0^1 x^e (1-x)^n \, dx$$

(n integral, e arbitrary) already considered by the latter. Euler obtained the value $\dfrac{n!}{(e+1)(e+2)\dots(e+n+1)}$ for it, using the Binomial expansion; a change of variables then showed him that $n!$ is the limit, as z tends to 0, of the integral $\int_0^1 \left(\frac{1-x^z}{z}\right)^n dx$, whence the "second Euler integral"

$$n! = \int_0^1 \left(\log \frac{1}{x}\right)^n dx;$$

by the same method, and using Wallis' formula, he obtained the formula $\int_0^1 \sqrt{\log 1/x}\, dx = \frac{1}{2}\sqrt{\pi}$. In his later works Euler often returned to these integrals; he thus discovered the complements' relation ((I), t. XV, p. 82 and t. XVII, p. 342), the formula $\mathbf{B}(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ ((I), t. XVII, p. 355), and the particular case of the Legendre-Gauss formula corresponding to $x = 1$ ((I), t. XIX, p. 483); all this of course without worrying about questions of convergence.

Gauss pursued the study of the Γ function in connection with his research on the hypergeometric function, of which the Γ function is a limit case (II); it was in the course of this research that he obtained the general multiplication formula (already noted by Legendre a little earlier for $p = 2$). The later work on Γ was mainly concerned with extending this function to the complex domain. Only recently has it been appreciated that the property of logarithmic convexity characterises $\Gamma(x)$ (in the real domain), up to a constant factor, among the solutions of the functional equation $f(x+1) = x\,f(x)$ (III); and Artin showed (IV) how one can link all the classical results on $\Gamma(x)$ simply to this property. We have followed his exposition quite closely.

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INDEX OF NOTATION

Index