Eigen

a c++ linear algebra library

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[http://eigen.tuxfamily.org]

Inria

Outline

- 9:00 → 9:30 – Discovering Eigen
  - motivations
  - API preview

- 9:30 → 10:30 – Eigen's internals
  - design choices
  - meta-programming
  - expression templates

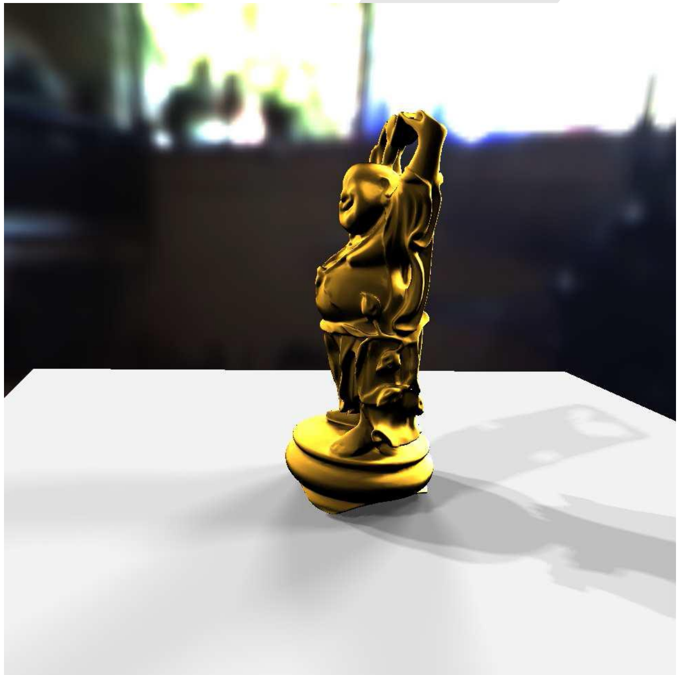
– coffee break –

Outline

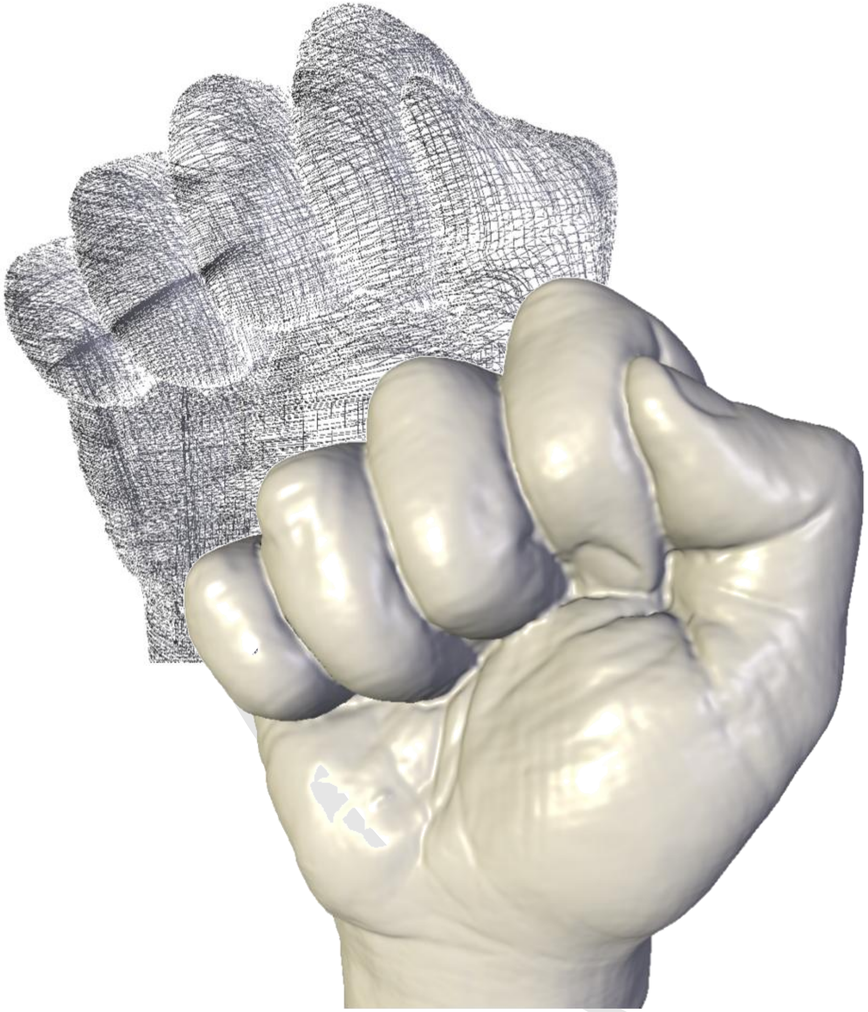
- 11:00 → 12:30 – Use cases
  - Geometry & 3D algebra
  - RBFs (dense solvers)
  - (bi-)Harmonic interpolation (sparse solvers)
- 12:30 → 13:00
  - Roadmap
  - Open-source community
  - Open discussions

Presentations

Real-time Soft Shadows



Surface Reconstruction



Skinning & Vector Graphics

Computer Graphics
&
Linear Algebra?

Computer Graphics & Linear Algebra

- points & vectors
  - 2D to 4D vectors (+, -, *, dot, cross, etc.)
- space transformations
  - 2x2 to 4x4 matrices (including non squared matrices as 3x4)
  - inverse (Cramer's rule, Div. & Conq.)
  - polar decomposition

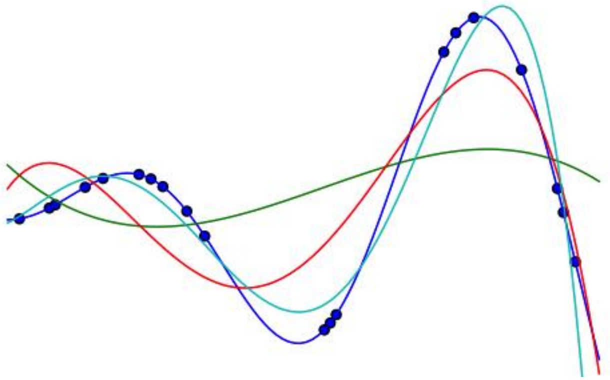
    → Singular Value Decomposition (SVD)
- point sets: normals, oriented bounding boxes

    → Eigen-value decomposition (EVD)

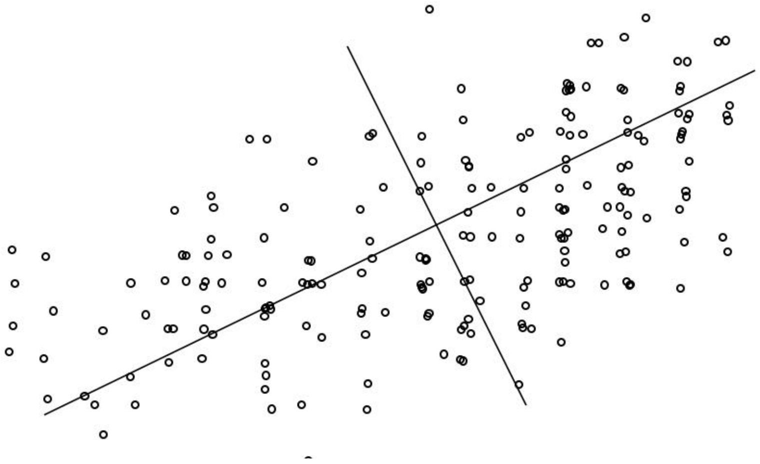
small fixed size linear algebra

Computer Graphics & Linear Algebra

- Linear Least-square
  - Polynomial fit,
    Curvature estimation, MLS
  - Radial Basis Functions



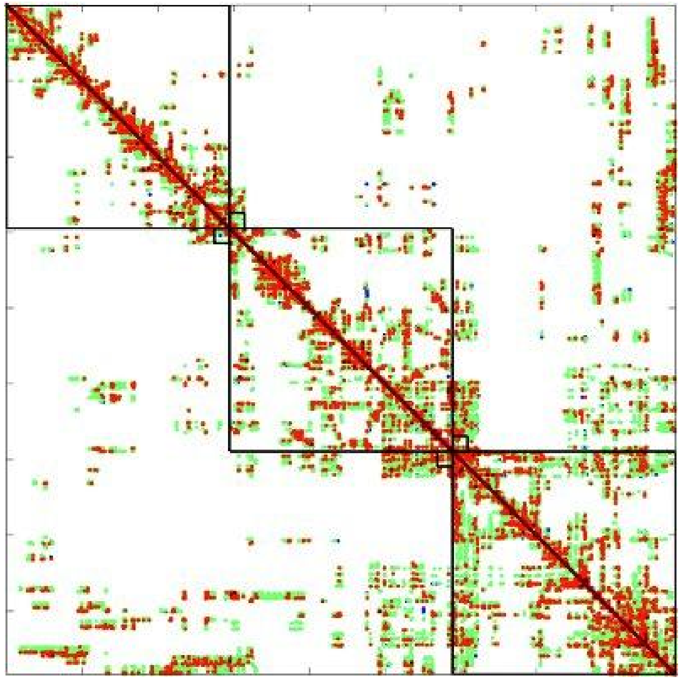
- Spectral analysis



from small to large dense linear algebra

Computer Graphics & Linear Algebra

- Differential equations, FEM
  - physical simulation
  - mesh processing
  - surface reconstruction
  - interpolation



large sparse linear algebra

Context summary

- Matrix computation are everywhere
  - Various applications:
    - simulators/simulations, video games, audio/image processing, design, robotic, computer vision, augmented reality, etc.
  - Need various tools:
    - numerical data manipulation, space transformations
    - inverse problems, PDE, spectral analysis
  - Need performance:
    - on standard PC, smartphone, embedded systems, etc.
    - real-time performance

Matrix computation?

MatLab

+ friendly API
+ large set of features
- math only
- extremely slow for small objects

→ **Prototyping**

?

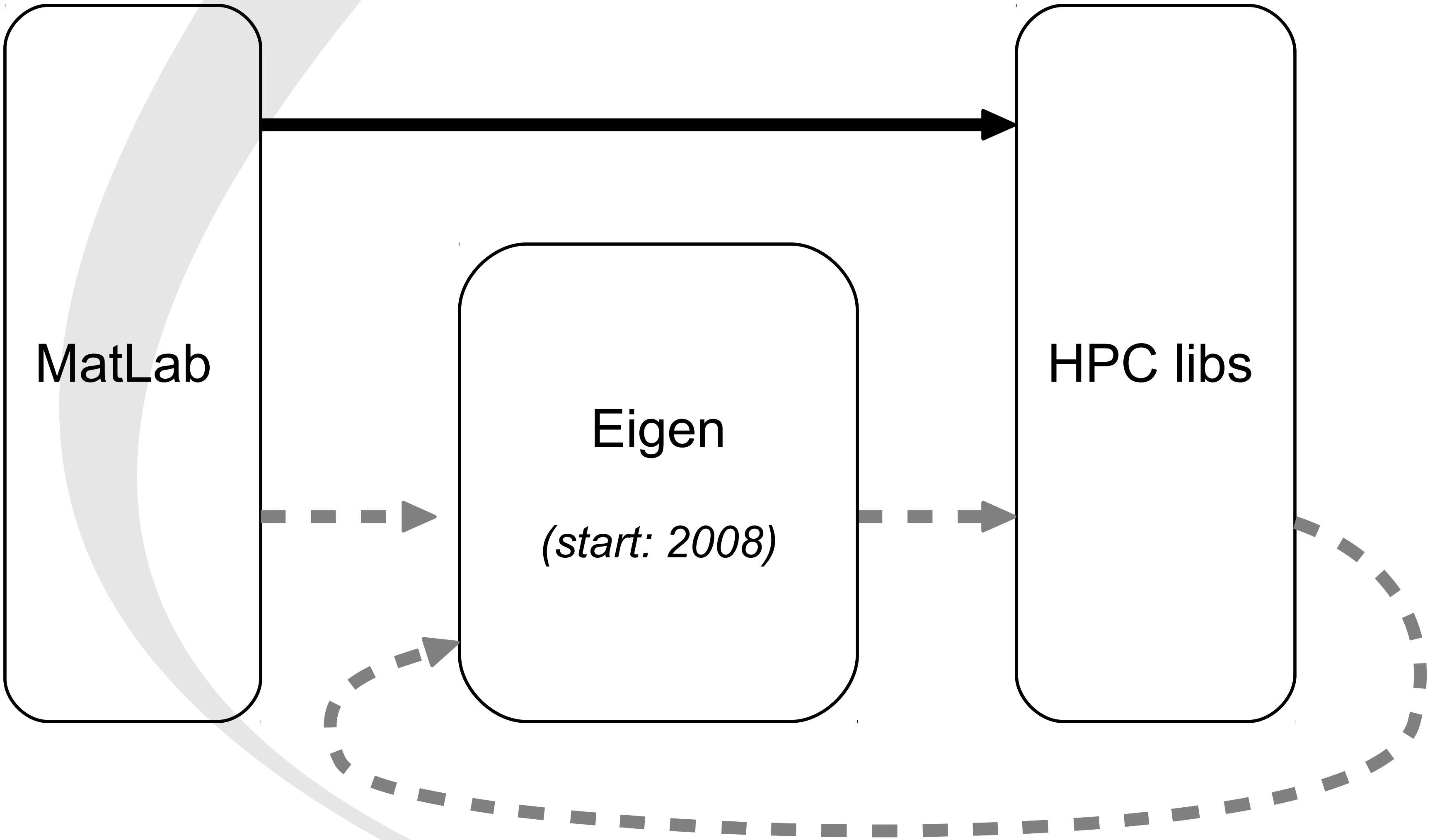
Zoo of libs

+ highly optimized
- 1 feature = 1 lib

+/- tailored for advanced user / clusters
- slow for small objects

→ **Advanced usages**

Matrix computation?



Facts

- Pure C++ template library
  - header only
  - no binary to compile/install
  - no configuration step
  - no dependency *(optional only)*

```
#include <Eigen/Eigen>

using namespace Eigen;

int main() {
  Matrix4f A = Matrix4f::Random();
  std::cout << A << std::endl;
}
```

```
$ g++ -O2 example.cpp -o example
```

Facts

- Pure C++ template library
  - header only
  - no binary to compile/install
  - no configuration step
  - no dependency *(optional only)*

- Packaged by all Linux distributions *(incl. macport)*

- Opensource: MPL2

→ **easy to install & distribute**

Multi-platforms

- Supported compilers:
  - GCC *(≥4.2)*, MSVC *(≥2008)*, Intel ICC, Clang/LLVM, ~~old apple's compilers~~
- Supported systems:
  - x86/x86_64, ARM, PowerPC
  - Linux, Windows, OSX, IOS
- Supported SIMD vectorization engines:
  - SSE{2,3,4}
  - NEON *(ARM)*
  - Altivec *(PowerPC)*

Large feature set

- Core
  - Matrix and array manipulation (~MatLab, 1D & 2D)
  - Basic linear algebra (~BLAS)
    - incl. triangular & self-adjoint matrix
- LU, Cholesky, QR, SVD, Eigenvalues
  - Matrix decompositions and linear solvers (~Lapack)
- Geometry (transformations, …)
- Sparse
  - Manipulation
  - Solvers (LLT, LU, QR & CG, BiCGSTAB, GMRES)
- WIP modules (autodiff, non-linear opt., FFT, etc.)

→ **"unified API" - "all-in-one"**

Optimized for both small and large objects

- Small objects
  - means fixed sizes:

```
Matrix<float,4,4>
```

  - malloc-free
  - meta unrolling
  - specialized algo

- Large objects
  - means dynamic sizes

```
Matrix<float,Dynamic,1>
```

  - cache friendly kernels
  - multi-threading *(OpenMP)*

- Vectorization (SIMD)
- Unified API → write generic code
- Mixed fixed/dynamic dimensions

Generic code (1/2)

- Non-generic code:

```
class     Sphere {
  float[3]                    center;
  float                       radius;
  /* … */
};
```

Generic code (1/2)

- Write generic code:

```
template <                          int AmbientDim=Eigen::Dynamic>
class HyperSphere {
  Eigen::Matrix<float ,AmbientDim,1> center;
  float                              radius;
  /* … */
};

typedef HyperSphere<2>          HyperSphere2;
typedef HyperSphere<3>          HyperSphere3;
typedef HyperSphere<3>          Sphere;
typedef HyperSphere<>           HyperSphereX;
```

  - Eigen takes care of the low level optimizations

Generic code (2/2)

- Write fully generic code:

```
template <typename Scalar, int AmbientDim=Eigen::Dynamic>
class HyperSphere {
  Eigen::Matrix<Scalar,AmbientDim,1> center;
  Scalar                             radius;
  /* … */
};

typedef HyperSphere<float, 2>  HyperSphere2f;
typedef HyperSphere<double,3>  HyperSphere3d;
```

Custom scalar types

- Can use custom types everywhere
  - Exact arithmetic (rational numbers)
  - Multi-precision numbers (e.g., via mpfr++)
  - Auto-diff scalar types
  - Interval
  - Symbolic

- Example:

```
typedef Matrix<mpreal,Dynamic,Dynamic> MatrixMP;
MatrixMP A, B, X;
// init A and B
// solve for A.X=B using LU decomposition
X = A.lu().solve(B);
```

Communication with the world

→ standard matrix representations

- to Eigen

```
float* raw_data = malloc(...);
Map<MatrixXd> M(raw_data, rows, cols);
// use M as a MatrixXd
M = M.inverse();
```

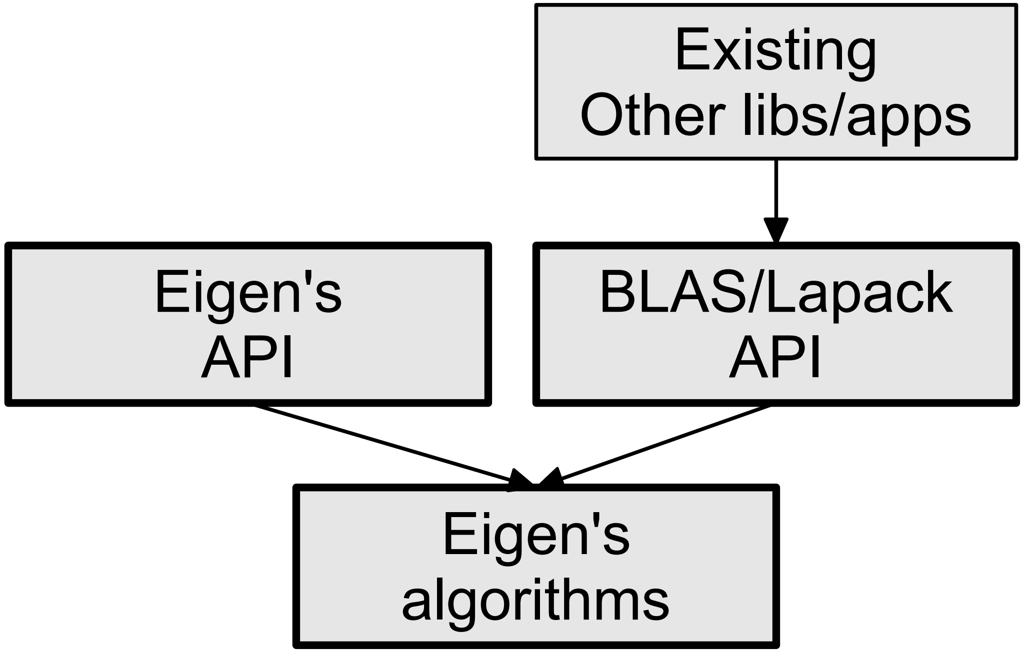
- from Eigen

```
MatrixXd M;
float* raw_data = M.data();
int stride = M.outerStride();
raw_data[i+j*stride]
```

→ same for sparse matrices

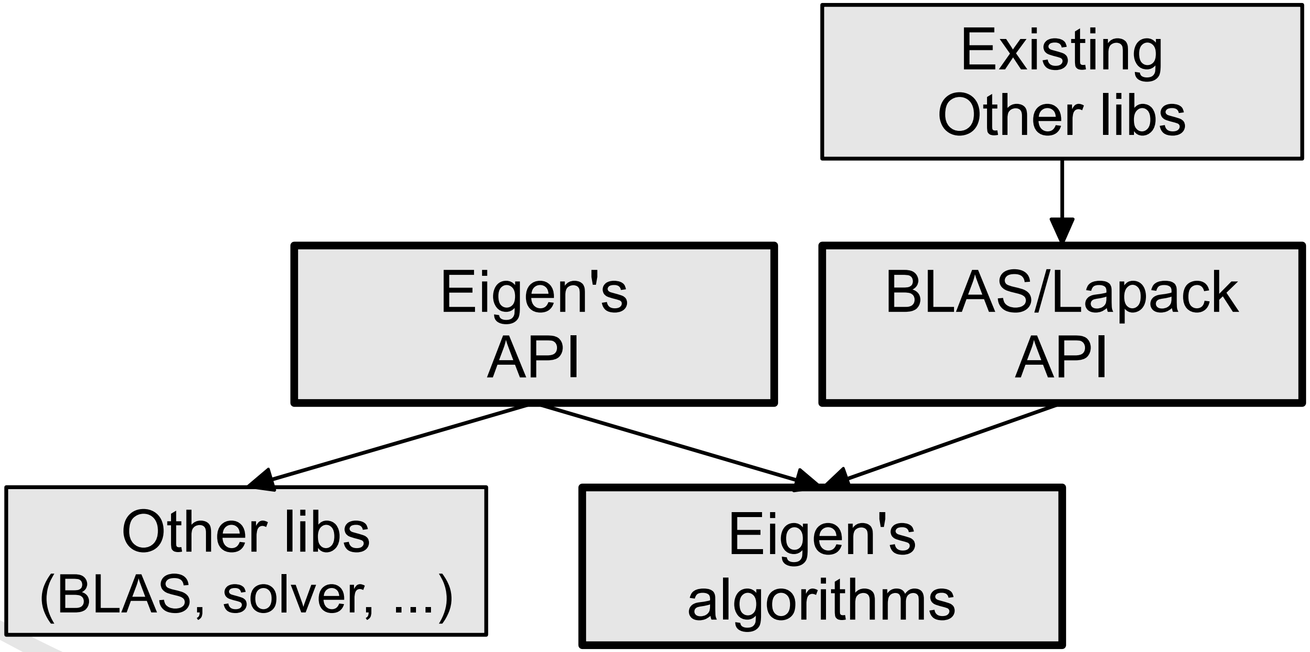
Eigen & BLAS

- Call Eigen's algorithms through a BLAS/Lapack API
  - Alternative to ATLAS, OpenBlas, Intel MKL
    - e.g., sparse solvers, Octave, Plasma, etc.
  - Run the Lapack test suite on Eigen



External backends

- External backends
  - Fallback to existing BLAS/Lapack/etc. (done by Intel)
  - Unified interface to many sparse solvers:
    - UmfPack, Cholmod, PaSTiX, Pardiso



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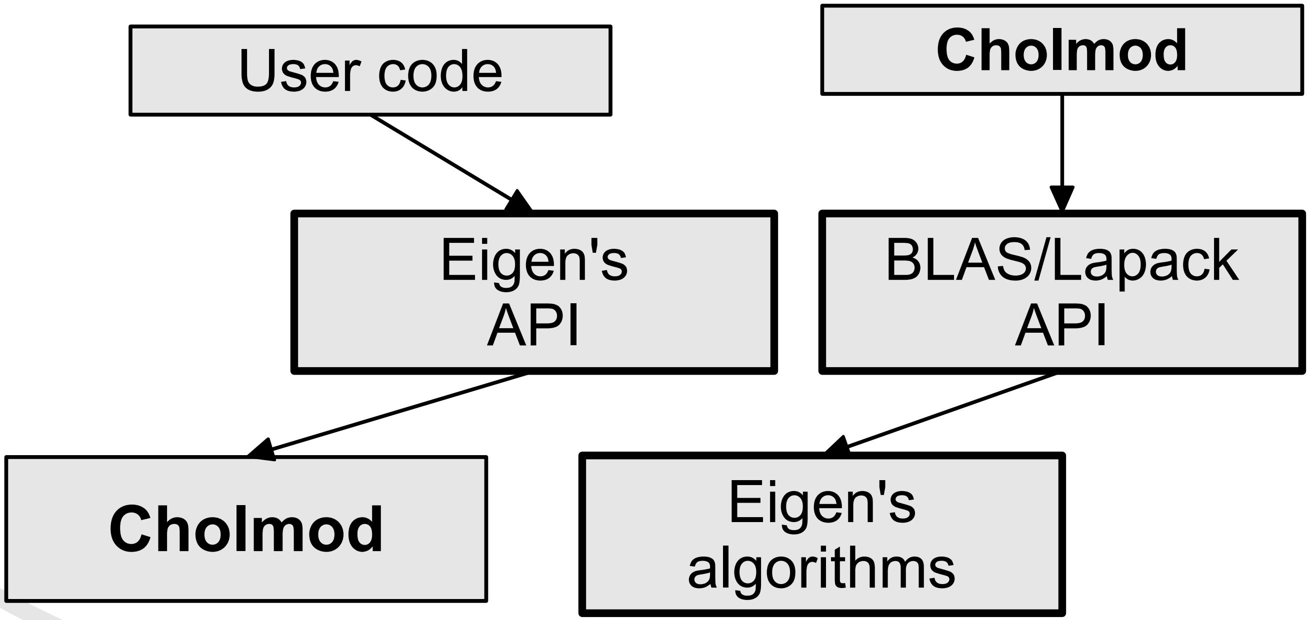


Documentation

- Documentation

- Support
  - Forum, IRC, Mailing-List
  - Bugzilla

API demo

Internals

Technical aspects

~~Matrix factorizations?~~

~~Matrix products?~~

~~Sparse algebra?~~

Expression templates

Meta-programming

Vectorization

Preliminaries 1/3 - C++

- Template programming & Inheritance:

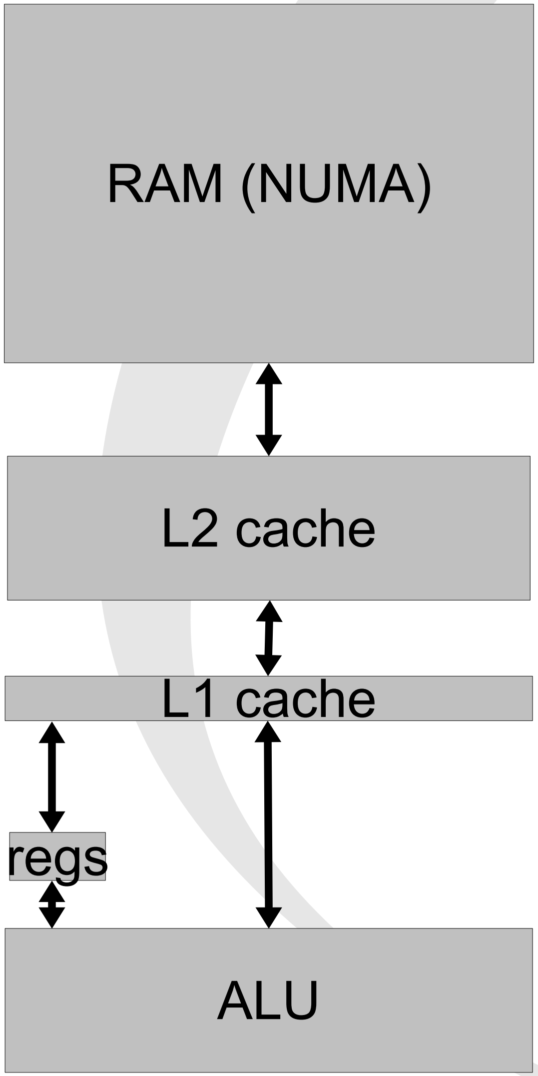
```
template<typename Scalar, int N>
class Vector : public SomeBaseClass {
  Scalar m_data[N];
  /* … */
};
```

- Partial template specialization:

```
template<typename Real, int N>
class Vector<complex<Real>,N> : public AnotherBaseClass {
  Real m_real[N];
  Real m_imag[N];
  /* … */
};

Vector<double,3>          v1;
Vector<complex<float>,3>  v2;
```

→ pattern matching

Preliminaries 2/3 – Memory Hierarchy



x1000 bigger ; ~400 cycles

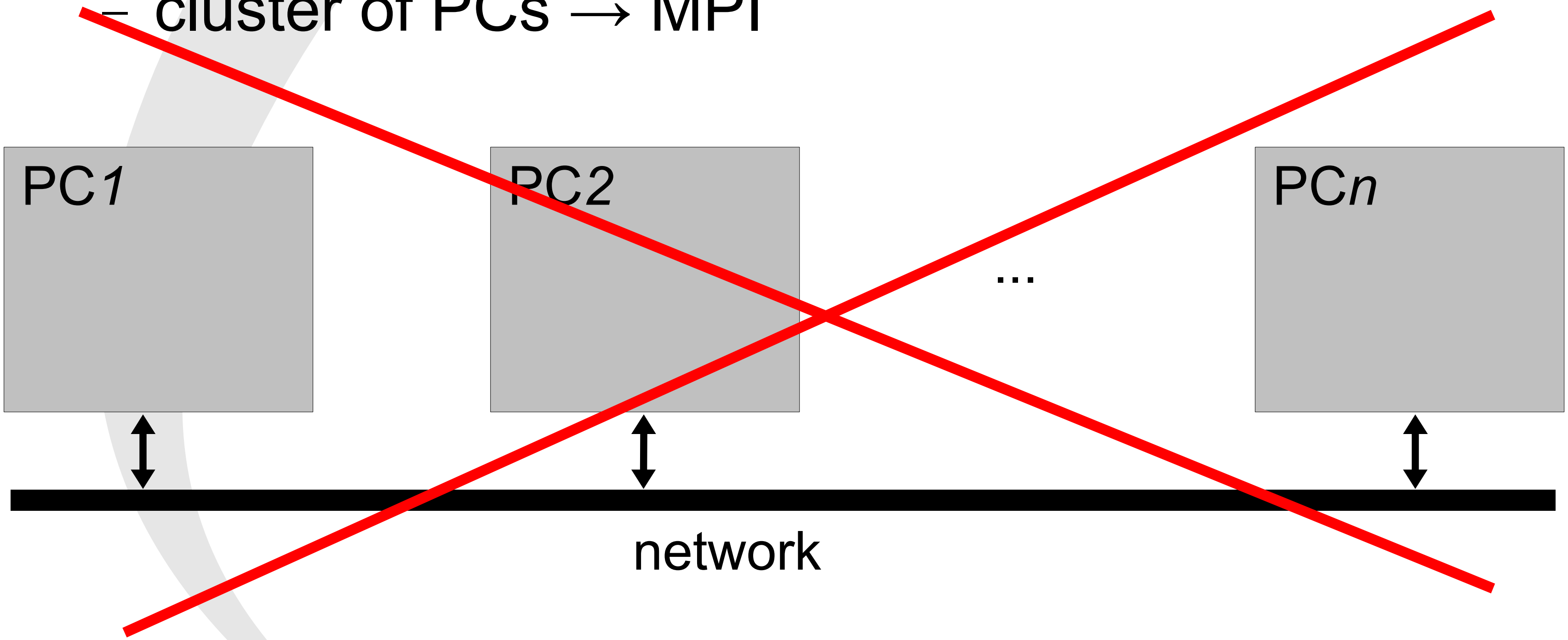
x100 bigger (900x900 floats) ; 40-100 cycles

x100 bigger (90x90 floats) ; 1-4 cycles

small (8x8 floats) ; 1 cycle

Preliminaries 3/3 - Parallelism

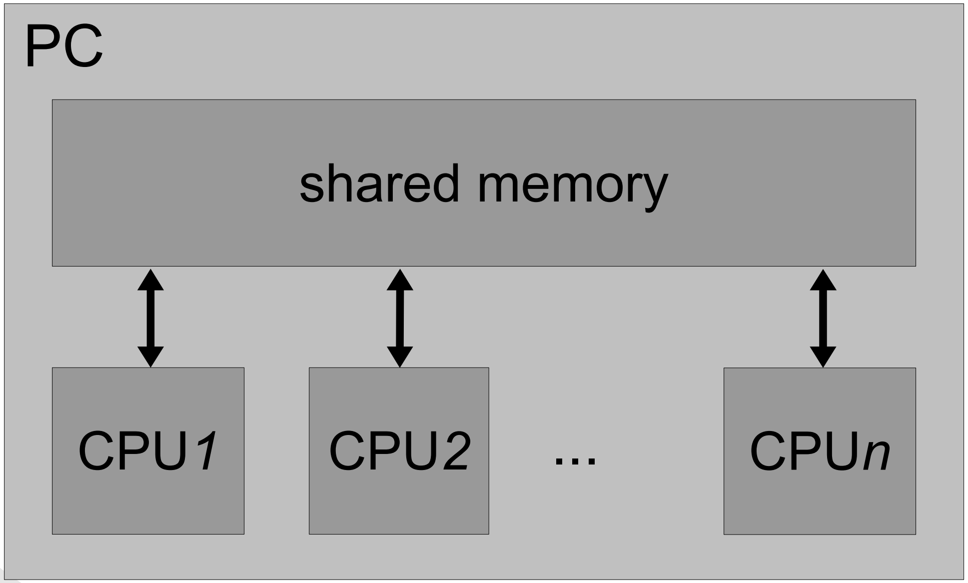
- 4 levels of parallelism:
  - cluster of PCs → MPI



out of the scope of Eigen

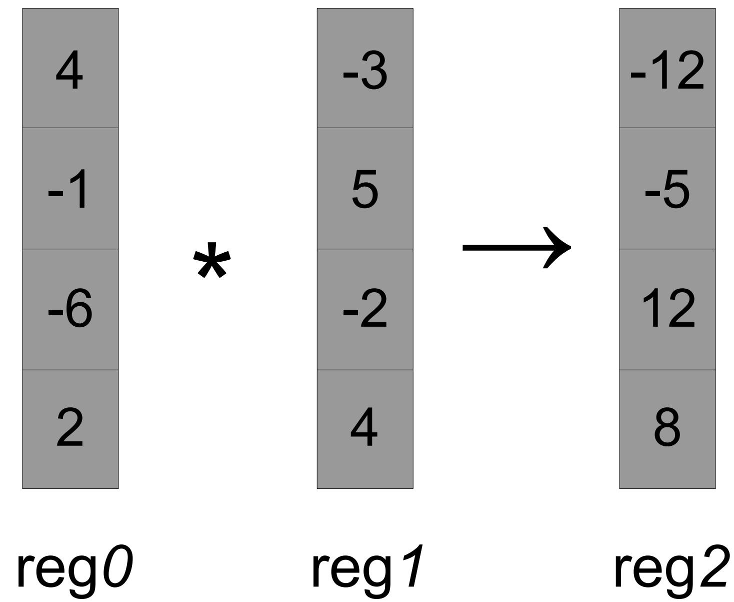
Preliminaries 3/3 – Parallelism

- 4 levels of parallelism:
  - ~~cluster of PCs → MPI~~
  - multi/many-cores → OpenMP



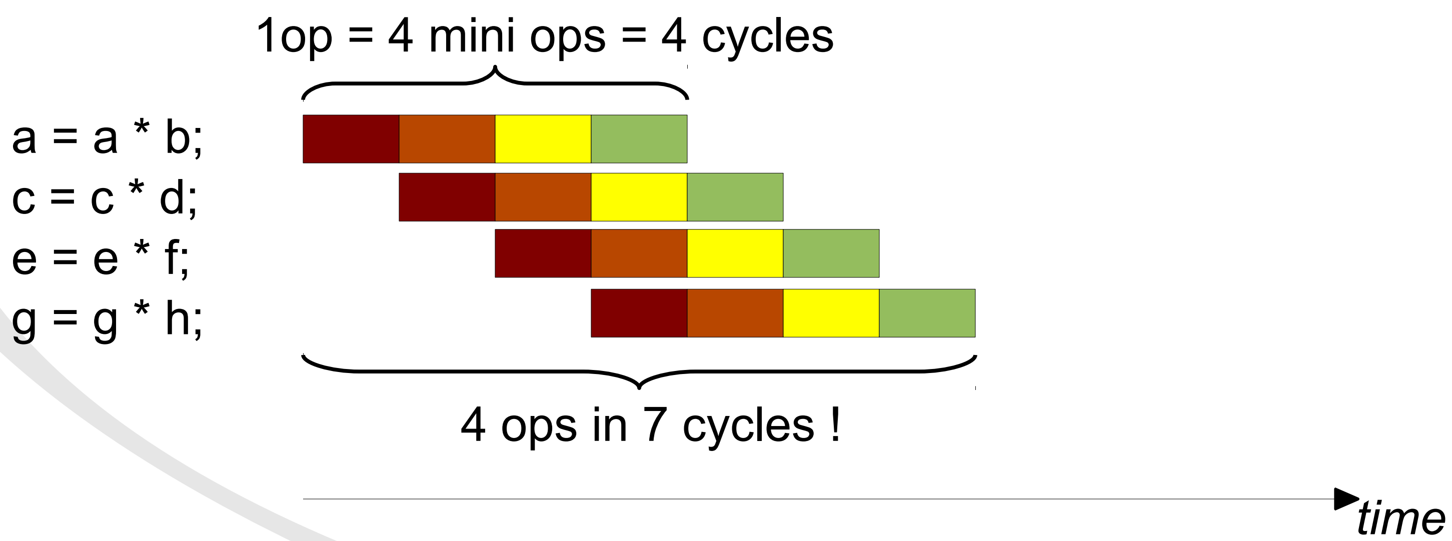
Preliminaries 3/3 – Parallelism

- 4 levels of parallelism:
  - ~~cluster of PCs → MPI~~
  - multi/many-cores → OpenMP
  - SIMD → intrinsics for vector instructions (SSE, AVX, …)



Preliminaries 3/3 – Parallelism

- 4 levels of parallelism:
  - ~~cluster of PCs → MPI~~
  - multi/many-cores → OpenMP
  - SIMD → intrinsics for vector instructions (SSE, AVX, …)
  - pipelining → needs non dependent instructions



Peak performance

- Example
  - Intel Core2 Quad CPU Q9400 @ 2.66GHz (x86_64)
    - pipelining → 1 mul + 1 add / cycle (**ideal case**)
    - SSE → x **4 single precision** ops at once
    - frequency → x **2.66G**
    - **peak performance: 21,790 Mflops** (for 1 core)

that's our goal!

Problem statement

- Example:

```
m3 = m1 + m2 + m3;
```

- Standard C++ way:

```
class Matrix {
  float m_data[M*N];
  float& operator()(int i, int j) { return m_data[i+j*M]; }
};

Matrix operator+(const Matrix& A, const Matrix& B) {
  Matrix res;

  for(int j=0; j<N; ++j)
    for(int i=0; i<M; ++i)
      res(i,j) = A(i,j) + B(i,j);

  return res;
}
```

Problem statement

- Example:

```
m3 = m1 + m2 + m3;
```

- Standard C++ way, result:

```
tmp1 = m1 + m2;
tmp2 = tmp1 + m3;
m3   = tmp2;
```

→ 3 loops :(
→ 2 temporaries :(
→ **8***M*N memory accesses :(

Expression templates

- Example:

```
m3 = m1 + m2 + m3;
```

- Expression templates:
  - "+" returns an expression:

```
Sum<Matrix,Matrix>
operator+(const Matrix& A, const Matrix& B) {
    return Sum<Matrix,Matrix>(A,B);
}


template<typename type_of_A, typename type_of_B>
class Sum {
    const type_of_A &A;
    const type_of_B &B;
};
```

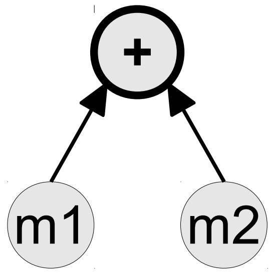
Expression templates

- Example:

```
m3 = m1 + m2 + m3
```

→ "expression tree"

```
Sum<Matrix,Matrix>
```



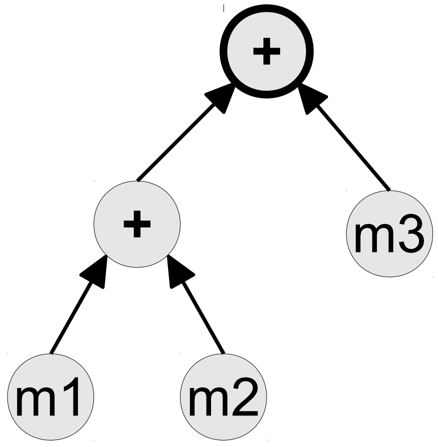
Expression templates

- Example:

```
m3 = m1 + m2 + m3
```

→ “expression tree”

```
Sum< Sum<Matrix,Matrix>,
     Matrix >
```



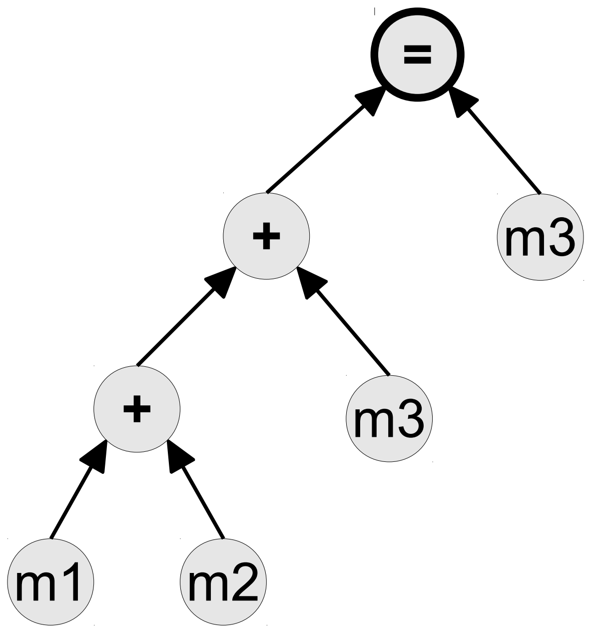
Expression templates

- Example:

```
m3 = m1 + m2 + m3
```

→ “expression tree”

```
Assign<Matrix,
       Sum< Sum<Matrix,Matrix>,
            Matrix > >
```



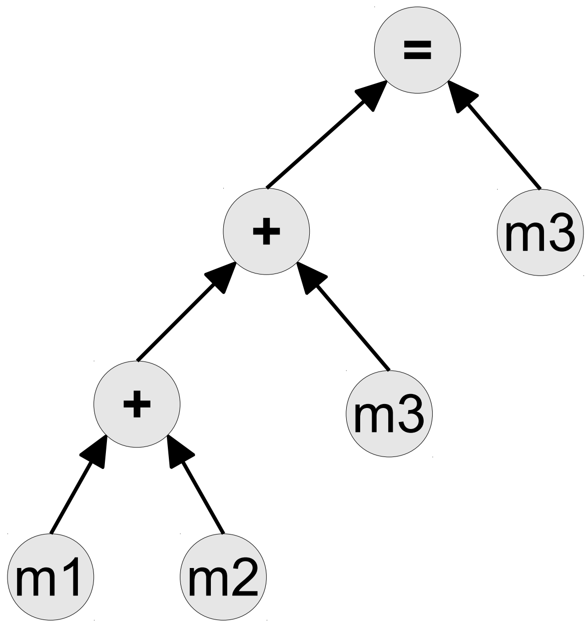
Expression templates

- Example:

```
m3 = m1 + m2 + m3;
```

→ “expression tree”

```
Assign<Matrix,
       Sum< Sum<Matrix,Matrix>,
            Matrix > >
```



- Immediate question:
  - How to evaluate this?

Evaluation of expressions

- Bottom-up approach:

```
template<type_of_A, type_of_B>
class Sum {
    const type_of_A &A;
    const type_of_B &B;

    Scalar coeff(i, j) {
        return A.coeff(i,j) + B.coeff(i,j);
    }
};
```

→ simple, can specialize the implementation based on the operand types...

… but not on the **whole expression** :(

Evaluation of expressions

- Solution: **top-down** creation of an evaluator
  - evaluator:

```
template<ExprType> class Evaluator;
```

  - partial specialization for each operation, e.g.:

```
template<type_of_A, type_of_B>
class Evaluator< Sum<type_of_A,type_of_B> > {
  Evaluator<type_of_A> evalA(A);
  Evaluator<type_of_B> evalB(B);
  Scalar coeff(i,j) {
    return evalA.coeff(i,j) + evalB.coeff(i,j);
  }
};
```

Evaluation of expressions

- Matrix evaluator:

```
class Evaluator<Matrix> {
  const Matrix &mat;
  Scalar coeff(i) { return mat.data[i]; }
};
```

Evaluation of expressions

- Solution: **top-down** creation of an evaluator
  - assignment evaluator (dest ← source):

```
template<Dest, Source>
class Evaluator< Assign<Dest,Source> > {
  Evaluator<Dest>   evalDst(dest);
  Evaluator<Source> evalSrc(source);
  void run() {
    for(int i=0; i<evalDst.size(); ++i)
      evalDst.coeff(i) = evalSrc.coeff(i);
  }
};
```

- Example: `m3 = m1 + m2 + m3;`
  - compiles to:

```
for(i=0; i<m3.size(); ++i)
    m3[i] = "Evaluator(m1+m2+m3)".coeff(i);
```

Template Instantiations

```
for(i=0; i<m3.size(); ++i)
    m3[i] = “Evaluator(m1+m2+m3)”.coeff(i);

class Evaluator< Sum< Sum<Matrix,Matrix>, Matrix > > {
  Evaluator<Sum<Matrix,Matrix> >  evalA(“m1+m2”);
  Evaluator<Matrix>               evalB(“m3”);
  Scalar coeff(i) {
    return evalA.coeff(i) + evalB.coeff(i);
  }
};
```

m3[i]

```
class Evaluator< Sum<Matrix,Matrix> > {
  Evaluator<Matrix>   evalA(“m1”);
  Evaluator<Matrix>   evalB(“m2”);
  Scalar coeff(i) {
    return evalA.coeff(i) + evalB.coeff(i);
  }
};
```

m1[i] m2[i]

generated by the compiler!

Template Instantiations

```
m3 = m1 + m2 + m3;
```

↓

- After inlining:

```
for(i=0; i<m3.size(); ++i)
    m3[i] = m1[i] + m2[i] + m3[i];
```

→ 1 loop
→ no temporaries
→ /2 memory accesses

Expression templates

- Generalize to any coefficient-wise operations
  - example:

```
m3.block(1,2,rows,cols) = 2*m1 – m2.transpose();
```

  - expression type:

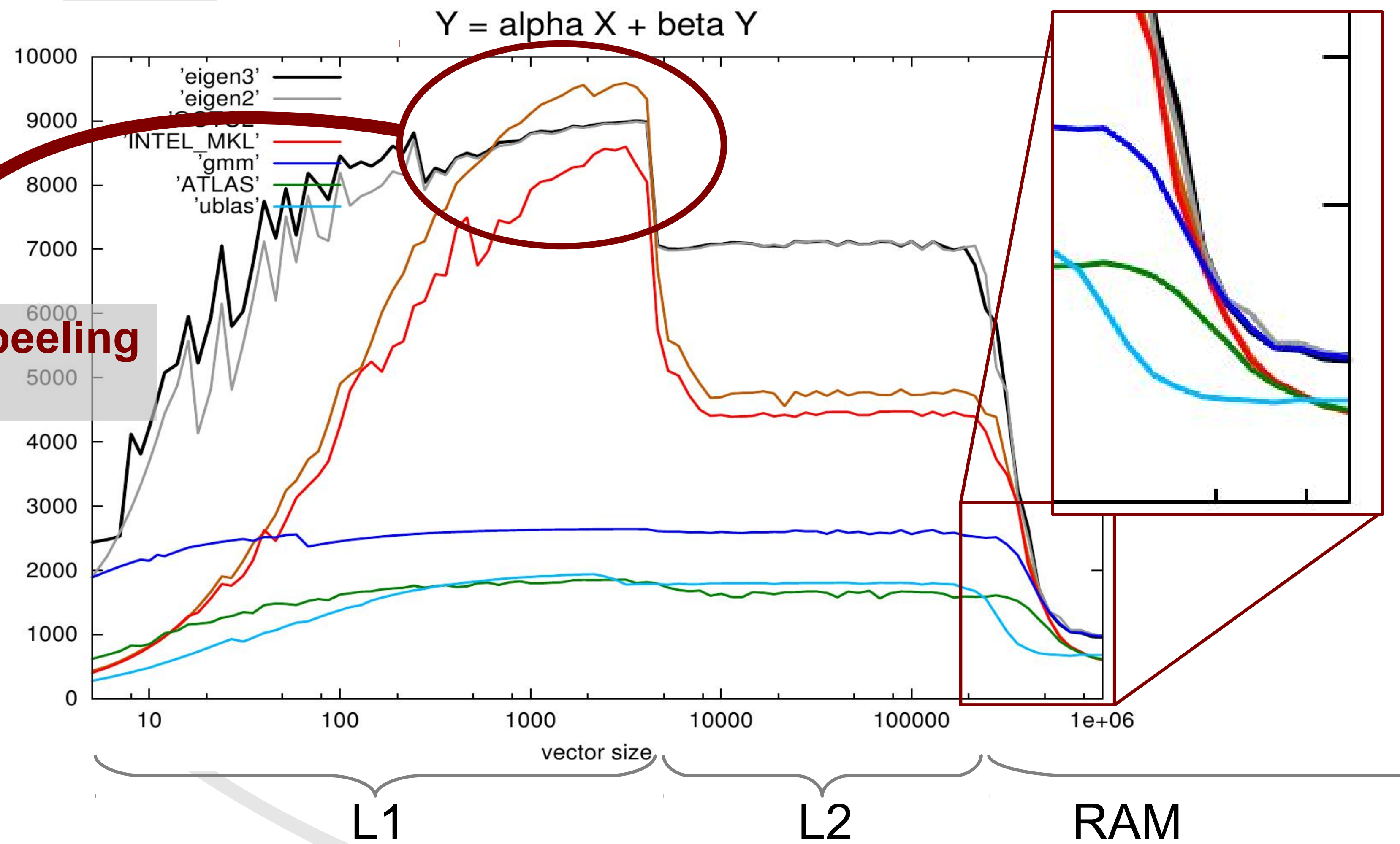
```
Assign<Block<Matrix>,
       Difference< ScalarMultiple<Matrix>,
                   Transpose<Matrix> > >
```

  - compiles to:

```
for(int j=0; j<cols; ++j)
  for(int i=0; i<rows; ++i)
    m3(i+1,j+2) = 2*m1(i,j) – m2(j,i);
```

Expr. templates: Fused operations

- reduce temporaries, memory accesses, cache misses



no loop peeling for Eigen

Expr. templates: Better API

- Better API
  - more examples:

```
x.col(4) = A.lu().solve(B.col(5));

x = b * A.triangularView<Lower>().inverse();
```

Combinatorial complexity

- Explosion of types and possible combinations

```
Sum<.,.> operator+(const Matrix& A, const Matrix& B);
Sum<.,.> operator+(const Sum<.,.>& A, const Matrix& B);
Sum<.,.> operator+(const Sum<.,.>& A, const Sum<.,.>& B);
Sum<.,.> operator+(const Sum<.,.>& A, const Transpose<.>& B);
Sum<.,.> operator+(const Transpose<.>& A, const Matrix& B);
...
```

→ need a common base class
+ polymorphism

Combinatorial complexity

- Common base class:

```
class Evaluator<Sum<A,B> >
        : EvaluatorBase {
   /* … */
   virtual Scalar coeff(i,j);
};
```

```
class Matrix    : MatrixBase {...};
class Sum<A,B> : MatrixBase {...};
```

```
class MatrixBase {


  Sum<MatrixBase,MatrixBase>
  operator+(const MatrixBase& other) {
    return Sum<MatrixBase,MatrixBase>(*this, other);
  }
```

cannot work this way!

```
};
```

→ need **compile-time** polymorphism

→ CRTP (Curiously Recurring Template Pattern)

CRTP

- base class:

```
class Matrix   : MatrixBase {...};
class Sum<A,B> : MatrixBase {...};


class MatrixBase {


  Sum<MatrixBase,MatrixBase>
  operator+(const MatrixBase& other) {
    return Sum<MatrixBase,MatrixBase>(*this, other);
  }



};
```

CRTP

- base class + static polymorphism:

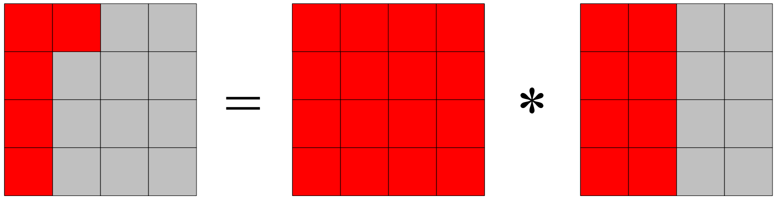
```
class Matrix   : MatrixBase< Matrix > {...};
class Sum<A,B> : MatrixBase< Sum<A,A> > {...};

template<typename Derived>
class MatrixBase {

  template<typename OtherDerived>
  Sum<Derived,OtherDerived>
  operator+(const MatrixBase<OtherDerived>& other) {
    return Sum<Derived,OtherDerived>(derived(), other.derived());
  }

  Derived& derived() { return static_cast<Derived&>(*this); }

};
```

Product-like operations?

- Expression templates
  - very good for any coefficient-wise operations
  - what about matrix products?

```
class Evaluator< Product<type_of_A,type_of_B> > {
  Scalar coeff(i,j) {
    return (A.row(i).cwiseProduct(B.col(i).transpose()).sum();
  }
};
```

  - what's wrong?



→ OK for very very small matrices only

How to make products efficient?

→ cache-aware product algorithm

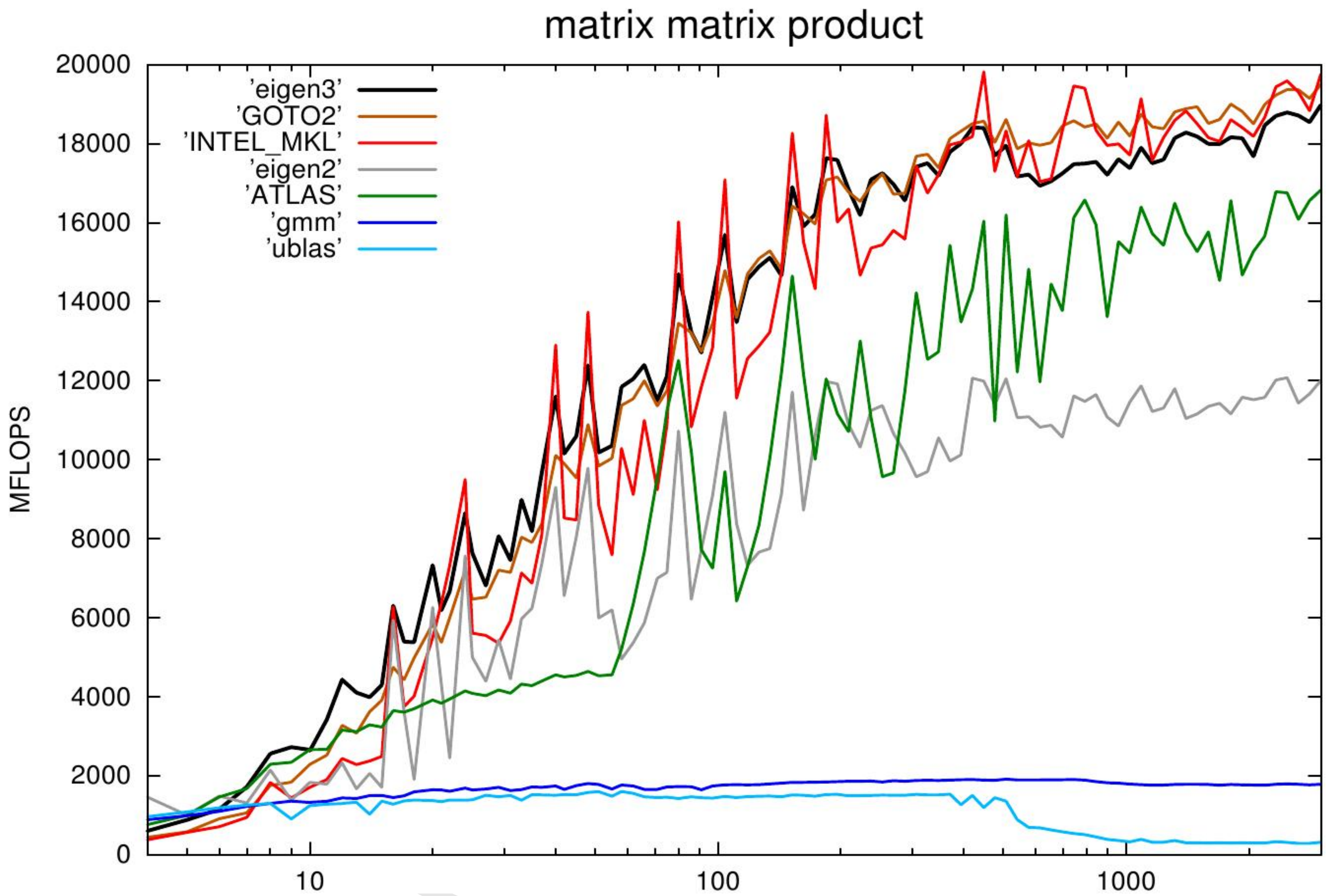
- optimize L2, L1 and register reuses
  - needs access the data of the result

```
D = C + A * B ;
```

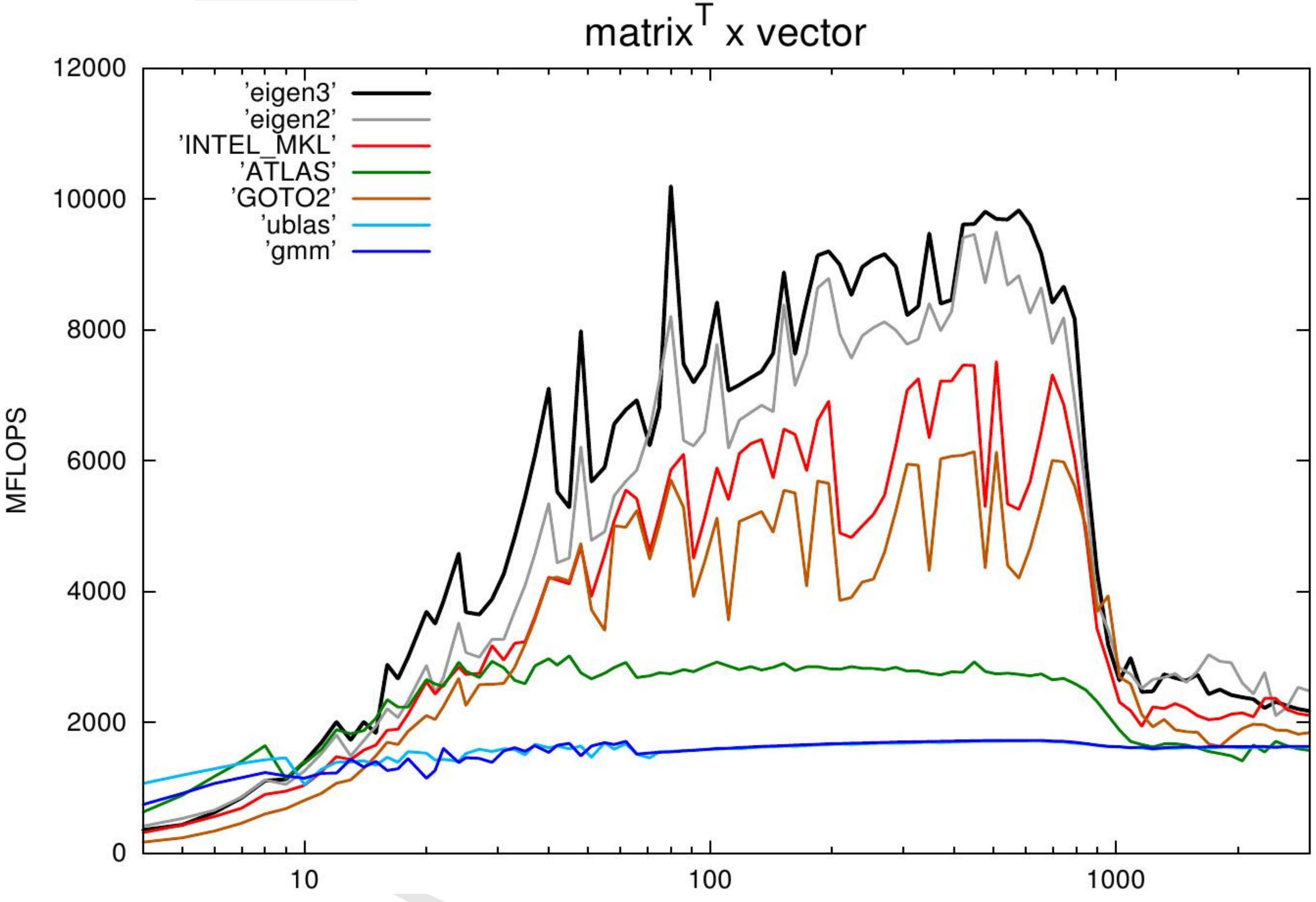
(A * B) → needs to be evaluated into a temporary:

```
Matrix tmp;
gemm(A, B, tmp);
D = C + tmp;
```

Performance



Performance



How to make products efficient?

- Combinatorial complexity

```
      A * B;            2*A * B.adjoint();
              A.col(j).transpose() * B;
A.transpose() * B;
           -A.block(i,j,r,c) * (2*B).transpose();
```

  - one product version is very complex (lot of instructions)

    → handle only one generic version:

    ```
    gemm<op1,op2>(A,B,s,C);
    → C += s * op1(A) * op2(B);
    ```

    - op_ = nop, conjugate, transpose, adjoint
    - A, B, C → reference to block of memory with strides

Top-down expression analysis

- Products
  - detect & evaluate product sub expressions
    - e.g.: `... 3*m1 + (2*m2).adjoint() * m3 + ...`

      → `gemm<Adj,Nop>(m2, m3, -2, tmp);`

```
Evaluator<Product<type_of_A,type_of_B> > : Evaluator<Matrix> {
  Evaluator(A,B) : Evaluator<Matrix>(tmp) {

    EvaluatorForProduct<type_of_A> evalA(A);
    EvaluatorForProduct<type_of_B> evalB(B);

    gemm<evalA.op,evalB.op>(  evalA.data, evalB.data,
                              evalA.scale*evalB.scale,
                              tmp);
  }
  Matrix tmp;
};
```

Top-down expression analysis

- Products
  - avoid temporary when possible
    - e.g.: `m4 -= (2 * m2).adjoint() * m3;`

      → `gemm<Adj,Nop>(m2, m3, -2, m4);`

```
Evaluator<Assign<type_of_C,Product<type_of_A,type_of_B> > {
  Evaluator(C,P) {

    EvaluatorForProduct<type_of_A> evalA(P.A());
    EvaluatorForProduct<type_of_B> evalB(P.B());

    gemm<evalA.op,evalB.op>(  evalA.data, evalB.data,
                              evalA.scale*evalB.scale,
                              C);
  }
};
```

Top-down expression analysis (cont.)

- More complex example:

```
m4 -= m1 + m2 * m3;
```

  - so far:

```
tmp = m2 * m3;
m4 -= m1 + tmp;
```

  - better:

```
m4 -= m1;
m4 -= m2 * m3;
```

```
// catch R = A + B * C
Evaluator<Assign<R,Sum<A,Product<B,C> > > { … };
```

Tree optimizer

- Even more complex example:

```
res -= m1 + m2 + m3*m4 + 2*m5 + m6*m7;
```

  - Tree optimizer

→
```
res -= ((m1 + m2 + 2*m5) + m3*m4) + m6*m7;
```

  - yields:

```
res -= m1 + m2 + 2*m5;
res -= m3*m4;
res -= m6*m7;
```

  - Need only two rules:

```
// catch A * B + Y and builds Y' + A' * B'
TreeOpt<Sum<Product<A,B>,Y> > { … };

// catch X + A * B + Y and builds (X' + Y') + (A' * B')
TreeOpt<Sum<Sum<X,Product<A,B> >,Y> > { … };
```

→ *demo*

Tree optimizer

- Last example:

```
res += m1 * m2 * v;
```

  - Tree optimizer

→
```
res += m1 * (m2 * v);
```

  - Rule:

```
TreeOpt<Product<Product<A,B>,C> > { … };
```

Vectorization

- How to exploit SIMD instructions?
  - Evaluator:

```
class Evaluator< Sum<type_of_A,type_of_B> > {

  Scalar coeff(i,j) {
    return evalA.coeff(i,j) + evalB.coeff(i,j);
  }

  Packet packet(i,j) {
    return padd(evalA.packet(i,j), evalB.packet(i,j));
  }
};
```

padd → unified wrapper to intrinsics (SSE, NEON, AVX)

Vectorization

- How to exploit SIMD instructions?
  - Assignment:

```
class Evaluator< Assign<Dest,Source> > {
  void run() {
    for(int i=0; i<evalDst.size(); ++i)
      evalDst.coeff(i,j) = evalSrc.coeff(i,j);
  }

  void run_simd() {
    for(int i=0; i<evalDst.size(); i+=PacketSize)
      evalDst.writePacket(i,j, evalSrc.packet(i,j));
  }
};
```

Vectorization: result

```
#include<Eigen/Core>
using namespace Eigen;

void foo(Matrix2f& u,
         float a, const Matrix2f& v,
         float b, const Matrix2f& w)
{
  u = a*v + b*w - u;
}
```

```
movl 8(%ebp), %edx
movss 20(%ebp), %xmm0
movl 24(%ebp), %eax
movaps %xmm0, %xmm2
shufps $0, %xmm2, %xmm2
movss 12(%ebp), %xmm0
movaps %xmm2, %xmm1
mulps (%eax), %xmm1
shufps $0, %xmm0, %xmm0
movl 16(%ebp), %eax
mulps (%eax), %xmm0
addps %xmm1, %xmm0
subps (%edx), %xmm0
movaps %xmm0, (%edx)
```

Unrolling

- Small sizes

  → cost dominated by loop logic

  → remove the loop... yourself!

(don't overestimate compiler's abilities)

```
for(int i=n-1; i>=0; --i)
  foo(i,args);
```

```
void foo_impl(int i,args) {
  foo(i,args);
  if(i>0)
    foo_impl(i-1,args);
}

foo_impl(n-1,args);
```

functional approach

```
template<int I> struct foo_impl {
  static void run(args) {
    foo(I,args);
    foo_impl<I-1>::run(args);
  }
};

template<> struct foo_impl<-1> {
  static void run(args) {}
};

foo_impl<N-1>::run(args);
```

Controls

- Still many questions:
  - which loops have to be unrolled?
  - which sub-expressions have to be evaluated?
  - is vectorization worth it?
- Depend on many parameters:
  - scalar type
  - expression complexity
  - kind of operations
  - architecture

→ **need an evaluation cost model**

Cost model

- Cost Model
  - Track an approximation of the cost to evaluate one coefficient
    - each scalar type defines: *ReadCost, AddCost, MulC*ost
    - combined for each expression by the evaluator, e.g.:

```
class Evaluator< Sum<A,B> > {
  enum {
    Cost = NumTraits<Scalar>::AddCost
         + Evaluator<A>::Cost
         + Evaluator<B>::Cost;
  };
  …
};
```

Cost model

- Examples:
  - loop unrolling (partial)

```
// somewhere in Evaluator<Assign>:
assign_impl<    (Evaluator<Src>::Cost*N > threshold)
              ? NoUnrolling : Unrolling >::run(dst,src);
```

  - evaluation of sub expressions
    - *(**a**+**b**)* * *c* → *(a+b)* is evaluated into a temporary
    - enable vectorization of sub expressions

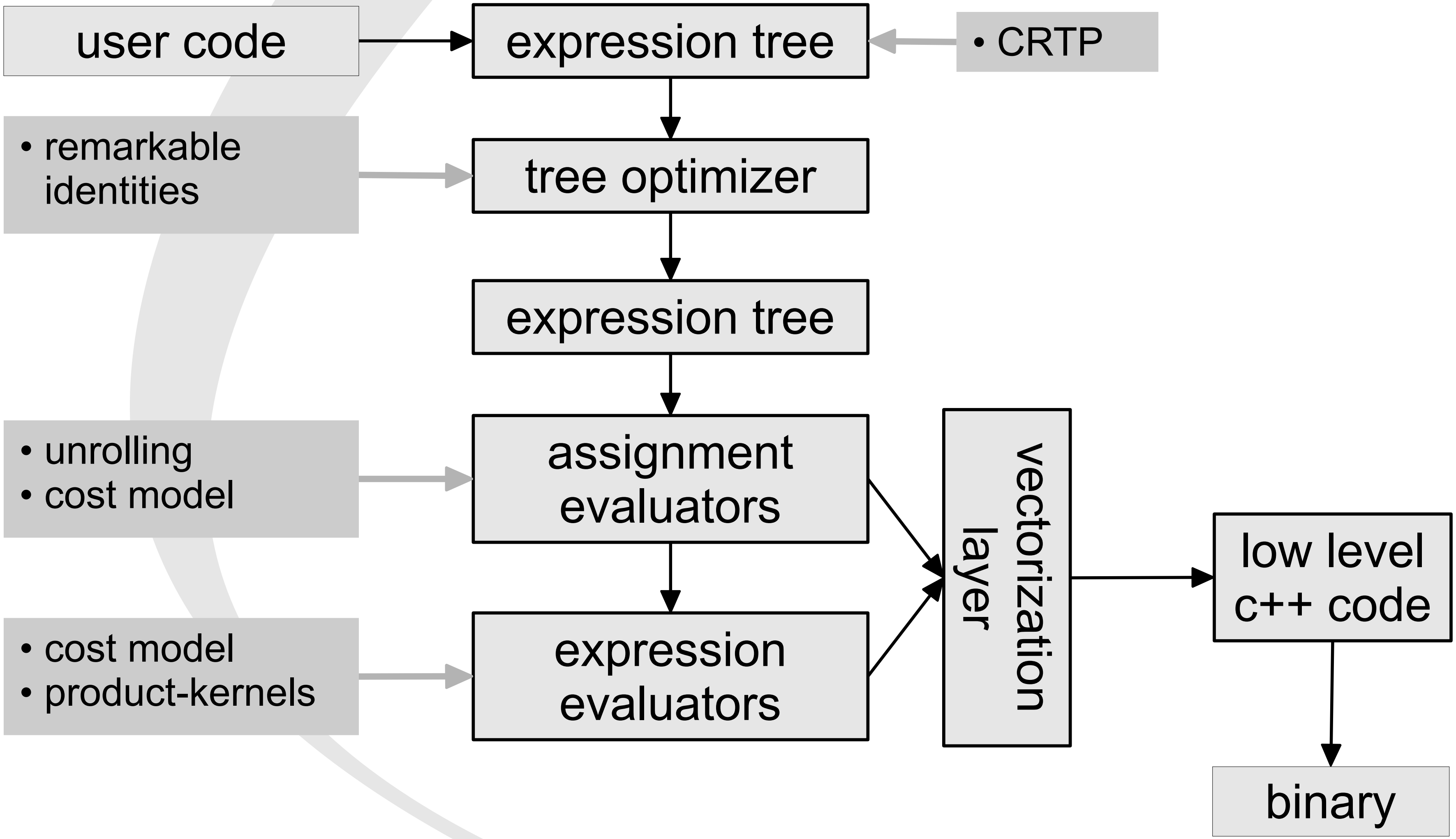
```
(2*A+B).log() + C.abs()/4
```

1 loop, but
no vectorization

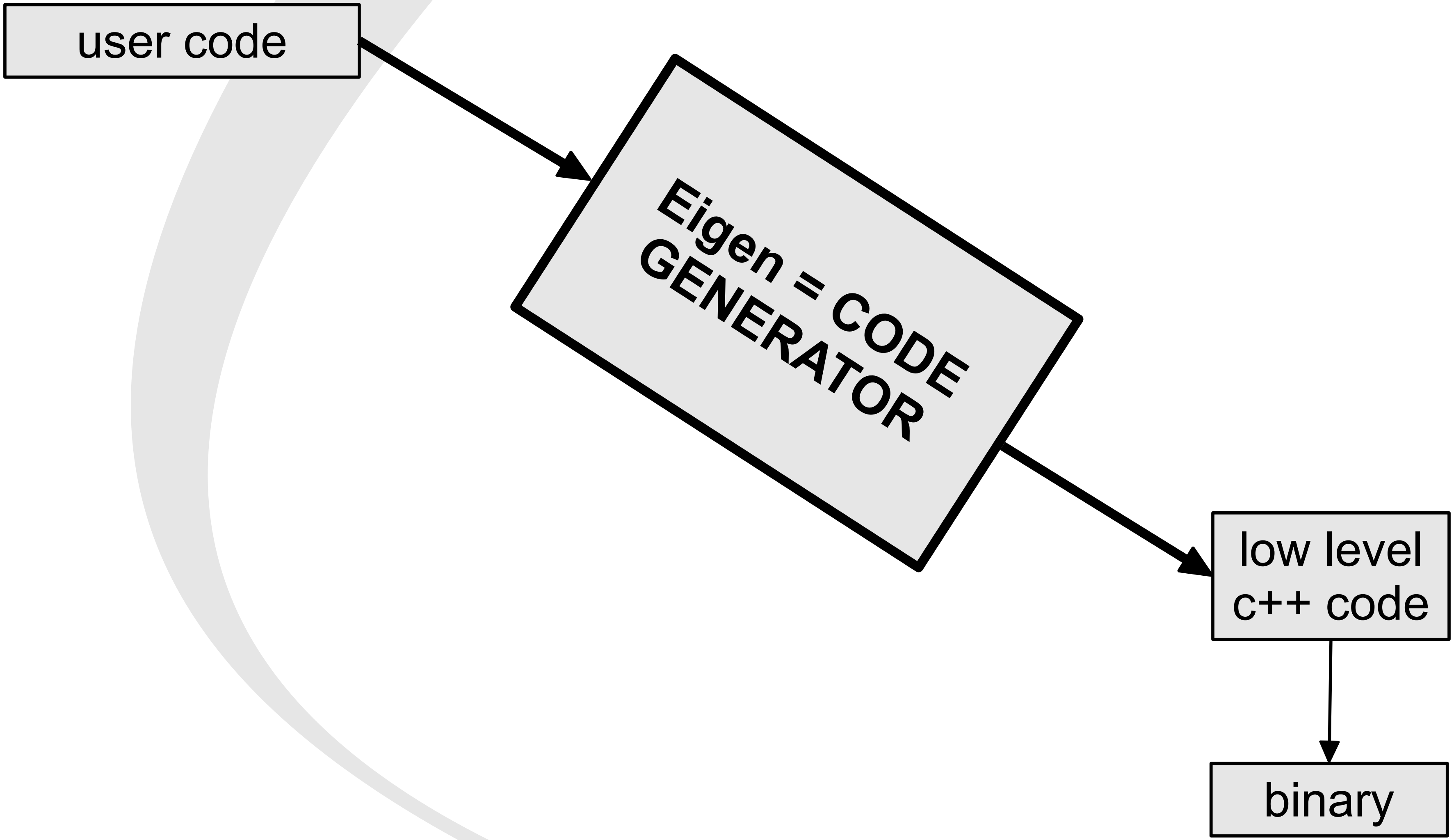
?

```
t1 = 2*A+B;       // vec on
t1 = t1.log();    // no vec
t1 + C.abs()/4;   // vec on
```

Putting everything together



Putting everything together



Reductions

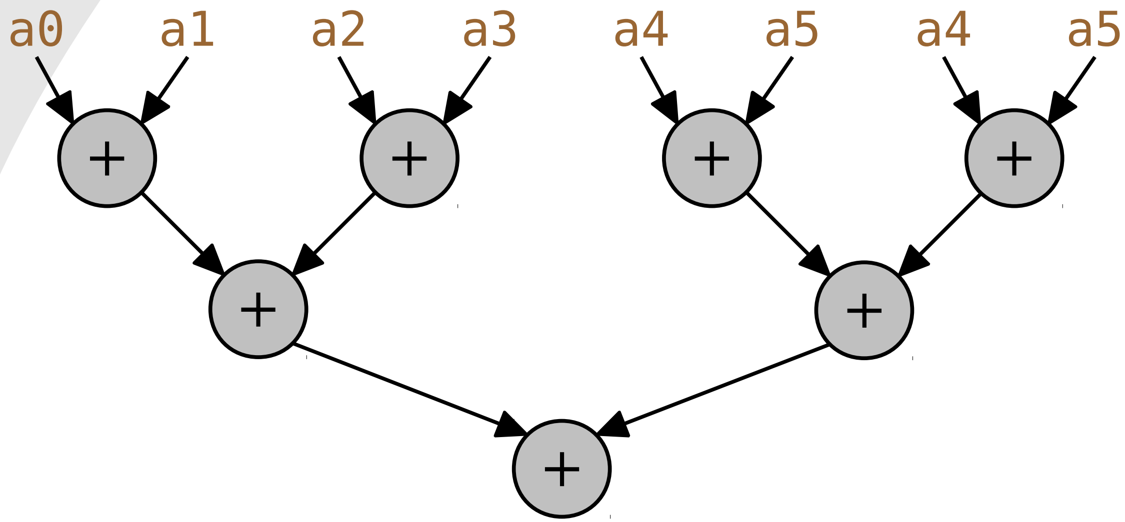
- Example: `dot = (v1.array() * v2.array()).sum();`
- Naive way:

```
class MatrixBase {
  …
  Scalar sum() const {
    Evaluator<Derived> eval(this->derived());
    Scalar acc = 0;
    for(int i=0; i<size(); ++i)
      acc = acc + eval.coeff(i);
    return acc;
  }
  …
};
```

cannot exploit
instruction level parallelism (:

Reductions

→ divide & conquer:



- Exercise: write a generic D&C meta-unroller!
  - solution in *`Eigen/src/Core/Redux.h`*

→ *demo*

Eigen tutorial

– coffee break –

Eigen tutorial

Use cases

Space Transformation & OpenGL

Space Transformations

- Needs
  - Translations, Scaling, Rotations,
  - Isometry, Affine/Projective transformation
  - … in arbitrary dimensions
- Many different approaches

Transformations
Own cooking

- Low level math:

```
Matrix3f rot = ???;
v' = rot * (2*v-p) + p;
```

```
v' = 2*rot*v + (p-rot*p)
```

- Directly manipulate a D+1 matrix:
  - form the matrix:

```
Matrix<float,3,4> T;
T.leftCols<3>() = 2*rot;
T.col(3) = p – rot * p;
```

  - apply the transformation:

```
Vector4f v1;
v1 << v, 1;
v' = T * v1;
```

```
v' = T.leftCols<3>() * v + T.col(3);
```

```
v' = T * v.homogeneous();
```

Transformations
The procedural approach

- OpenGL1 inspired

```
Transform<float,3,Affine> T;  // wrap a Matrix4f
```

```
T.setIdentity();
T.translate(p);
T.rotate(angle,axis);
T.translate(-p);
T.scale(2);
```

```
T.setIdentity();
T.preScale(2);
T.preTranslate(-p);
T.preRotate(angle,axis);
T.preTranslate(p);
```

```
v' = T * v;
```

  - cons:
    - hide the concatenation logic
      - how to concatenate on the left?
      - error prone, does help to understand transformations
      - far away to what people write on paper

Transformations
The "natural" approach

- Example:

```
Transform<float,3,Affine> T;

T = Translation3f(p) * rot * Translation3f(-p) * Scaling(2);

v' = T * v;
```

- Unified concatenate/apply → "*"

```
Translation3f(p) * v;   // compiles to "p+v"
Isometry3f T1;
T1 * Scaling(-1,2,2);   // returns an Affine3f
```

Transformations
The “natural” approach

- 3D rotations:
  - AngleAxis, Unit quaternion, Unitary matrix

```
… * AngleAxis3f(M_PI/2, Vector3f::UnitZ()) * …
```

- Unified conversions → “=”

```
Quaternionf q; q = AngleAxis3f(...);
```

- Unified inversion → .inverse()

```
Translation3f(p).inverse()    // → Translation3f(-p)
Isometry3f T1;
T1.inverse()      // →   T1.linear().transpose()
                        *Translation3f(-T1.translation())
```

Transformations
Generic programming

- Write generic optimized functions:

```
template<typename TransformationType>
void foo(const TransformationType& T) {
  T * v
  T.inverse() * v
  Projective3f(...) * T * Translation3f(...) * T.inverse()
  …
}
```

Eigen & OpenGL

- Expression templates & OpenGL

```
Vector3f p;
glUniform3fv(p.data());
```

we already know that from "p"!

```
Vector3f p1, p2;
glUniform3fv((0.5*(p1+p2)).data());
```

not available on expressions!

```
Matrix4f A, B;
glUniformMatrix4fv((A*B).eval().data());
```

Are we sure the storage order match?

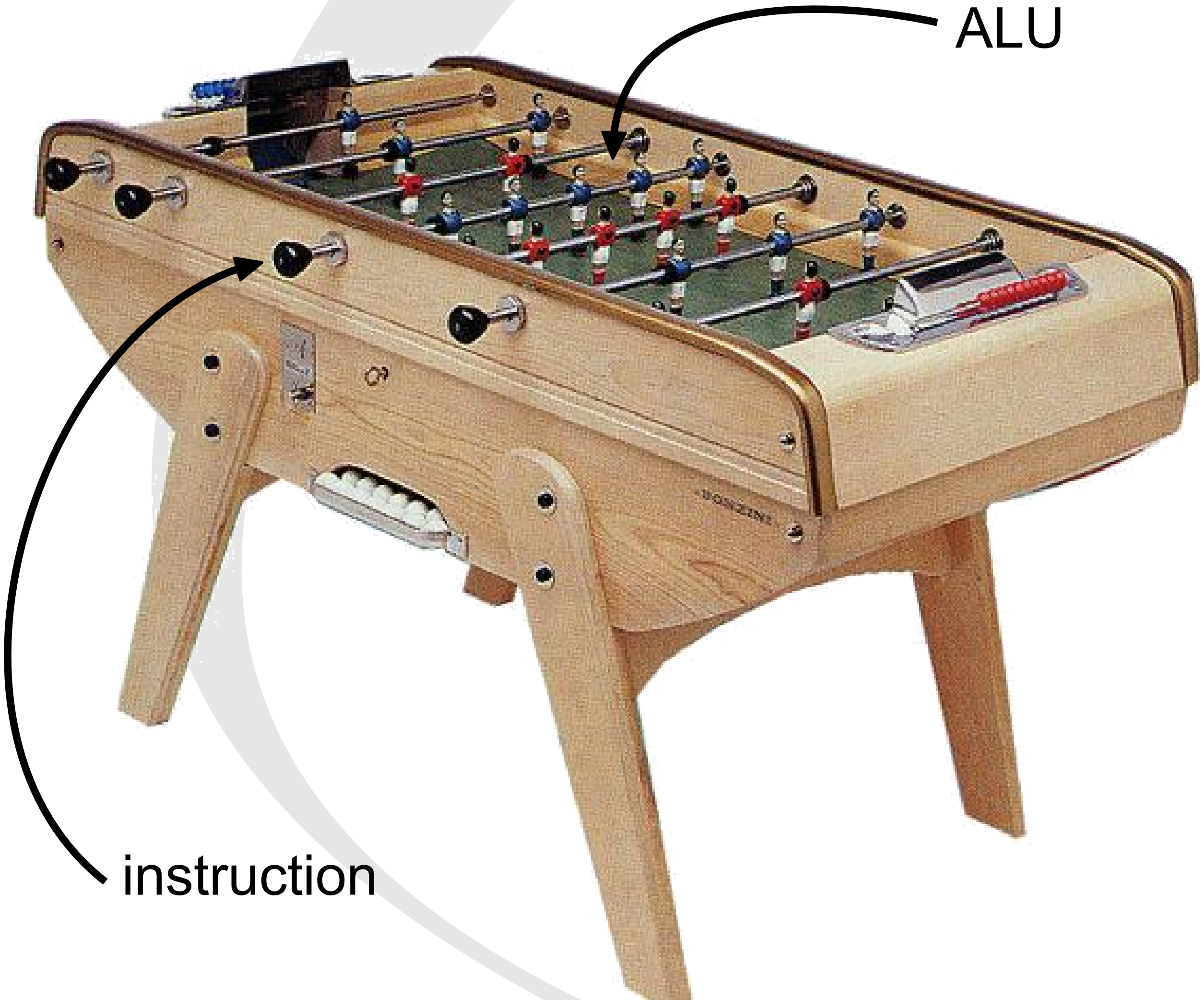
Eigen & OpenGL

- <Eigen/OpenGLSupport>

```
#include <Eigen/OpenGLSupport>
using Eigen::glUniform;

Vector3f p1, p1;
Matrix3f A;
glUniform(p1+p2);
glUniform(A*p1);
glUniform(A.topRows<2>().transpose());
…
```

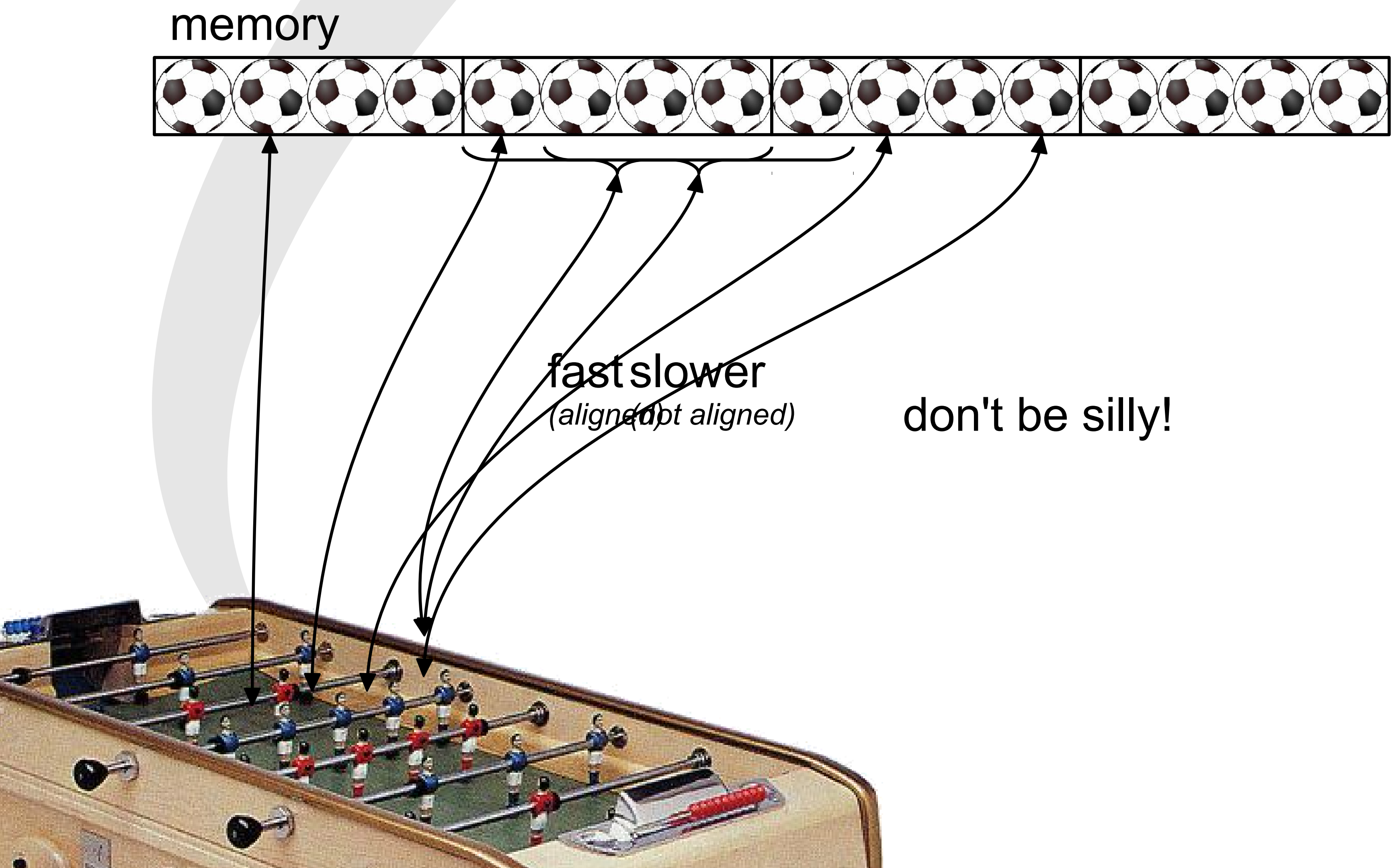
Vectorization of 3D vectors

Vectorization: difficulties?



Challenge:
put 4 balls in front of each player!

Vectorization: difficulties?



AoS versus SoA

- Array of Structure

```
std::vector<Vector3f> points_aos;
```

- Structure of Array

```
struct SoA {
  VectorXf x, y, z;
};

SoA points_soa;
```

- Example: compute the mean

```
for(int i, …) mean_aos += points_aos[i];

mean_soa <<  points_soa.x.mean(),
             points_soa.y.mean(),
             points_soa.z.mean();
```

AoS versus SoA

- Eigen's way

```
Matrix<float,3,Dynamic,ColMajor> points_aos;
Matrix<float,3,Dynamic,RowMajor> points_soa;
```

  - Highly flexible:

```
points_xxx.col(i) = Vector3f(...);

Affine3f T;
point_xxx = T * points_xxx;

mean = points_xxx.rowwise().mean();
```

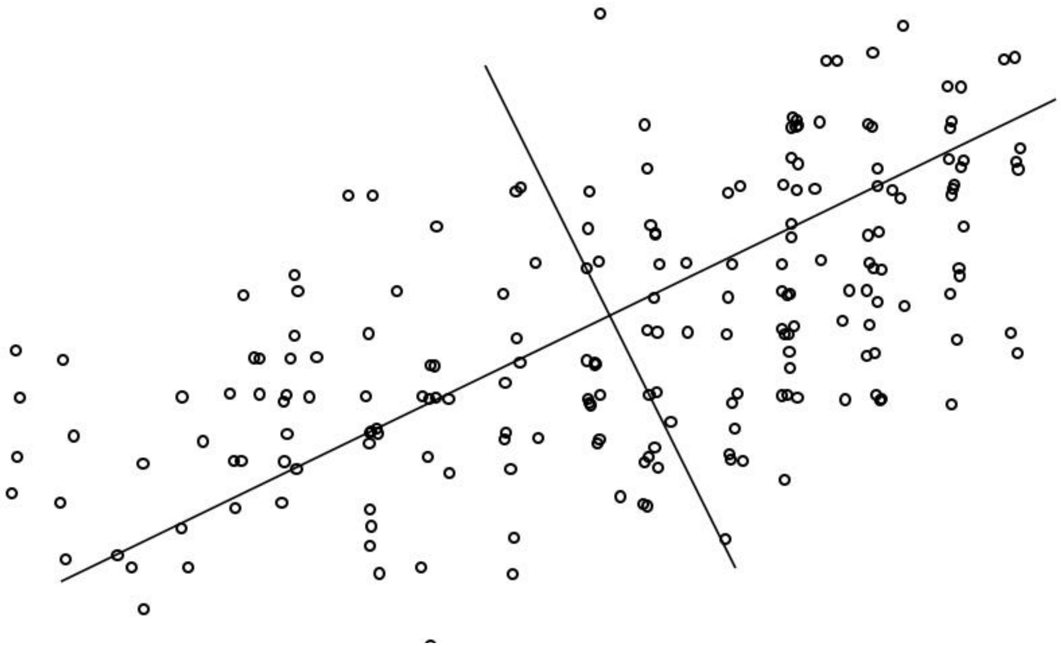
→ *demo*

Covariance Analysis

Covariance Analysis

- Example on 3D point clouds

```
Matrix3Xf points = …;
Matrix3xf c_points = points.colwise() - points.rowwise().mean();
Matrix3f cov = c_points * c_points.transpose();
SelfAdjointEigenSolver<Matrix3f> eig(cov);
```



Covariance Analysis

- Example on 3D point clouds

```
Matrix3Xf points = …;
Matrix3xf c_points = points.colwise() - points.rowwise().mean();
Matrix3f cov = c_points * c_points.transpose();
SelfAdjointEigenSolver<Matrix3f> eig(cov);
```

  - normal estimations

```
Vector3f normal = eig.eigenvectors().col(0);
```

  - local planar parameterization

```
Matrix3Xf l_points = eig.eigenvectors().transpose() * points;
```

  - compute (cheap) oriented bounding boxes

```
AlignedBox3f bbox;
for(int i=0; i<points.cols(); ++i)
  bbox.extend(l_points.col(i));
Quaternionf q(eig.eigenvectors().transpose());
```

Covariance Analysis

- Tips for 2D and 3D matrices
  - default iterative algorithm:

```
SelfAdjointEigenSolver<Matrix3f> eig;
eig.compute(cov);
```

  - closed form algorithms:
    (fast but lack a few bits of precision)

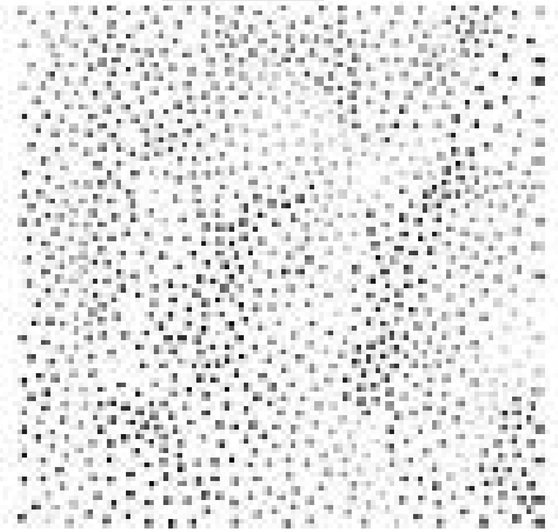
```
SelfAdjointEigenSolver<Matrix3f> eig;
eig.computeDirect(cov);
```

Linear Regression

(Dense Solvers)

Scattered Data Approximation

- Example in the functional settings



input:
- sample positions $\mathbf{p}_i$
- with associated values f_i

output:
- a smooth scalar field $f : \mathbb{R}^d \rightarrow \mathbb{R}$
  s.t., $f(\mathbf{p}_i) \approx f_i$

Basis Functions Decomposition

- Express the solution as:

$$f(\mathbf{x}) = \sum_j \alpha_j \phi_j(\mathbf{x})$$

- Radial Basis Functions

$$f(\mathbf{x}) = \sum_j \alpha_j \phi(\|\mathbf{x} - \mathbf{q}_j\|)$$

  - example:
    - $\phi(t) = t^3$
    - $\mathbf{q}_j$? $\rightarrow$ evenly distributed

Matrix formulation

- Least square minimization: $\boldsymbol{\alpha} = \underset{\boldsymbol{\alpha}}{argmin} \sum_i \left(f(\mathbf{p}_i) - f_i \right)^2$
- Matrix form:

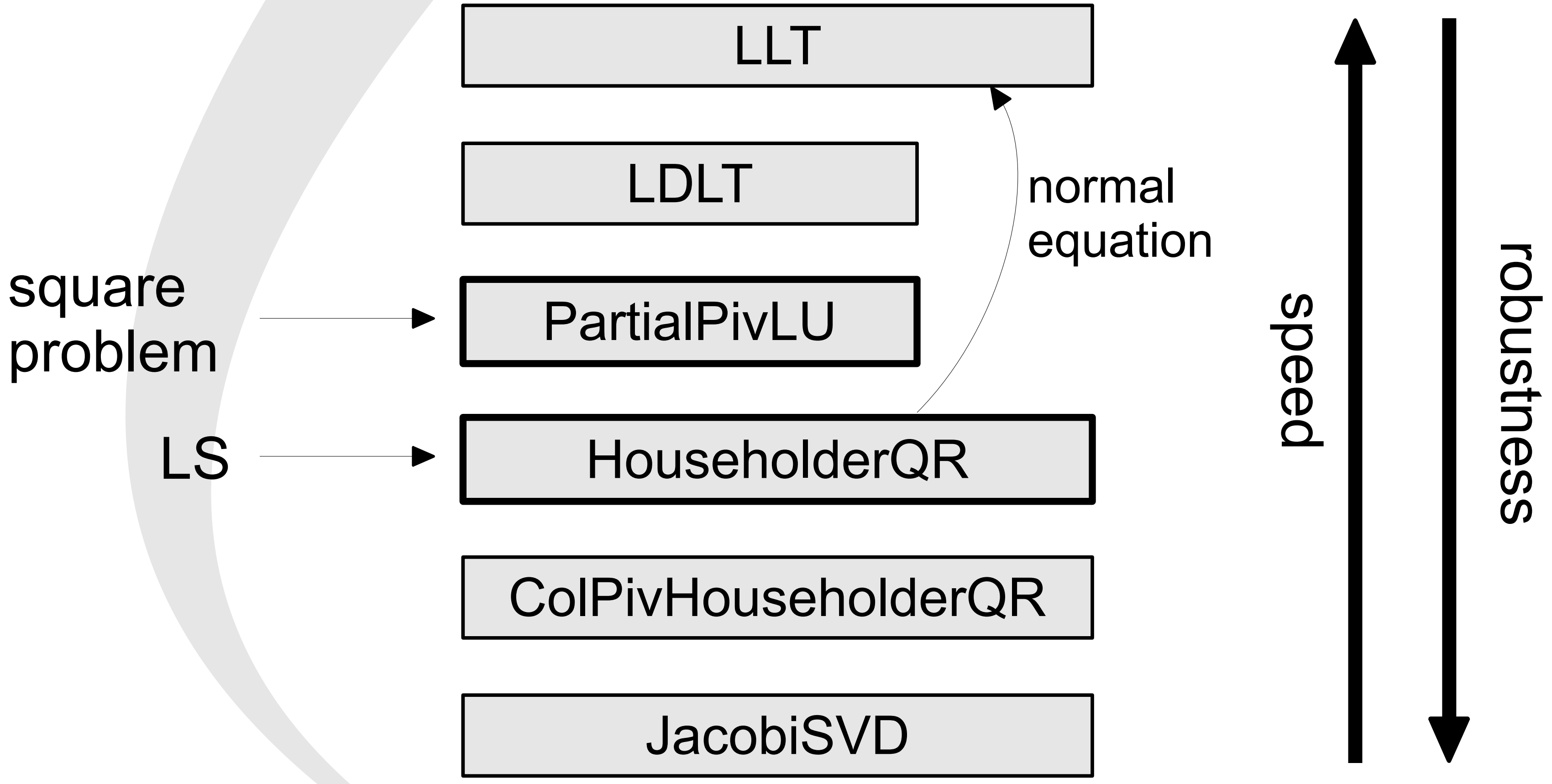
$$\begin{bmatrix} & \vdots & \\ \cdots & \phi\left(\|\mathbf{p}_i - \mathbf{q}_j\|\right) & \cdots \\ & \vdots & \end{bmatrix} \cdot \boldsymbol{\alpha} = \begin{bmatrix} \vdots \\ f_i \\ \vdots \end{bmatrix} \quad \rightarrow \quad \boldsymbol{\alpha} = \underset{\boldsymbol{\alpha}}{arg\,min} \left\| \mathrm{A}\,\boldsymbol{\alpha} - \mathbf{b} \right\|^2$$

  - as many unknowns as constraints: $\Leftrightarrow \quad \mathrm{A}\,\boldsymbol{\alpha} = \mathbf{b}$

    → interpolation

→ *demo*

Dense Solvers



(bi-)Harmonic interpolation

(Sparse Solvers)

Laplacian equation

$$\Delta f = 0$$

$$\Delta f = \nabla \cdot \nabla f = \frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \cdots$$

- Many applications
  - interpolation
  - smoothing
  - regularization
  - deformations
  - parametrization
  - etc.

[courtesy of A. Jacobson et al.]

Laplacian equation

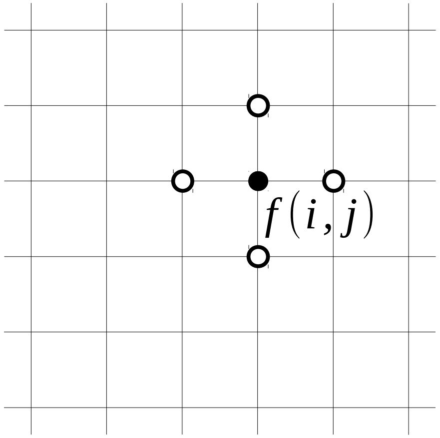
$$\Delta f = 0$$

[courtesy of A. Jacobson et al.]

Discretization

- Example on a 2D grid:

$$\Delta \Leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



$$\Delta f(i,j) = \frac{\left(f(i-1,j) + f(i+1,j) + f(i,j-1) + f(i,j+1)\right)}{4} - f(i,j) \;=\; 0$$

  - Matrix form:

$$\mathbf{L}\,\mathbf{f} = 0$$

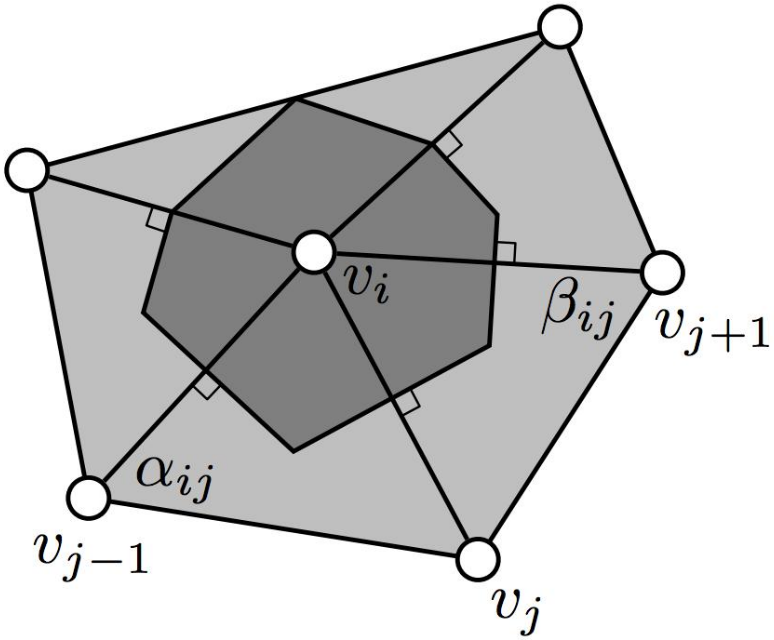
Discretization

- On a 3D mesh:

$$\Delta f(v_i) = \sum_{v_j \in N_1(v_i)} L_{i,j}\left(f(v_j) - f(v_i)\right)$$

$$L_{i,j} = \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{A_i + A_j}$$

$$L_{i,i} = -\sum_{v_j \in N_1(v_i)} L_{i,j}$$



[courtesy of M. Botsch and O. Sorkine]

→ *demo*

Sparse Representation

- Naive way:

```
std::map<pair<int,int>, double>
```

- Eigen::SparseMatrix
  - Matrix:

0	3	0	0	0
22	0	0	0	17
7	5	0	1	0
0	0	0	0	0
0	0	14	0	8

  - Compressed Column-major Storage:

Values:	22	7	3	5	14	1	17	8
InnerIndices:	1	2	0	2	4	2	1	4

OuterStarts:	0	2	4	5	6	*8*

Constraints

- Dirichlet boundary conditions
  - fix a few (or many) values: $f(v_i) = \bar{f}_i \,, \quad v_i \in \Gamma$
  - updated problem:

$$\begin{bmatrix} \mathrm{L}_{00} & \mathrm{L}_{01} \\ \cancel{\mathrm{L}_{10}} & \cancel{\mathrm{L}_{11}} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{f}} \\ \bar{\mathbf{f}} \end{bmatrix} = \begin{bmatrix} 0 \\ \cancel{0} \end{bmatrix} \quad \Rightarrow \mathrm{L}_{00} \cdot \hat{\mathbf{f}} = -\mathrm{L}_{01} \cdot \bar{\mathbf{f}}$$

→ *demo*

Bi-harmonic interpolation

- Continuous formulation:

$$\Delta \cdot \Delta f = 0$$

- Discrete form:

$$\mathbf{L} \cdot \mathbf{L} \cdot \mathbf{f} = 0$$

→ *demo*

Solver Choice

- Questions:
  - Solve multiple times with the same matrix?
    - yes → direct methods
  - Dimension of the support mesh
    - 2D → direct methods
    - 3D → iterative methods
  - Can I trade the performance? Good initial solution?
    - yes → iterative methods
  - Hill conditioned?
- Still lost? → sparse benchmark

→ *demo*

What next?

Coming soon: 3.2

- Already in 3.2-beta1
  - SparseLU
  - SparseQR
  - GeneralEigenSolver (Ax=lBx)
  - Ref<>
    - write generic but non-template function!

WIP: AVX

- AVX
  - SIMD on 256bits register (8 floats, 4 doubles)
  - … or 128bits (4 floats, 2 doubles)
- Challenge
  - select the best register-width

WIP: CUDA

- Why only now?
  - CUDA 5 made it possible
- Roadmap
  - call Eigen from CUDA kernel
    - useful for small fixed size algebra
  - add a CudaMatrix class
    - coefficient-wise ops
      - special assignment evaluator
    - products & solvers
      - wrap optimized CUDA libraries (ViennaCL, CuBlas, Magma, etc.)

WIP: SparseMatrixBlock

- SparseMatrixBlock
  → “SparseMatrix<Matrix4f>”
  - useful with high-order elements
  - classic with iterative methods (→ ViennaCL)
  - would be a first with direct methods!
    - huge speed-up expected!

WIP: Non-Linear Optimization

- Non-linear least square
  - Generic Levenberg-Marquart
    - Dense & Sparse
- Quadratic Programming
  - linear least-square + inequalities

WIP: utility modules

- Auto-diff
- Polynomials
  - differentiation & integration

Concluding remarks

License

- Initially:
  - LGPL3+
    - → default choice
    - → not as liberal as it might look...
- Now:
  - MPL2 (Mozilla Public License 2.0)
    - same spirit but with tons of advantages:
      - accepted by industries
      - do work with header only libraries
      - versatile (apply to anything)
      - a lot simpler
      - good reputation

Developer Community

- Jan 2008: start of Eigen2
  - part of KDE
    - packaged by all Linux distributions
  - open repository
  - open discussions on mailing/IRC
    - 300 members, 300 messages/month

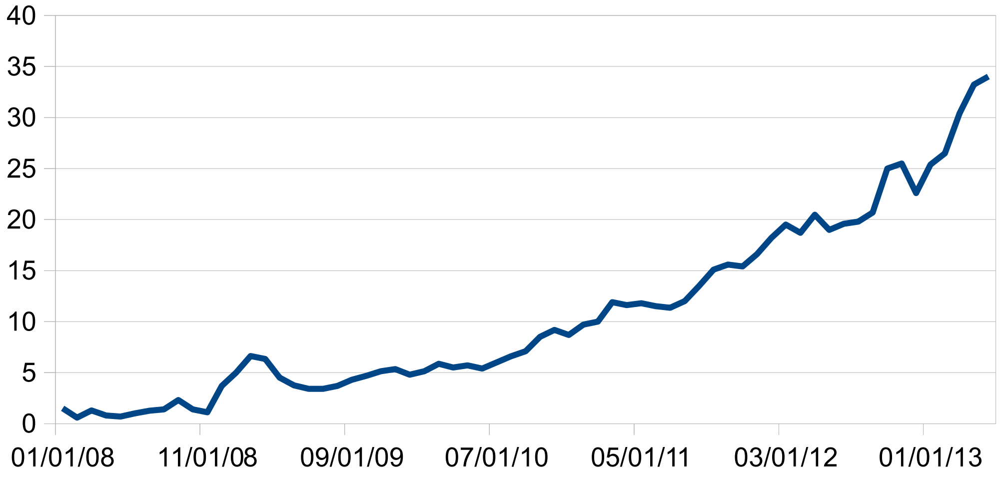
      → good quality API
- Today
  - most development @ Inria (Gaël + full-time engineer)
- Future

  → consortium... ??

User community

- Active project with many users
  - Website:

    ~30k unique visitors/months



- Major domains
  - Geometry processing, Robotics, Computer vision, Graphics

Summary

- **Many unique features**:
  - C++ friendly API
  - Easy to use, install, distribute, etc.
  - Versatile
    - small, large, sparse
    - custom scalar types
    - large set of tools
  - No compromise on performance
    - static allocation, temporary removal, unrolling, auto vectorization, cache-aware algorithms, multi-threading, etc.
  - Multi-platforms

Acknowledgements

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  - Benoit Jacob,
  - Jitse Niesen,
  - Hauke Heibel,
  - Désiré Nuentsa,
  - Christoph Hertzberg,
  - Thomas Capricelli

  + 100 others

- You're welcome to join!
  - documentation
  - bug report/patches
  - write unit tests
  - discuss the future on ML
  - ...

  - don't be shy!